§ 1.1 Q. 12: Find all solutions of the linear system $\Delta$ represent graphically, as intersections of lines in the plane.

\begin{align*}
  x - 2y &= 3 \\
  2x - 4y &= 6 \\

\text{so we have} \quad & 2(x - 2y = 3) \\
\text{and take} \quad & y \text{ to be free, let } y = t.
\end{align*}

Then $x = 3 + 2t$, so

\begin{align*}
  x &= 3 + 2t \\
  y &= t
\end{align*}

is the solution set to this system of linear equations.

\begin{align*}
  x - 2y &= 3 \\
  2x - 4y &= 6
\end{align*}

\begin{align*}
  (0, \frac{3}{2}) & \quad \text{(0, 1\frac{1}{2}) line} \\
  (3, 0) & \quad \text{on the graph of this line} \\
  2x - 4y &= 6
\end{align*}

\begin{align*}
  x &= 3 + 2t \\
  (3, 0) & \quad \text{are on the} \\
  \text{graph of this line.} \\

\text{These lines are the same line.}
\end{align*}
Example 0.14, 16 Find all solutions and describe in terms of intersecting planes. (You do not need to sketch these planes.)

14: \( x + 4y + z = 0 \)
\( 4x + 13y + 7z = 0 \) (double this is \( 8x + 26y + 14z = 0 \))
\( 7x + 22y + 13z = 1 \)
\( -(x + 4y + z = 0) \)
\( 7x + 22y + 13z = 0 \)

Subtracting one of these from the other, we find that \( 0 = 1 \); the system of equations is inconsistent; it has no solutions.

The planes described by the first and second equations intersect in a line, but this line does not intersect the plane described by the third equation. Instead, the line is parallel to the plane.

(This is a rough picture of what this kind of situation looks like; it is not intended to be an accurate graph of the plane & line from this problem.)
20) First write the system as an augmented matrix:

\[
\begin{pmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 1 & 3 \\
1 & 1 & k^2 - 5 & k
\end{pmatrix}
\]

Although we don’t know what \( k \) is, let’s try row reducing. Subtract the top row from those below it. Afterwards, subtract the second row from the first.

\[
\begin{pmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 2 & 1 \\
0 & 0 & k^2 - 4 & k - 2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & k^2 - 4 & k - 2
\end{pmatrix}
\]

To proceed, we need to make the first entry of the bottom row a 1. This can be done by dividing through by \( k^2 - 4 \). So, assuming that \( k^2 - 4 \neq 0 \), we get the first matrix below. A little more cleaning up then yields the RREF

\[
\begin{pmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 1/(k^2 + 2)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & 0 & (k + 5)/(k + 2) \\
0 & 1 & 0 & k/(k + 2) \\
0 & 0 & 1 & 1/(k + 2)
\end{pmatrix}
\]

As we can clearly see, our system of solutions has a unique solution in terms of \( k \). To finish our analysis, we need to go back to before we divided by \( k^2 - 4 \) and consider what happens in the case that \( k^2 - 4 = 0 \). Of course, this happens whenever \( k = 2 \) or \( k = -2 \).

If we plug in \( k = 2 \), we get

\[
\begin{pmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & k^2 - 4 & k - 2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

This matrix is already in RREF, and we can see that there are infinitely many solutions. The general solution is of the form \((x, y, z) = (1 + 3z, 1 - 2z, z)\).

On the other hand, if we plug in \( k = -2 \), we get

\[
\begin{pmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & k^2 - 4 & k - 2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

The last line as an equation reads \( 4x + 2y + 0z = -4 \), so the system is inconsistent and has no solutions.
Solution for Question 43, part (a):

For each fixed value of $t \in \mathbb{R}$, the solutions to the equation $x + \frac{t}{2}y = t$ is the line $L_t$ (which depends on $t$) in the $(x,y)$-plane whose $x$-intercept $t$ and $y$-intercept is 2. The family of lines $\{L_t : t \in \mathbb{R}\}$ are exactly the lines which pass through $(0,2)$ and intersect the $x$-axis. The parameter $t$ is a way to name these lines using real numbers (that is, we call them by their $x$-intercept). Another way of parameterizing these lines is by their slope. The slope of $L_t$ is $-2/t$ (where $L_t$ is regarded as having slope $\pm \infty$).

The solution set of $x + y = 1$ is a line with $x$-intercept 1 and $y$-intercept 1. Let’s denote this line by $L$. It’s distinct from $L_t$ for all $t$ and its slope is $-1$.

