Solutions for Homework 10. Due 4/22

Section 7.1

72. (a). From the text. # of new branches at year $t = n(t) = 2a(t-1)$, and # of old branches at year $t-1 = a(t) = n(t-1) + a(t-1)$. i.e. we have the recursive formula:

$$\begin{cases} n(t) = 2a(t-1) \\ a(t) = n(t-1) + a(t-1) \end{cases} \Rightarrow \begin{bmatrix} n(t+1) \\ a(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} n(t) \\ a(t) \end{bmatrix} \quad (*)$$ so the matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(b). For $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue 2

For $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. $A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector of eigenvalue -1.

Now if we use the eigenbasis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and set $S = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, $S^{-1}AS = B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

(c). From the formula $(*)$. $\begin{bmatrix} n(t) \\ a(t) \end{bmatrix} = A^t \begin{bmatrix} n(0) \\ a(0) \end{bmatrix} = A^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but $A = SBS^{-1}$. $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

$\Rightarrow A^t = (SBS^{-1})^t = S \cdot B^t \cdot S^{-1}$ note $S = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, $B^t = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 2^t & 0 \\ 0 & (-1)^t \end{bmatrix}$, $S^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$.

plug in these back, $\begin{bmatrix} n(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & (-1)^t \end{bmatrix} \cdot -\frac{1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3\,(2^t + (-1)^t \cdot 2) \\ 1/3\,(2^t - (-1)^t) \end{bmatrix}$.

this is just the closed formula for $n(t)$ & $a(t)$.

Section 7.4

Find closed formulas for A^t. t a positive integer.

4. $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ $P_A(t) = \det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$.

A has two eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$. $E_{\lambda_1} = \ker(A - \lambda_1 I) = \ker\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = \text{Span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$E_{\lambda_2} = \ker(A - 3I) = \ker\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} = \text{Span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. hence $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenbasis. set $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

then $S^{-1}AS = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. $\Rightarrow A = S\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}S^{-1} \Rightarrow A^t = S\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^t S^{-1}$ (see theorem 7.4.2).

$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^t & 0 \\ 0 & 1 \end{bmatrix} \cdot -1 \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2^t & 2 \\ 2^t & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2-2^t & 2^{t+1}-2 \\ 1-2^t & 2^{t+1}-1 \end{bmatrix}$.

6. $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. $P_A(t) = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda$. eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$.

$E_{\lambda_1} = \ker(A - \lambda_1 I) = \ker\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \text{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $E_{\lambda_2} = \ker(A - \lambda_2 I) = \ker\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} = \text{Span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

an eigenbasis $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. $S^{-1}AS = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$. hence $A^t = S \cdot \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}^t S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 3^t \end{bmatrix}$

$\cdot \frac{1}{3}\begin{bmatrix} 2 & -1 \\ +1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3^t \\ 0 & 2 \cdot 3^t \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 3^t & 3^t \\ 2 \cdot 3^t & 2 \cdot 3^t \end{bmatrix} = \begin{bmatrix} 3^{t-1} & 3^{t-1} \\ 2 \cdot 3^{t-1} & 2 \cdot 3^{t-1} \end{bmatrix}$.

10. $A=\begin{bmatrix}1&1&1\\0&0&1\\0&0&2\end{bmatrix}$. $P_A(t)=\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda&1&1\\0&-\lambda&1\\0&0&2-\lambda\end{bmatrix}=-\lambda(\lambda-1)(\lambda-2)$. eigenvalues

$\lambda_1=0$. $\lambda_2=1$. $\lambda_3=2$. $E_{\lambda_1}=\ker A=\ker\begin{bmatrix}1&1&1\\0&0&1\\0&0&2\end{bmatrix}=\operatorname{Span}\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $E_{\lambda_2}=\ker(A-I)=\ker\begin{bmatrix}0&1&1\\0&-1&1\\0&0&1\end{bmatrix}$

