Review Problems for the Final

The problems below are based on the sections of the textbook that have been covered in this course, with an emphasis on more recent material.

1. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line spanned by $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

2. Consider the linear transformation that reflects a vector about the line by $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Can you find an orthonormal eigenbasis of this transformation? What are its eigenvalues, with multiplicity? (You do not need to compute the matrix of this transformation.)

3. Find vectors that span the image of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$. Give as few vectors as possible.

4. Find vectors that span the kernel of the matrix $A = \begin{bmatrix} 2 & 3//6 & 9 \end{bmatrix}$, giving as few vectors as possible.

5. Is the set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + y = 1 \right\}$ a subspace of $\mathbb{R}^2$?

6. Is the set $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = y \right\}$ a subspace of $\mathbb{R}^3$?

7. Is the set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0 \right\}$ a subspace of $\mathbb{R}^2$?

8. Let L be the line in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$:

(a) Find a matrix A so that $\operatorname{im}(A) = L$.

(b) Find a matrix B so that $\ker(B) = L$.

(c) Verify the rank-nullity theorem for the matrices A, B you found in parts (a) and (b).

9. For which values of a, b, c, d, e, f are the following vectors linearly independent?

$$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} d \\ e \\ f \\ 0 \end{bmatrix}$$

10. Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent in $\mathbb{R}^n$. Are the vectors $\vec{v}_1, \vec{v}_1+\vec{v}_2, \vec{v}_1+\vec{v}_2+\vec{v}_3$ linearly independent or not? How can you tell?

11. Suppose that A is a 5×5 matrix that can be written as $A = BC$, where B is a 5×4 matrix and C is a 4×5 matrix. Is A invertible? How do you know?

12. Given an example of a 4×5 matrix A with $\dim(\ker A) = 3$. What is the dimension of the image of A?

13. Find the coordinates of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} \right\}$.

14. Find the matrix of the transformation $T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \vec{x}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

15. Consider the matrix $A = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}$, and the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find constants c_1, c_2, c_3, c_4 such that:

$$\begin{aligned} A\vec{v}_1 &= c_1\vec{v}_1 + c_2\vec{v}_2 \\ A\vec{v}_2 &= c_3\vec{v}_1 + c_4\vec{v}_2. \end{aligned}$$

Interpret your answer in terms of coordinates.

16. Is the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ similar to the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$? Is A similar to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$?

17. Compute the angle between $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

18. Can you find a line L in $\mathbb{R}^n$ and a vector $\vec{x}$ in $\mathbb{R}^n$ such that

$$\vec{x} \cdot \operatorname{proj}_L(\vec{x})$$

is negative?

19. Find an orthonormal basis for the image of A, where A is the matrix:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}.$$

20. Find an orthonormal basis for the kernel of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

21. If A, B are any $n \times n$ matrices, which of the following matrices must be symmetric?

(a) $A^T A$

(b) AA^T

(c) $A - A^T$

(d) $B(A + A^T)B^T$

22. Are the rows of an orthogonal matrix always orthonormal?

23. If A is a symmetric $n \times n$ matrix, what is the relationship between $\text{im}(A)$ and $\ker(A)$?

24. Find the trigonometric function of the form $f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)$ that best fits the data points $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$, using least squares. Sketch the solution together with the function $g(t) = t$.

25. Consider an inconsistent system $A\vec{x} = \vec{b}$, where A is a 3×2 matrix. We know that the least-squares solution of this system is $\vec{x}^* = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$. If S is an orthogonal 3×3 matrix, what is the least-squares solution of the system $SA\vec{x} = S\vec{b}$?

26. Find the determinant of $A = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$. Is A invertible? What is the kernel of A? What is the image of A? Is zero an eigenvalue of A?

27. Use Gaussian elimination to find the determinant of the matrix:

$$\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

28. A matrix is called *skew-symmetric* when $A^T = -A$. Suppose that A is an $n \times n$ skew-symmetric matrix, and n is odd. Show that A is not invertible, by showing that its determinant is zero.

29. Let $a \neq b$ be real numbers. Consider the function:

$$f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}$$

(a) Show that f is a quadratic polynomial. What is the coefficient of t^2?

(b) Explain why $f(a) = f(b) = 0$. Conclude that $f(t) = k(t-a)(t-b)$, and find k using the work in (a).

(c) For which values of t is this matrix invertible?

30. Consider a triangle with vertices $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 10 \\ 1 \end{bmatrix}$. What is its area?

31. Consider two positive numbers a, b. Solve the following system:

$$\begin{aligned} ax - by &= 1 \\ bx + ay &= 0 \end{aligned}$$

What are the signs of the solutions x and y? How does x change as b increases?

32. What are the possible real eigenvalues of an orthogonal matrix?

33. If a vector $\vec{v}$ is an eigenvector of both A and B, with eigenvalues λ_A, λ_B respectively, is $\vec{v}$ necessarily an eigenvector of AB? If so, what is its eigenvalue as an eigenvector of AB?

34. If $\vec{v}$ is an eigenvector of the $n \times n$ matrix A with associated eigenvalue λ, what can you say about $\ker(A - \lambda I_n)$? Is the matrix $A - \lambda I_n$ invertible?

35. Consider the linear transformation $proj_L$ that orthogonally projects onto a line in $\mathbb{R}^3$. Find all eigenvalues and all eigenvectors of this transformation, and find an eigenbasis if you can. Is this transformation diagonalizable?

36. For each of the following matrices, find all (complex) eigenvalues, with their algebraic multiplicity:

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

37. For each of the following matrices, find all eigenvalues, with their algebraic multiplicity, and find an eigenbasis for each eigenspace:

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$

38. Is the matrix $\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$ similar to the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

39. For each of the following matrices, find a closed formula for A^t, where t is allowed to be any positive integer:

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$

40. Find $\lim_{t\to\infty}(A^t\vec{x}_0)$, where $A = \begin{bmatrix} 0 & 0.5 & 0.4 \\ 1 & 0 & 0.6 \\ 0 & 0.5 & 0 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \end{bmatrix}$.

41. Find an invertible matrix S such that $S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$.

42. (a) If $2i$ is an eigenvalue of a real 2×2 matrix A, find A^2.

(b) Find a real 2×2 matrix A with all nonzero entries such that $2i$ is an eigenvalue of A, and compute A^2. (This should agree with your results from (a).)

43. Find an orthonormal eigenbasis for each of the following matrices:

(a) $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

44. If A is invertible and orthogonally diagonalizable, is A^{-1} orthogonally diagonalizable as well?

45. Sketch the curve defined by the following equation:

$$6x_1^2 + 4x_1x_2 + 4x_2^2 = 1$$

Draw and label its principal axes, label the intercepts of the curve with these axes, and give the formula of the curve in the coordinate system defined by the principal axes.

46. Let A be the matrix of an orthogonal projection onto an m-dimensional subspace of $\mathbb{R}^n$. Determine the definiteness of A. (Compare your answer here to your answer to Problem 18 of this review sheet.)

47. If A is a positive-definite invertible symmetric matrix, what can you say about the definiteness of A^{-1}?

48. Find the singular values of the matrix $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$. Explain your answer geometrically by considering the image of the unit circle under A. Then, find a singular value decomposition of A.

49. Find the singular values of an orthogonal matrix A.

50. Find the singular value decomposition of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.