§ 1.1 Q. 12: Find all solutions of the linear system $A$ represent graphically as intersections of lines in the plane.

\[
\begin{align*}
    x - 2y &= 3 \\
    2x - 4y &= 6 \\
    2x - 4y &= 6 - 2(x - 2y = 3) \\
    0 &= 0
\end{align*}
\]

So we take $y$ to be free.

Then $x = 3 + 2t$, so let $y = t$.

$x = 3 + 2t$ is the solution set to this system of linear equations.

\[
\begin{align*}
    x - 2y &= 3 \\
    (0, \frac{-3}{2}) &\text{ are on the graph of this line} \\
    2x - 4y &= 6 \\
    (3, 0) &\text{ are on the graph of this line,} \\
    \text{These lines are the same line.}
\end{align*}
\]
Find all solutions and describe in terms of intersecting planes. (You do not need to sketch these planes.)

14: \( x + 4y + z = 0 \)
\[ 4x + 13y + 7z = 0 \] (double this is \( 8x + 26y + 14z = 0 \))
\[ 7x + 22y + 13z = 1 \]

Subtracting one of these from the other, we find that \( 0 = 1 \); the system of equations is inconsistent; it has no solutions.

The planes described by the first and second equations intersect in a line, but this line does not intersect the plane described by the third equation. Instead, the line is parallel to the plane.

(This is a rough picture of what this kind of situation looks like; it is not intended to be an accurate graph of the plane & line from this problem.)
§ 1.1 Q 16: \[ \begin{align*}
    x + 4y + z &= 0, \\
    4x + 13y + 7z &= 0, \\
    7x + 22y + 13z &= 0
    \end{align*} \]
\[ 8x + 76y + 44z = 0 \]
\[ -(x + 4y + z = 0) \]
\[ 7x + 22y + 13z = 0 \]

So, we can instead solve:
\[ \begin{align*}
    x + 4y + z &= 0, \\
    7x + 22y + 13z &= 0, \\
    7x + 22y + 13z &= 0
    \end{align*} \]
\[ \downarrow \text{using the 2nd row to reduce the 3rd} \]
\[ \begin{align*}
    x + 4y + z &= 0, \\
    7x + 22y + 13z &= 0, \\
    0x + 0y + 0z &= 0
    \end{align*} \]
\[ \rightarrow \text{using the 1st row to reduce the 2nd} \]
\[ x + 4y + z = 0, \quad -6y + 6z = 0, \quad 0 = 0 \]

We now take \( z \) to be a free variable; from the 2nd equation we see that \( y = z \), and then \( x = -5z \). Thus, the solutions to this system of equations are of the form \( \begin{bmatrix} -5z \\ z \\ z \end{bmatrix} \).

Geometrically, these three planes intersect in a line.
Solution for Question 43, part (a):

For each fixed value of \( t \in \mathbb{R} \), the solutions to the equation \( x + \frac{1}{t} y = t \) is the line \( L_t \) (which depends on \( t \)) in the \((x, y)\)-plane whose \( x \)-intercept and \( y \)-intercept is 2. The family of lines \( \{ L_t : t \in \mathbb{R} \} \) are exactly the lines which pass through \((0, 2)\) and intersect the \( x \)-axis. The parameter \( t \) is a way to name these lines using real numbers (that is, we call them by their \( x \)-intercept). Another way of parameterizing these lines is by their slope. The slope of \( L_t \) is \(-2/t\) (where \( L_0 \) is regarded as having slope \( \pm \infty \)).

The solution set of \( x + y = 1 \) is a line with \( x \)-intercept 1 and \( y \)-intercept 1. Let's denote this line by \( L \). It’s distinct from \( L_t \) for all \( t \) and its slope is \(-1\).

We are interested in the intersection of \( L \) with \( L_t \) (considered as a function of \( t \)). Two distinct lines in the plane intersect if and only if they aren’t parallel. The slope of \( L_t \) is \(-1\) exactly when \( t = 2 \). Hence, \( L_t \) and \( L \) intersect for all values of \( t \neq 2 \). Let \((x(t), y(t))\) denote the point of intersection between the lines \( L_t \) and \( L \).

The lines \( L_t \) vary continuously with \( t \). One way to interpret this is: if one considers a bounded coordinate plane with a fixed scale (i.e. the type of coordinate axes we’d draw to graph a line), for suitably close values \( t', t \in \mathbb{R} \) the graphs of the solution sets \( L_t \) and \( L_{t'} \) (on this bounded plane) are very close to one another. In this sense, one may also regard the line of slope 0 which we’d use to graph a line), then the graph of the solution set \( L_t \) for \(|t| > 0\) is (qualitatively) indistinguishable from the graph of the line of slope 0 which passes through \((0, 2)\). We will denote this limiting line \( L_{\pm \infty} \).