$=\operatorname{Span}\begin{bmatrix}1\\0\\0\end{bmatrix}$. $E_{\lambda_3}=\ker(A-2I)=\ker\begin{bmatrix}-1&1&1\\0&-2&1\\0&0&0\end{bmatrix}=\operatorname{Span}\begin{bmatrix}3\\1\\2\end{bmatrix}$. an eigenbasis $S=\begin{bmatrix}1&1&3\\-1&0&1\\0&0&2\end{bmatrix}$

(For convenience of computation, set $S=\begin{bmatrix}1&1&3\\0&-1&1\\0&0&2\end{bmatrix}$ instead, then $S^{-1}AS=\begin{bmatrix}1&0&0\\0&0&0\\0&0&2\end{bmatrix}$).

Now $A^t=S\cdot\begin{bmatrix}1&0&0\\0&0&0\\0&0&2\end{bmatrix}^t S^{-1}=S\cdot\begin{bmatrix}1&0&0\\0&0&0\\0&0&2^t\end{bmatrix}S^{-1}$. by row-reducing we have $S^{-1}=\begin{bmatrix}1&1&-2\\0&-1&\frac12\\0&0&\frac12\end{bmatrix}$.

$\Rightarrow A^t=\begin{bmatrix}1&1&3\\0&-1&1\\0&0&2\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&0&0\\0&0&2^t\end{bmatrix}\cdot\begin{bmatrix}1&1&-2\\0&-1&\frac12\\0&0&\frac12\end{bmatrix}=\begin{bmatrix}1&0&3\cdot2^t\\0&0&2^t\\0&0&2^{t+1}\end{bmatrix}\cdot\begin{bmatrix}1&1&-2\\0&-1&\frac12\\0&0&\frac12\end{bmatrix}=\begin{bmatrix}1&1&3\cdot2^{t-1}-2\\0&0&2^{t-1}\\0&0&2^t\end{bmatrix}$

12. $A=\begin{bmatrix}0.3&0.1&0.3\\0.4&0.6&0.4\\0.3&0.3&0.3\end{bmatrix}$ $P_A(t)=\det(A-\lambda I)=\det\begin{bmatrix}0.3-\lambda&0.1&0.3\\0.4&0.6-\lambda&0.4\\0.3&0.3&0.3-\lambda\end{bmatrix}\overset{\text{Laplace expansion (on the first row)}}{=}$

$(0.3-\lambda)\det\begin{bmatrix}0.6-\lambda&0.4\\0.3&0.3-\lambda\end{bmatrix}-0.1\det\begin{bmatrix}0.4&0.4\\0.3&0.3-\lambda\end{bmatrix}+0.3\det\begin{bmatrix}0.4&0.6-\lambda\\0.3&0.3\end{bmatrix}=-\lambda(\lambda-0.2)(\lambda-1)$.

eigenvalues are $0, 0.2, 1$. Then $E_0=\ker(A)=\operatorname{Span}\begin{bmatrix}\overset{1}{\cancel{4}}\\0\\\underset{-1}{\cancel{-3}}\end{bmatrix}$. (solve $\begin{bmatrix}0.3&0.1&0.3\\0.4&0.6&0.4\\0.3&0.3&0.3\end{bmatrix}\vec{x}=0$)

$E_{0.2}=\ker(A-0.2I)=\ker\begin{bmatrix}0.1&0.1&0.3\\0.4&0.4&0.4\\0.3&0.3&0.1\end{bmatrix}=\operatorname{Span}\begin{bmatrix}1\\-1\\0\end{bmatrix}$. $E_3=\ker(A-I)=\ker\begin{bmatrix}-0.7&0.1&0.3\\0.4&-0.4&0.4\\0.3&0.3&-0.7\end{bmatrix}$

$=\operatorname{Span}\begin{bmatrix}2\\+5\\+3\end{bmatrix}$. eigenbasis $S=\begin{bmatrix}1&1&2\\0&-1&5\\-1&0&3\end{bmatrix}$ then $S^{-1}AS=B=\begin{bmatrix}0&0&0\\0&0.2&0\\0&0&1\end{bmatrix}\Rightarrow A^t=SB^tS^{-1}$.