We are interested in the intersection of $L$ with $L_t$ (considered as a function of $t$). Two distinct lines in the plane intersect if and only if they aren’t parallel. The slope of $L_t$ is $-1$ exactly when $t = 2$. Hence, $L_t$ and $L$ intersect for all values of $t \neq 2$. Let $(x(t), y(t))$ denote the point of intersection between the lines $L_t$ and $L$.

The lines $L_t$ vary continuously with $t$. One way to interpret this is: if one considers a bounded coordinate plane with a fixed scale (i.e. the type of coordinate axes we’d draw to graph a line), for suitably close values $t', t \in \mathbb{R}$ the graphs of the solution sets $L_{t'}$ and $L_t$ (on this bounded plane) are very close to one another. In this sense, one may also regard the line of slope 0 which passes through $(0,2)$ as the limit of the family $L_t$ as $t \to \infty$ or $t \to -\infty$, that is if one considers a bounded coordinate plane with a fixed scale (i.e. the type we’d use to graph a line), then the graph of the solution set $L_t$ for $|t| \gg 0$ is (qualitatively) indistinguishable from the graph of the line of slope 0 which passes through $(0,2)$. We will denote this limiting line $L_{\pm \infty}$.

When $t$ is very large in magnitude, the line $L_t$ is almost horizontal and has a graph that looks very similar to the line $L_{\pm \infty}$. It follows as $t \to \infty$ (or $t \to -\infty$), the intersection point of $L_t$ and $L$ gets closer and closer to the intersection between $L_{\pm \infty}$ and $L$. This intersection is the point $(-1,2)$. As a consequence, $x(t)$ has a horizontal asymptote at $-1$ as $t \to \pm \infty$ and $y(t)$ has a horizontal asymptote at $2$ as $t \to \pm \infty$.

Geometrically, the effect of increasing $t$ from $-\infty$ to $\infty$ rotates the line $L_t$ counterclockwise around the point $(0,2)$. Since $(0,2)$ is to the right of the line $L$, starting with the horizontal line $L_{\pm \infty}$ and rotating around $(0,2)$ has the effect of moving the point of intersection $(x(t), y(t))$ down and to the right along the entire length of the ray in $L$ starting from $(-1,2)$. It follows that $x(t)$ approaches its horizontal asymptote at $-1$ from above as $t \to -\infty$, is increasing on the range $\infty < t < 2$, and has a vertical asymptote at $t = 2$. Similarly, $y(t)$ approaches its horizontal asymptote at $2$ from below as $t \to -\infty$, is decreasing on the range $\infty < t < 2$, and has a vertical asymptote at $t = 2$. A similar explanation (rotating clockwise) shows that $x(t)$ approaches its horizontal asymptote at $-1$ from below as $t \to \infty$, and is increasing on the range $2 < t < \infty$, whereas $y(t)$ approaches its horizontal asymptote at $-1$ from above as $t \to \infty$, and is decreasing on the range $2 < t < \infty$.

Graphs of the $x$- and $y$-coordinates of the intersection of $L_t$ and $L$, in terms of $t$, are on the next page.
Based on the previous arguments, the graphs should be as shown here above.
§1.1 § 43. (6) \( x + y = 1 \) \( \rightarrow \) \( x + y = 1 \)
\( x + \frac{1}{2} y = t \) \( \rightarrow \) \( \frac{1}{2} y = t - 1 \)

If \( t = 2 \), the 2nd eqn. is \( dy = 1 \), and the system is inconsist.

\( t \neq 2 \), we divide by \( \frac{1}{2} y = \frac{t - 2}{2} \):

\( x + y = 1 \)
\( y = \frac{2(t - 1)}{t - 2} \) \( \rightarrow \) \( x = 1 - \frac{2(t - 1)}{t - 2} \)

Now, taking derivatives:
\[
\frac{dy}{dt} = \frac{2(t - 2) - 2(t - 1)}{(t - 2)^2} = \frac{-2}{(t - 2)^2}, \text{ so whenever } t \neq 2,
\]
y is decreasing as \( t \) increases, and \( y \) approaches \(-\infty\) from the left and \(+\infty\) from the right as \( t \to 2 \).

\[
\frac{dx}{dt} = \frac{2}{(t - 2)^2}, \text{ so whenever } t \neq 2, x \text{ increases as } t \text{ does, and } x \text{ approaches } \infty \text{ from the left and } -\infty \text{ from the right as } t \to 2.
\]

This all agrees with what is shown in the graphs above.
Section 1.2

2) As a matrix, the system of equations is
\[
\begin{pmatrix}
3 & 4 & -1 \\
6 & 8 & -2
\end{pmatrix}
\begin{pmatrix}
8 \\
3
\end{pmatrix}
\]

Subtract twice the first row from the second to get
\[
\begin{pmatrix}
3 & 4 & -1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
8 \\
-13
\end{pmatrix}
\]

This system is inconsistent.