When \( t \) is very large in magnitude, the line \( L_t \) is almost horizontal and has a graph that looks very similar to the line \( L_{\pm \infty} \). It follows as \( t \to \infty \) (or \( t \to -\infty \)), the intersection point of \( L_t \) and \( L \) gets closer and closer to the intersection between \( L_{\pm \infty} \) and \( L \). This intersection is the point \((-1, 2)\). As a consequence, \( x(t) \) has a horizontal asymptote at \(-1\) as \( t \to \pm \infty \) and \( y(t) \) has a horizontal asymptote at 2 as \( t \to \pm \infty \).

Geometrically, the effect of increasing \( t \) from \(-\infty \) to \( \infty \) rotates the line \( L_t \) counterclockwise around the point \((0, 2)\). Since \((0, 2)\) is to the right of the line \( L \), starting with the horizontal line \( L_{\pm \infty} \) and rotating around \((0, 2)\) has the effect of moving the point of intersection \((x(t), y(t))\) down and to the right along the entire length of the ray in \( L \) starting from \((-1, 2)\). It follows that \( x(t) \) approaches its horizontal asymptote at \(-1\) from above as \( t \to -\infty \), is increasing on the range \(-\infty < t < 2\), and has a vertical asymptote at \( t = 2 \). Similarly, \( y(t) \) approaches its horizontal asymptote at 2 from below as \( t \to -\infty \), is decreasing on the range \(-\infty < t < 2\), and has a vertical asymptote at \( t = 2 \). A similar explanation (rotating clockwise) shows that \( x(t) \) approaches its horizontal asymptote at \(-1\) from below as \( t \to \infty \), and is increasing on the range \( 2 < t < \infty \), whereas \( y(t) \) approaches its horizontal asymptote at \(-1\) from above as \( t \to \infty \), and is decreasing on the range \( 2 < t < \infty \).

Graphs of the \( x \)- and \( y \)-coordinates of the intersection of \( L_t \) and \( L \), in terms of \( t \), are on the next page.
Based on the previous arguments, the graphs should be as shown here above.
§1.1 A43. (b) \( x + y = 1 \) \( \implies \) \( x + y = 1 \)

\[ x + \frac{1}{2}y = t \implies \frac{1}{2}y = t - 1 \]

If \( t = 2 \), the 2nd eqn. is \( dy = 1 \), and the system is inconsistent.

\( t \neq 2 \), we divide by \( \frac{1}{2} = \frac{t - 2}{2} \):

\[ x + y = 1 \]
\[ y = \frac{2(t-1)}{t-2} \implies x = 1 - \frac{2(t-1)}{t-2} \]

Now, taking derivatives:

\[ \frac{dy}{dt} = \frac{2(t-2) - 2(t-1)}{(t-2)^2} = \frac{-2}{(t-2)^2} \]
so whenever \( t \neq 2 \),

\( y \) is decreasing as \( t \) increases, and \( y \) approaches

\(-\infty \) from the left & \( \infty \) from the right as \( t \to 2 \).

\[ \frac{dx}{dt} = \frac{2}{(t-2)^2} \]
so whenever \( t \neq 2 \), \( x \) increases

as \( t \) does, and \( x \) approaches \( \infty \) from the left &

\(-\infty \) from the right as \( t \to 2 \).

This all agrees with what is shown in the graphs above.
\[\begin{align*}
X_1 - 7X_2 + X_5 &= 3 \\
X_3 - 2X_5 &= 2 \\
X_4 + X_5 &= 1
\end{align*}\]

Notice that \[
\begin{bmatrix}
1 & -7 & 0 & 0 & 1 \\
0 & 1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
the matrix corresponding to the linear system on the left-hand side is already in reduced row echelon form.

So, all that remains is to solve for the leading variables \((X_1, X_3, X_4)\) in terms of the other variables \((X_2, X_5)\):

\[\begin{align*}
X_1 &= 3 + 7X_2 - X_5 \\
X_3 &= 2 + 2X_5 \\
X_4 &= 1 - X_5
\end{align*}\]