by row-reducing S we get $S^{-1}=\begin{bmatrix}0.3&0.3&\overset{-0.7}{\cancel{0.3}}\\0.5&-0.5&0.5\\0.1&0.1&0.1\end{bmatrix}$. now $A^t=\begin{bmatrix}1&1&2\\0&-1&5\\-1&0&3\end{bmatrix}\begin{bmatrix}0&0&0\\0&(0.2)^t&0\\0&0&1\end{bmatrix}\begin{bmatrix}0.3&0.3&-0.7\\0.5&-0.5&0.5\\0.1&0.1&0.1\end{bmatrix}$

$=\begin{bmatrix}0&(0.2)^t&2\\0&-(0.2)^t&5\\0&0&3\end{bmatrix}\cdot\begin{bmatrix}0.3&0.3&-0.7\\0.5&-0.5&0.5\\0.1&0.1&0.1\end{bmatrix}=\begin{bmatrix}0.5(0.2)^t+0.2&0.2-0.5(0.2)^t&0.5\cdot(0.2)^t+0.2\\0.5-0.5(0.2)^t&0.5+0.5(0.2)^t&0.5-(0.5)(0.2)^t\\0.3&0.3&0.3\end{bmatrix}$

26. Find $\lim_{t\to\infty}A^t\vec{x}_0$. $A=\begin{bmatrix}0.4&0.5\\0.6&0.5\end{bmatrix}$. $\vec{x}_0=\begin{bmatrix}0.54\\0.46\end{bmatrix}$

$P_A(t)=\det(A-\lambda I)=\det\begin{bmatrix}0.4-\lambda&0.5\\0.6&0.5-\lambda\end{bmatrix}=\lambda^2-0.9\lambda-0.1=(\lambda-1)(\lambda+0.1)$. hence A has two eigenvalues

$\lambda_1=1$. $\lambda_2=-0.1$.

For $\lambda=1$. $E_1=\ker(A-I)=\ker\begin{bmatrix}-0.6 & 0.5\\ 0.6 & -0.5\end{bmatrix}=\mathrm{Span}\begin{bmatrix}5\\6\end{bmatrix}$. For $\lambda=-0.1$, $E_2=\ker(A+0.1I)$

$=\ker\begin{bmatrix}0.5 & 0.5\\ 0.6 & 0.6\end{bmatrix}=\mathrm{Span}\begin{bmatrix}1\\-1\end{bmatrix}$. Hence an eigenbasis $S=\begin{bmatrix}5 & 1\\ 6 & -1\end{bmatrix}$. $S^{-1}AS=\begin{bmatrix}1 & 0\\ 0 & -0.1\end{bmatrix}$.

$\Rightarrow A^t=S\begin{bmatrix}1 & 0\\ 0 & -0.1\end{bmatrix}^t S^{-1}$. $\Rightarrow \lim\limits_{t\to\infty}A^t\vec{x_0}=\lim\limits_{t\to\infty}S\cdot\begin{bmatrix}1 & 0\\ 0 & (-0.1)^t\end{bmatrix}S^{-1}\vec{x_0}=S\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}S^{-1}\vec{x_0}$.

(Since $\lim\limits_{t\to\infty}(-0.1)^t=0$). $=\begin{bmatrix}5 & 1\\ 6 & -1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\cdot -\frac{1}{11}\begin{bmatrix}-1 & -1\\ -6 & 5\end{bmatrix}\cdot\begin{bmatrix}0.54\\ 0.46\end{bmatrix}=\begin{bmatrix}5 & 0\\ 6 & 0\end{bmatrix}\cdot -\frac{1}{11}\begin{bmatrix}-1 & -1\\ -6 & 5\end{bmatrix}$

$\begin{bmatrix}0.54\\ 0.46\end{bmatrix}=-\frac{1}{11}\cdot\begin{bmatrix}-5 & -5\\ -6 & -6\end{bmatrix}\begin{bmatrix}0.54\\ 0.46\end{bmatrix}=\frac{1}{11}\cdot\begin{bmatrix}5 & 5\\ 6 & 6\end{bmatrix}\begin{bmatrix}0.54\\ 0.56\end{bmatrix}=\begin{bmatrix}5/11\\ 6/11\end{bmatrix}$.