4) As a matrix, the system of equations is
\[
\begin{pmatrix}
1 & 1 & 1 \\
2 & -1 & 5 \\
3 & 4 & 2
\end{pmatrix}
\]

Subtracting multiples of the first row from those below it we get
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & -3 & 3 \\
0 & 1 & -1
\end{pmatrix}
\]

Finally, divide the middle row by -3 and subtract it from the other rows to get
\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}
\]

Reinterpreting this matrix as equations, we see that we have a unique solution with \(x = 2, y = -1\).
\[ 8 \]

\[ \begin{align*}
\xi_{12} \quad & x_1 - 7x_2 + x_5 = 3 \\
& x_3 - 2x_5 = 2 \\
& x_4 + x_5 = 1
\end{align*} \]

Notice that \[
\begin{bmatrix}
1 & -7 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

is already in reduced row echelon form.

So, all that remains is to solve for the leading variables \((x_1, x_3, x_4)\) in terms of the other variables \((x_2, x_5)\)

\[ x_1 = 3 + 7x_2 - x_5 \]
\[ x_3 = 2 + 2x_5 \]
\[ x_4 = 1 - x_5 \]

Letting \(x_2 = s\) and \(x_5 = t\), the solutions to this linear system are:

\[
\begin{bmatrix}
3 + 7s - t \\
5 \\
2 + 2t \\
1 - t \\
t
\end{bmatrix}
\]

where \(s, t\) can be any real numbers.
§ 1.2 10. \[ 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \]
\[ 5x_1 + 4x_2 + 3x_3 - x_4 = 4 \]
\[-2x_1 - 2x_2 - x_3 + 2x_4 = -3 \]
\[ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \]

Start by clearing downwards:
get leading 1's below them to be 0.

\[ R_2 - R_1 \Rightarrow R_2 : \]
\[ 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \]
\[ x_1 + x_2 + x_3 = 0 \]
\[ R_4 , \quad 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \]
\[-2x_1 - 2x_2 - x_3 + 2x_4 = -3 \]
\[ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \]

\[ x_1 + x_2 + x_3 = 0 \]

\[ R_2 - 4R_1 \Rightarrow R_2 : \]
\[-x_2 - 2x_3 - x_4 = 4 \]
\[-5x_2 - 7x_3 + x_4 = 11 \]

\[ R_4 - 5R_2 \Rightarrow R_4 : \]
\[ 3x_3 + 6x_4 = -9 \]

\[ x_1 + x_2 + x_3 = 0 \]

\[ R_2 + 1 \Rightarrow R_2 : \]
\[ x_2 + 2x_3 + x_4 = -4 \]

\[ R_4 , \frac{1}{3} \Rightarrow R_4 : \]
\[ x_3 + 2x_4 = -3 \]

\[ R_1 - R_3 \Rightarrow R_1 : \]
\[ x_1 + x_2 = 3 \]
\[ x_1 - R_2 \Rightarrow R_1 : \]
\[ x_1 + x_4 = 1 \]

\[ R_2 - 2R_3 \Rightarrow R_2 : \]
\[ x_2 - 3x_4 = 2 \]
\[ x_3 + 2x_4 = -3 \]

\[ 0 = 0 \]

is in reduced row echelon form.

\[ \text{Solutions are} \]
\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t \\ 2+3t \\ -3-2t \\ t \end{bmatrix} \]

For any real number \( t \).
20) The matrix under consideration is
\[
\begin{pmatrix}
0 & a & 2 & 1 & b \\
0 & 0 & 0 & c & d \\
0 & 0 & c & 0 & 0
\end{pmatrix}
\]

For this matrix to be in RREF, the first nonzero entry of each row must be a 1. It follows that \(a, c, e\) are each either 0 or 1. Next we observe that \(a\) cannot be 0, or the first nonzero entry of row 1 would be a 2. Therefore, \(a = 1\). On the other hand, \(e\) cannot be 1. This is because \(e\) would then be a pivot and so all other entries in its column would need to be 0. Therefore, \(e = 0\). By the exact same reasoning, \(c = 0\) as well. Our matrix now looks like
\[
\begin{pmatrix}
0 & 1 & 2 & 1 & b \\
0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

At this point, we observe that \(d\) is either 0 or 1. If \(d = 1\), then \(d\) is a pivot and so everything else in its column must be 0. In particular, we would get \(b = 0\):
\[
\begin{pmatrix}
0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