Letting \(X_2 = s\), \(X_5 = t\), the solutions to this linear system are:

\[
\begin{bmatrix}
3 + 7s - t \\
5 \\
2 + 2t \\
1 - t \\
t
\end{bmatrix}
\]

where \(s, t\) can be any real numbers.
§1.2 10. \[4x_1 + 3x_2 + 2x_3 - x_4 = 4\]
\[5x_1 + 4x_2 + 3x_3 - x_4 = 4\]
\[-2x_1 - 2x_2 - x_3 + 2x_4 = -3\]
\[11x_1 + 6x_2 + 4x_3 + x_4 = 11\]

\[R_2 - R_1 \rightarrow R_2: \quad 4x_1 + 3x_2 + 2x_3 - x_4 = 4\]
\[x_1 + x_2 + x_3 = 0\]
\[-2x_1 - 2x_2 - x_3 + 2x_4 = -3\]
\[11x_1 + 6x_2 + 4x_3 + x_4 = 11\]

\[R_2 - 4R_1 \rightarrow R_2: \quad x_2 - 2x_3 - x_4 = 4\]
\[R_3 + 2R_1 \rightarrow R_3: \quad x_3 + 2x_4 = -3\]
\[R_4 - 11R_1 \rightarrow R_4: \quad -5x_2 - 7x_3 + x_4 = 11\]

\[x_1 + x_2 + x_3 = 0\]
\[x_2 + 2x_3 + x_4 = -4\]
\[x_3 + 2x_4 = -3\]

\[R_2 - 1 \rightarrow R_2\]
\[R_4 : \frac{1}{3} \rightarrow R_4\]
\[R_1 - R_3 \rightarrow R_1\]
\[R_2 - 2R_3 \rightarrow R_2\]

\[R_1 - R_2 \rightarrow R_1: \quad x_1 + x_2 = 3\]
\[x_2 - 3x_4 = 2\]
\[x_3 + 2x_4 = -3\]

\[0 = 0\]

Let \[x_4 = t\].

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} = \begin{bmatrix}
  1 - t \\
  2 + 3t \\
  -3 - 2t \\
  t
\end{bmatrix}
\]

For any real number \[t\].
3.1.2 Q12. Convert the system into an augmented matrix:

\[
\begin{bmatrix}
2 & 0 & -3 & 0 & 7 & 7 & 0 \\
-2 & 1 & 6 & 0 & -6 & -12 & 0 \\
0 & 1 & -3 & 0 & 1 & 5 & 0 \\
0 & 0 & -2 & 0 & 1 & 1 & 0 \\
2 & 1 & -3 & 0 & 8 & 7 & 0
\end{bmatrix}
\]

First, we row-reduce this.

Start by clearing columns below leading 1s.

\[
R_1 + R_2 \rightarrow R_2 \\
R_5 - R_2 \rightarrow R_5
\]

\[
\begin{bmatrix}
2 & 0 & -3 & 0 & 7 & 7 & 0 \\
0 & 1 & 3 & 0 & 1 & 5 & 0 \\
0 & 1 & -3 & 0 & 1 & 5 & 0 \\
0 & 1 & -6 & 0 & 1 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

It is now easy to clear the entire third column as this affects no other columns:

\[
R_3 - R_2 \rightarrow R_3 \\
R_4 + 2R_2 \rightarrow R_4 \\
R_5 - R_2 \rightarrow R_5
\]

\[
\begin{bmatrix}
2 & 0 & 0 & 0 & 7 & 7 & 0 \\
0 & 1 & 0 & 0 & 1 & 5 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

6th column:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 7/2 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

and the \( x_5 \) variable is free. Setting \( x_5 = t \), the solutions to this

System are

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6
\end{bmatrix}
= \begin{bmatrix}
-7/2 \\
-t \\
0 \\
-3t \\
tl \\
0
\end{bmatrix}
\]
Suppose we have some solution to our original system, and the two equations we're working with are

\[ a_1 x_1 + a_2 x_2 + \ldots + a_m x_m = b, \]
\[ a_1 x_1 + a_2 x_2 + \ldots + a_m x_m = b. \]

Since we assume we have a solution \( \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \), plugging that solution in will solve these equations, the left and right hand sides will agree.

Subtracting \( t \) times the 2nd equation from the first:

\[ (a_{11} - ta_{21}) x_1 + \ldots + (a_{1m} - ta_{2m}) x_m = b - tb, \]
\[ a_{11} x_1 + \ldots + a_{1m} x_m = b. \]

We see that \( \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \) is still a solution; because

\[ a_{21} x_1 + \ldots + a_{2m} x_m = tb, \]

we're subtracting equal things from the first equation, so it is still solved. Also, all other equations in the system are solved since they haven't changed. So, any solution to the original system solves the transformed system.

Going the other way, if \( \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \) solves the new system, then

\[ (a_{11} - ta_{21}) y_1 + \ldots + (a_{1m} - ta_{2m}) y_m = b - tb, \]
\[ a_{11} y_1 + \ldots + a_{1m} y_m = b. \]

So adding \( t \) times the 2nd equation to the first leaves it as an equality (for the same reason as before), so \( \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \) solves the old system of equations, too.