34. (a). $\vec{x}(t+1)=\begin{bmatrix}0.7 & 0 & 0\\ 0.1 & 0.6 & 0\\ 0 & 0.2 & 0.8\end{bmatrix}\vec{x}(t)$. 0.7: the decreasing rate of pollutant in Lake Sils.

0.1: the rate of pollutant transported to Lake Silvaplana from Lake Sils.

0.6: the decreasing rate of pollutant in Lake Silvaplana.

0.2: the rate of pollutant transported to Lake St. Moritz from Lake Silvaplana.

0.8: the decreasing rate of pollutant in Lake St. Moritz.

(b). For $A=\begin{bmatrix}0.7 & 0 & 0\\ 0.1 & 0.6 & 0\\ 0 & 0.2 & 0.8\end{bmatrix}$ it has 3 eigenvalues $\lambda_1=0.7$, $\lambda_2=0.6$, $\lambda_3=0.8$.

$E_{0.7}=\ker(A-0.7I)=\ker\begin{bmatrix}0 & 0 & 0\\ 0.1 & -0.1 & 0\\ 0 & 0.2 & 0.1\end{bmatrix}=\mathrm{Span}\begin{bmatrix}1\\1\\-2\end{bmatrix}$. $E_{0.6}=\ker(A-0.6I)=\ker\begin{bmatrix}0.1 & 0 & 0\\ 0.1 & 0 & 0\\ 0 & 0.2 & 0.2\end{bmatrix}$

$=\mathrm{Span}\begin{bmatrix}0\\1\\-1\end{bmatrix}$. $E_{0.8}=\ker(A-0.8I)=\ker\begin{bmatrix}-0.1 & 0 & 0\\ 0.1 & -0.2 & 0\\ 0 & 0.2 & 0\end{bmatrix}=\mathrm{Span}\begin{bmatrix}0\\0\\1\end{bmatrix}$ hence if $S=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ -2 & -1 & 1\end{bmatrix}$.

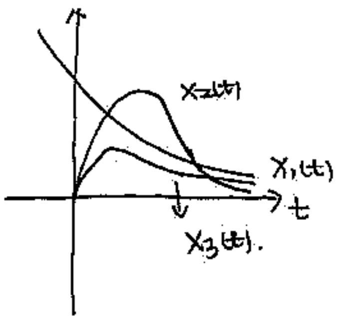
$S^{-1}AS=\begin{bmatrix}0.7 & 0 & 0\\ 0 & 0.6 & 0\\ 0 & 0 & 0.8\end{bmatrix}\Rightarrow A^t=S\begin{bmatrix}(0.7)^t & 0 & 0\\ 0 & (0.6)^t & 0\\ 0 & 0 & (0.8)^t\end{bmatrix}S^{-1}$, and $\vec{x}(0)=\begin{bmatrix}100\\0\\0\end{bmatrix}$ by condition.