On the other hand, if \(d = 0\) then \(b\) can be anything!
\[
\begin{pmatrix}
0 & 1 & 2 & 1 & b \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
When does the system below have a unique solution, no solution, or only many solutions?

\[
\begin{align*}
\gamma + 2kz &= 0 \\
x + (6-4k)z &= 2 \\
x + 2y + 6z &= 2 \\
kx + 2z &= 1
\end{align*}
\]

(switching the top 2 rows & clearing the 2nd column).

\[
\begin{align*}
x + (6-4k)z &= 2 \\
y + 2kz &= 0 \\
(-k(6-4k)+2)z &= 1-2k
\end{align*}
\]

declearing the 1st column.

\[
\begin{align*}
k = \frac{1}{2}: \\
x + (6-4k)z &= 2 \\
y + 2kz &= 0 \\
z &= 0
\end{align*}
\]

z is free; the system has infinitely many solutions,

\[
\begin{align*}
x &= 2 - 4z \\
y &= -z \\
z &= \text{can be any real number}
\end{align*}
\]

\[
k = 1: \text{The last equation becomes} \\
0z = -1, \text{so the system has no solutions in this case.}
\]

\[
k \neq \frac{1}{2}, k \neq 1: \\
x + (6-4k)z = 2 \\
y + 2kz = 0 \\
z = \frac{-((2k-1)-(2k-2))}{(2k-1)(2k-2)} = \frac{-1}{2k-2}
\]

clearing the third column.

\[
\begin{align*}
x &= 2 + \frac{(6-4k)}{2k-2} \\
y &= \frac{2k}{2k-2} \\
z &= \frac{-1}{2k-2}
\end{align*}
\]

so the system has a unique solution for these values of k.

Thus, if \( k = \frac{1}{2} \) there are infinitely many solutions (part (a)), if \( k = 1 \) there are no solutions (part (c)), and otherwise, the system has a unique solution (part (a)).
A quick solution here is to note that \( x_k = (1/2)(x_{k-1} + x_{k+1}) \) is equivalent to \( x_{k+1} - x_k = x_k - x_{k-1} \). Therefore, our equations say that all differences are the same. This means that a tuple of numbers \((x_1, x_2, \ldots, x_n)\) solves our equations if and only if it is an arithmetic progression. From this point, we can finish the problem without any linear algebra. This solution is very efficient, but also nonobvious.

Instead, a more reasonable approach may be to rewrite our equations in the form \( x_{k-1} - 2x_k + x_{k+1} = 0 \) and apply row reduction. Row reducing a general \( n - 2 \) by \( n \) matrix is rather difficult, so let’s start with the case \( n = 5 \). Then, our augmented matrix looks like

\[
\begin{pmatrix}
1 & -2 & 1 & 0 & 0 & | & 0 \\
0 & 1 & -2 & 1 & 0 & | & 0 \\
0 & 0 & 1 & -2 & 1 & | & 0 \\
\end{pmatrix}
\]

Add twice the second row to the first row:

\[
\begin{pmatrix}
1 & 0 & -3 & 2 & 0 & | & 0 \\
0 & 1 & -2 & 1 & 0 & | & 0 \\
0 & 0 & 1 & -2 & 1 & | & 0 \\
\end{pmatrix}
\]

Now add twice the last row to the middle row and three times the last row to the first row. We get

\[
\begin{pmatrix}
1 & 0 & 0 & -4 & 3 & | & 0 \\
0 & 1 & 0 & -3 & 2 & | & 0 \\
0 & 0 & 1 & -2 & 1 & | & 0 \\
\end{pmatrix}
\]

A pattern seems to be emerging. It looks like in the general case row reduction will yield the matrix

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & -(n-1) & n-2 & | & 0 \\
0 & 1 & \ldots & 0 & -(n-2) & n-3 & | & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & | & \vdots \\
0 & 0 & \ldots & 1 & -2 & 1 & | & 0 \\
\end{pmatrix}
\]

Converting back to equations, we see we can write all of the other variables in terms of \( x_{n-1} \) and \( x_n \). In particular we get equations

\[
x_1 = (n-1)x_{n-1} - (n-2)x_n, \quad x_2 = (n-2)x_{n-1} - (n-3)x_n, \quad \ldots \quad x_{n-2} = 2x_{n-1} - x_n
\]

We can see that the general pattern is \( x_k = (n-k)x_{n-1} - (n-k-1)x_n \). If we let \( x_{n-1} = s, \ x_n = t \), then our general answer is

\[
x_k = (n-k)s - (n-k-1)t \text{ if } 1 \leq k \leq n-2, \quad x_{n-1} = s, \ x_n = t
\]