Thus, since any solution to either system also solves the other, these systems have the same solutions.
When does the system below have a unique solution, no solution, or only many solutions?

\[ \begin{align*}
y + 2kz &= 0 \\
x + (6-4k)z &= 2 \\
x + 2y + 6z &= 2 \\
x + 2kz &= 0 \\
x + 2z &= 1
\end{align*} \]

(switching the top 2 rows & clearing the 2nd column).

\[ \begin{align*}
x + (6-4k)z &= 2 \\
y + 2kz &= 0 \\
(-k(6-4k)+2)z &= 1 - 2k
\end{align*} \]

- Clearing the 1st column.

Now, 
\[ (-k(6-4k)+2)z = 4k^2 - 6k + 2 \]
\[ = (2k-1)(2k-2) \]
and is 0 when \( k = \frac{1}{2} \) or when \( k = 1 \).

\( k \neq \frac{1}{2}, k \neq 1: \)

\[ \begin{align*}
x + (6-4k)z &= 2 \\
y + 2kz &= 0 \\
Z &= \frac{-(2k-1)}{(2k-1)(2k-2)} = \frac{-1}{2k-2}
\end{align*} \]

Clearing the third column.

\[ \begin{align*}
x &= 2 + \frac{(6-4k)}{2k-2} \\
y &= \frac{2k}{2k-2} \\
z &= \frac{-1}{2k-2}
\end{align*} \]

Thus, if \( k = \frac{1}{2} \) there are infinitely many solutions (part (c)), if \( k = 1 \), there are no solutions (part (b)), and otherwise, the system has a unique solution (part (a)).
§1.2 2.50 We rewrite these equations as
\[ x_1 + x_3 = 2x_2 \]
\[ x_2 + x_4 = 2x_3 \]
\[ \vdots \]
\[ x_{n-2} + x_n = 2x_{n-1} \] and rearrange:
\[ x_3 \cdot x_2 = x_2 - 2x_1 \]
\[ x_4 - x_3 = x_3 - x_2 \]
\[ x_5 - x_4 = x_4 - x_3 \]
\[ \vdots \]
\[ x_n - x_{n-1} = x_{n-1} - x_{n-2} \] and we note that this means all of these differences are equal!

Thus, if we know \( x_1 \) and \( (x_2 - x_1) \), we can solve for
\[ x_3 = (x_3 - x_2) + x_2 = (x_3 - x_2) + (x_2 - x_1) + x_1, \] because
\[ x_3 - x_2 = x_2 - x_1, \] and we can just keep going. Setting
\[ x_1 = S, \quad x_2 - x_1 = t, \] we see that
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  \vdots \\
  x_{n-1} \\
  x_n 
\end{bmatrix} =
\begin{bmatrix}
  S \\
  S + t \\
  S + 2t \\
  \vdots \\
  S + (n-2)t \\
  S + (n-1)t 
\end{bmatrix}
\]
is the only way for the above equations to hold, because
\[ t = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \cdots = x_n - x_{n-1} \]

Thus, these are the solutions to this linear system, where \( S, t \) can be any real numbers.
§ 1.3 1. The rref of the augmented matrix is given. How many solutions does the system have?

(a) \[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
No solutions. The last row represents the equation \(Ox + Ox_2 + Ox_3 = 1\), which cannot happen; the system is inconsistent.

(b) \[
\begin{bmatrix}
1 & 0 & 1 & 5 \\
0 & 1 & 1 & 6
\end{bmatrix}
\]
This has a unique solution. We can see that we have two unknowns (one for each column) and we see that \(x_1 = 5\), \(x_2 = 6\); this pair is the unique solution.

(c) \[
\begin{bmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]
Infinitely many solutions. We see that we have three columns & thus three unknowns; while we know that \(x_2 = 2\) & \(x_3 = 3\), \(x_1\) can take on any value and is thus a free variable, indicating that we have infinitely many solutions.

22. We have a system of 3 linear equations in 3 unknowns with a unique solution. What does the rref of the coefficient matrix of this system look like? Explain. This rref is \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

For a 3x3 matrix, if its rref was not the identity matrix, it would have a row of all zeroes (count the leading 1's; if there are 3, the rref is \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] otherwise the bottom row of the rref is all zeroes). Now, consider the corresponding row in the augmented matrix, taking the rref of that augmented matrix:
- If any row looks like \[
\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
\], the system is inconsistent.
- If all the rows that are all zeroes for the unaugmented matrix look like \[
\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\], the system has infinitely many solutions.
In neither case does the system have a unique solution, which we are told that it does! So, the rref of this matrix does not have any rows of all zeroes, and is thus \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\].