$S^{-1}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\Rightarrow\vec{x}(t)=A^t\vec{x}(0)=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ -2 & -1 & 1\end{bmatrix}\begin{bmatrix}(0.7)^t & 0 & 0\\ 0 & (0.6)^t & 0\\ 0 & 0 & (0.8)^t\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}100\\0\\0\end{bmatrix}$

$=\begin{bmatrix}(0.7)^t & 0 & 0\\ (0.7)^t & (0.6)^t & 0\\ -2(0.7)^t & -(0.6)^t & (0.8)^t\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}100\\0\\0\end{bmatrix}=\begin{bmatrix}(0.7)^t & 0 & 0\\ (0.7)^t-(0.6)^t & (0.6)^t & 0\\ (0.8)^t+(0.6)^t-2(0.7)^t & (0.6)^t(0.8)^t & (0.8)^t\end{bmatrix}\cdot\begin{bmatrix}100\\0\\0\end{bmatrix}=\begin{bmatrix}100\cdot(0.7)^t\\ 100\cdot((0.7)^t-(0.6)^t)\\ 100\cdot((0.8)^t+(0.6)^t-2(0.7)^t)\end{bmatrix}$

i.e. we have the closed formula $\begin{cases}x_1(t)=100\cdot(0.7)^t\\ x_2(t)=100\cdot((0.7)^t-(0.6)^t)\\ x_3(t)=100\cdot((0.8)^t+(0.6)^t-2(0.7)^t)\end{cases}$

a sketch for these three functions looks like:



(For a more accurate graph, you can use computer).

for $x_2(t) = 100\cdot\left((0.7)^t - (0.6)^t\right)$ by taking derivatives & finding roots. it attains maximum when $t \approx 2.33$.

38. (a). From the text, $a(t+1) = a(t) + j(t)$: the $j(t)$ juvenile pairs become adult pairs at t; $j(t+1) = a(t)$: new juvenile pairs from adult pairs at time t.

Hence if $\vec{x}(t) = \begin{bmatrix} a(t) \\ j(t) \end{bmatrix}$, then $\vec{x}(t+1) = \begin{bmatrix} a(t+1) \\ j(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a(t) \\ j(t) \end{bmatrix} = A \begin{bmatrix} a(t) \\ j(t) \end{bmatrix} = A\vec{x}(t)$. here

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the transition matrix.

(b). $P_A(\lambda) = \overset{\det}{\cancel{\ker}}(A - \lambda I) = \overset{\det}{\cancel{\ker}}\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1$, the two eigenvalues are therefore

$\lambda_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$. $E_{\frac{1+\sqrt5}{2}} = \ker\left(A - \frac{1+\sqrt5}{2} I\right) = \ker \begin{bmatrix} \frac{1-\sqrt5}{2} & 1 \\ 1 & -\frac{1+\sqrt5}{2} \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ \frac{\sqrt5 - 1}{2} \end{bmatrix}$

$E_{\frac{1-\sqrt5}{2} I} = \ker\left(A - \frac{1-\sqrt5}{2} I\right) = \ker \begin{bmatrix} \frac{1+\sqrt5}{2} & 1 \\ 1 & \frac{\sqrt5-1}{2} \end{bmatrix} = \operatorname{Span}\begin{bmatrix} \frac{\sqrt5-1}{2} \\ -1 \end{bmatrix}$. an eigenbasis $S = \begin{bmatrix} 1 & \frac{\sqrt5-1}{2} \\ \frac{\sqrt5-1}{2} & -1 \end{bmatrix}$.

$S^{-1}AS = \begin{bmatrix} \frac{1+\sqrt5}{2} & 0 \\ 0 & \frac{1-\sqrt5}{2} \end{bmatrix}$. hence $A^t = S \begin{bmatrix} \left(\frac{1+\sqrt5}{2}\right)^t & 0 \\ 0 & \left(\frac{1-\sqrt5}{2}\right)^t \end{bmatrix} S^{-1}$ note $\vec{x}(0) = \begin{bmatrix} a(0) \\ j(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. then

we have $\vec{x}(t) = A^t \vec{x}(0) = \begin{bmatrix} 1 & \frac{\sqrt5-1}{2} \\ \frac{\sqrt5-1}{2} & -1 \end{bmatrix} \begin{bmatrix} \left(\frac{\sqrt5+1}{2}\right)^t & 0 \\ 0 & \left(\frac{1-\sqrt5}{2}\right)^t \end{bmatrix} \cdot \frac{2}{\sqrt5 - 5} \cdot \begin{bmatrix} -1 & \frac{1-\sqrt5}{2} \\ \frac{1-\sqrt5}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} \left(\frac{\sqrt5+1}{2}\right)^t & \frac{\sqrt5-1}{2}\cdot\left(\frac{1-\sqrt5}{2}\right)^t \\ \frac{\sqrt5-1}{2}\left(\frac{\sqrt5+1}{2}\right)^t & -\left(\frac{1-\sqrt5}{2}\right)^t \end{bmatrix} \cdot \frac{2}{\sqrt5-5} \cdot \begin{bmatrix} -1 \\ \frac{1-\sqrt5}{2} \end{bmatrix} = \frac{2}{\sqrt5-5} \cdot \begin{bmatrix} -\left(\frac{\sqrt5+1}{2}\right)^t - \left(\frac{\sqrt5-1}{2}\right)^2 \cdot \left(\frac{1-\sqrt5}{2}\right)^t \\ -\frac{\sqrt5-1}{2}\left(\frac{\sqrt5+1}{2}\right)^t - \left(\frac{1-\sqrt5}{2}\right)^{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{t+1} - \cancel{+}\frac{1}{\sqrt5}\cdot\left(\frac{1-\sqrt5}{2}\right)^{t+1} \\ \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^t - \cancel{+}\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^t \end{bmatrix}$

Hence we get the closed formula $\begin{cases} a(t) = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{t+1} - \cancel{+}\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{t+1} \\ j(t) = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^t - \cancel{+}\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^t \end{cases}$

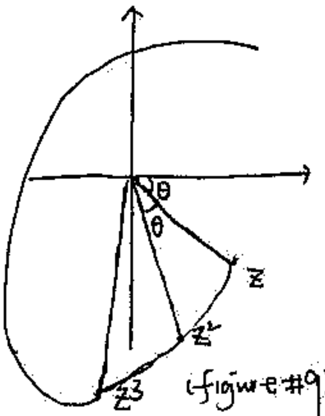
(c). $\lim\limits_{t\to\infty} \dfrac{a(t)}{j(t)} = \lim\limits_{t\to\infty} \dfrac{\left(\frac{1+\sqrt5}{2}\right)^{t+1} - \left(\frac{1-\sqrt5}{2}\right)^{t+1}}{\left(\frac{1+\sqrt5}{2}\right)^t - \left(\frac{1-\sqrt5}{2}\right)^t} = \lim\limits_{t\to\infty} \dfrac{\frac{1+\sqrt5}{2} - \frac{1-\sqrt5}{2}\cdot\left(\frac{1-\sqrt5}{1+\sqrt5}\right)^t}{1 - \left(\frac{1-\sqrt5}{1+\sqrt5}\right)^t}$ (divide both numerator & denominator by $\frac{1+\sqrt5}{2}$) note $\frac{(1-\sqrt5)}{1+\sqrt5} = -\frac{(1-\sqrt5)^2}{(\sqrt5)^2-1} = -\frac{3-\sqrt5}{2}$ and $\frac{3-\sqrt5}{2} < 1$, so $\lim\limits_{t\to\infty}\left(\frac{1-\sqrt5}{1+\sqrt5}\right)^t = 0 \Rightarrow$

the limit above $= \frac{1+\sqrt5}{2}$ (the golden section).

Section 7.5

9. $z = 0.8 - 0.7i$. compute $z^2, z^3, \cdots$ in the complex plane & explain their long-term behavior.

Note the modulus $|z| = \sqrt{(0.8)^2 + (0.7)^2} > 1$. if z is in the polar form $z = re^{i\theta}$, then $r > 1$. $\theta \in (\frac{3}{2}\pi, 2\pi)$, hence $z^n = r^n e^{in\theta}$. the modulus of z^n is strictly increasing. if we sketch them in the complex plane.



(figure #9)

We can see the modulus is increasing to infinity as $n \to \infty$.
So the figure looks like a helix
(or, using terms in the textbook, it's a spiraling trajectory).

For 20, 22, 24. find all complex eigenvalues.

20. $\begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$. $P_A(t) = \det \begin{bmatrix} 3-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} = \lambda^2 - 9 + 10 = \lambda^2 + 1$. eigenvalues $\pm i$.

22. $\begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}$. $P_A(t) = \det \begin{bmatrix} 1-\lambda & 3 \\ -4 & 10-\lambda \end{bmatrix} = \lambda^2 - 11\lambda + 10 + 12 = \lambda^2 - 11\lambda + 22$.

eigenvalues $= \dfrac{11 \pm \sqrt{11^2 - 22 \cdot 4}}{2} = \dfrac{11 \pm \sqrt{121 - 88}}{2} = \dfrac{11 \pm \sqrt{33}}{2}$.

24. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix}$ $P_A(t) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 5 & -7 & 3-\lambda \end{bmatrix} = \begin{pmatrix} \text{Laplace expansion} \\ \text{on first row} \end{pmatrix} = -\lambda \det \begin{bmatrix} -\lambda & 1 \\ -7 & 3-\lambda \end{bmatrix}$

$-1 \cdot \det \begin{bmatrix} 0 & 1 \\ 5 & 3-\lambda \end{bmatrix} = -\lambda(\lambda^2 - 3\lambda + 7) + 5 = -(\lambda^3 - 3\lambda^2 + 7\lambda - 5)$. note $\lambda = 1$ is one eigenvalue of this polynomial. so we can factorize $P_A(t) = -(\lambda - 1)(\lambda^2 - 2\lambda + 5)$. roots of $\lambda^2 - 2\lambda + 5$ are $\dfrac{2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$. so eigenvalues are 1 & $1 \pm 2i$.

30. (a). $2i$ an eigenvalue of a real 2×2 matrix A. find A^2.

Since $2i$ is ~~an~~ a complex eigenvalue of A. its conjugate $-2i$ is also an eigenvalue. i.e. A has eigenvalues $\pm 2i$. $\Rightarrow$ A can be diagonalized: $\exists$ an invertible matrix S.

s.t. $A = S \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} S^{-1}$. then $A^2 = S \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}^2 S^{-1} = S \cdot \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} S^{-1} = -4 \cdot S \cdot S^{-1} = -4I$.

here I is the 2×2 identity matrix.

b. Given an example of a real 2x2 matrix A s.t. all entries of $A \neq 0$. and $2i$ is an eigenvalue of A. also compute A^2 and check with part A.

$A=\begin{bmatrix}1 & 5\\ -1 & -1\end{bmatrix}$. then $P_A(t)=\det\begin{bmatrix}1-\lambda & 5\\ -1 & -1-\lambda\end{bmatrix}=\lambda^2-1+5=\lambda^2+4$. eigenvalues $\pm 2i$.

$A^2=\begin{bmatrix}1 & 5\\ -1 & -1\end{bmatrix}\cdot\begin{bmatrix}1 & 5\\ -1 & -1\end{bmatrix}=\begin{bmatrix}-4 & 0\\ 0 & -4\end{bmatrix}=-4I$. the same as our result from part (a).

33. (a). Let $B=(b_{ij})_{n\times n}$. then $D^{-1}=\mathrm{diag}(b_{11}^{-1}, b_{12}^{-1}, \cdots, b_{1n}^{-1})$. now $BD^{-1}=\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix}\cdot\begin{bmatrix}b_{11}^{-1} & & 0\\ & b_{12}^{-1} & \\ 0 & & \ddots\ b_{1n}^{-1}\end{bmatrix}=\begin{bmatrix}1 & 1 & & 1\\ b_{21}/b_{11} & b_{22}/b_{12} & \cdots & b_{2n}/b_{1n}\\ \vdots & \vdots & & \vdots\\ b_{n1}/b_{11} & b_{n2}/b_{22} & & b_{nn}/b_{1n}\end{bmatrix}$. So C is just dividing the i-th column of B by the element b_{1i}. (in our case, $n=5$)

(b). the first element in every column is 1.

(c). Suppose $\lambda_1, \lambda_2, \cdots, \lambda_5$ are the eigenvalues of A. the corresponding eigenvectors $\vec{v_i}$. then assume $\lambda_1 > |\lambda_j|$. $2\le j\le 5$. for $\forall \vec{e_i}$ $(1\le i\le 5)$ a standard basis. assume $\vec{e_i}=c_1\vec{v_1}+\cdots+c_5\vec{v_5}$. then the i-th column of $A^t = A^t\vec{e_i}=A^t(c_1\vec{v_1}+\cdots+c_5\vec{v_5})$. Note $A\vec{v_j}=\lambda_j\vec{v_j}$. so we have $A^t\vec{e_i}=A^t(c_1\vec{v_1}+\cdots+c_5\vec{v_5})=c_1\lambda_1^t\vec{v_1}+\cdots+c_5\lambda_5^t\vec{v_5}$. for t large enough, $(\lambda_j/\lambda_1)^t\to 0$ $(2\le j\le 5)$. So the i-th column $A^t\vec{e_i}=c_1\lambda_1^t\vec{v_1}+\cdots+c_5\lambda_5^t\vec{v_5}$ will be nearly parallel to the eigenvector $\vec{v_1}$ of λ_1.

(d). Combining (a) & (c). we know for t sufficiently large. columns of $B=A^t$ will be nearly the same as eigenvectors $\vec{v_1}$ of λ_1. From (a). the columns of C will now be eigenvectors of A with first entry $=1$. Hence $AC=[A\vec{c_1}\ A\vec{c_2}\ \cdots\ A\vec{c_5}]$. ($\vec{c_i}$ i-th column of C). Since $\vec{c_i}$ can be taken as our approximation to the eigenvector $\vec{v_1}$, $A\vec{c_i}=\lambda_1\vec{c_i}$. note first entry of $\vec{c_i}=1$. so the first entry of $A\vec{c_i}=\lambda_1$. In conclusion, entries in top row of AC can be our approximation of the eigenvalue λ of largest modulus. columns of AC can be our approximation of the eigenvectors corresponding to λ_1. (this is a method to find the dominant eigenvalue)

34. (a). $A-\lambda I_n$ has eigenvalues $\lambda_1-\lambda, \lambda_2-\lambda, \cdots, \lambda_n-\lambda$. by our assumption, the one with smallest modulus should be $\lambda_1-\lambda$. Now for $(A-\lambda I_n)^{-1}$, its eigenvalues are $(\lambda_i-\lambda)^{-1}$. So $(\lambda_1-\lambda)^{-1}$ is the eigenvalue with largest modulus. if $\vec{v_i}$ is an eigenvector for λ_i. then $A\vec{v_i}=\lambda_i\vec{v_i}\Rightarrow(A-\lambda I_n)\vec{v_i}=(\lambda_i-\lambda)\vec{v_i}\Rightarrow(A-\lambda I_n)^{-1}\vec{v_i}=(\lambda_i-\lambda)^{-1}\vec{v_i}$. i.e. $\vec{v_i}$ is then an eigenvector for $(\lambda_i-\lambda)^{-1}$, the eigenvalue

of the matrix $A-\lambda I_n$. Since $(\lambda_1-\lambda)^{-1}$ is the eigenvalue of $(A-\lambda I_n)^{-1}$ largest modulus, we can then use the method of Ex 33, to find $(\lambda_1-\lambda)^{-1}$ and the corresponding eigenvector $\vec{v}$, which will also be an eigenvector for λ_1. in this way we can then recover the eigenvalue λ_1 and the eigenvector $\vec{v}$.

(b). use computer to carry out the method mentioned in Ex 33 & Ex 34.