Math 201: Linear Algebra
Spring 2019
Practice Final
Exam date: 5/9/2019
Time Limit: 180 Minutes

Name (Print): ____________________

JHU-ID: ____________________

Teaching Assistant ____________________

This exam contains 17 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	35	
2	28	
3	25	
4	39	
5	20	
6	38	
7	15	
Total:	200	

1. Consider the system of linear equations:

$$\begin{aligned} 2x_1 + 8x_2 + 4x_3 &= \quad 2, \\ 2x_1 + 5x_2 + x_3 &= \quad 5, \\ 4x_1 + 10x_2 - x_3 &= \quad 1. \end{aligned}$$

   (a) (5 points) Express this system as a single matrix equation $Ax = b$.

   (b) (20 points) Is A invertible? If so, compute A^{-1}. Find $\det(A)$.

(c) (10 points) Find all solutions to the system of linear equations above.

2. (a) (25 points) Let V be the subspace of $\mathbf{R}^3$ consisting of all vectors v whose coordinate entries sum to 0. Find an orthonormal basis for V.

(b) (3 points) Let $v_1, \ldots, v_k$ be the basis for V which you found in part (a). How many orthogonal matrices A are there with the property that the first k columns of A are $v_1, \ldots, v_k$, respectively? Reason geometrically. You are not required to find these matrices explicitly.

3. (25 points) Consider the data set:

Input (i)	Experimental Output (y_i)
0	0
1	1
2	2

Find the quadratic polynomial $f(x)$ of the form $f(x) = ax^2 + b$ for which the quantity

$$\epsilon := \left\| \sum_{i=0}^{2} (f(i) - y_i)^2 \right\|$$

is minimized. Graph the best fit curve $f(x)$ and the data set together on a coordinate plane.

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4. Let $B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

   (a) (22 points) Find a basis for $\mathbf{R}^2$ consisting of eigenvectors for B.

(b) (11 points) Find a closed form expression for $B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for all integers t.

(c) (6 points) Find a positive integer t such that $\left\| B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| > 10^9$. Justify your answer. You do not need to find the smallest such t.

5. (20 points) Let

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 7 & 3 & 4 \\ 7 & 1 & 6 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 1 & 1 & 1 \end{bmatrix}.$$

Determine the rank of A.

6. Consider the quadratic form $q : \mathbf{R}^2 \to \mathbf{R}$ given by

$$q(x, y) = 6x^2 + 10xy + 6y^2.$$

(a) (5 points) Find a symmetric matrix S such that

$$q(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot S \begin{bmatrix} x \\ y \end{bmatrix}.$$

(b) (15 points) Sketch the curve defined by $q(x, y) = 1$. Draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

(c) (15 points) Find the singular value decomposition for S. Sketch the curve obtained by applying S to the set of unit vectors in $\mathbf{R}^2$ (i.e. sketch the image of the unit circle under S). Is this curve equal to the curve found in part (b)?

(d) (3 points) Determine which symmetric matrices S have the property that the image of the unit circle under S and the curve $\{v \in \mathbf{R}^2 : v \cdot Sv = 1\}$ are equal. How many such matrices are there?

7. (15 points) Determine if each of the statements below are *Always True* or *Possibly False.*

True False $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

True False Let A be a square matrix. The subspaces $\ker(A)$ and $\operatorname{im}(A)$ are orthogonal.

True False For any non-zero vector $v \in \mathbf{R}^n$ there is a basis β such that $[v]_\beta = e_1$.

True False $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$.

True False Let A be a square $n \times n$ matrix and $\mathbf{0}$ be the $n \times n$ zero matrix. If $A \neq \mathbf{0}$ but $A^2 = \mathbf{0}$, then A is not diagonalizable.

True False Given any $n-1$ dimensional subspace $V \subseteq \mathbf{R}^n$, there is an orthogonal basis $v_1, \ldots, v_n$ of $\mathbf{R}^n$ such that $v_1, \ldots, v_n \notin V$.

True False If A is a square matrix and $\det(A) = 9$, then A is not orthogonal.

True False Every subspace of $\mathbf{R}^n$ has an orthonormal basis.

True False The least squares solution x^* for the matrix equation $Ax = b$ is the solution to $Ax = b$ for which $\|Ax^* - b\|$ is minimized.

True False Let A be a square matrix. The matrix $I + kA$ is invertible for all but finitely many values of $k \in \mathbf{R}$.

True False Let A be a square matrix. The matrix A is diagonalizable if and only if the matrix A^2 is diagonalizable.

True False If a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ preserves area, then T is orthogonal.

True False If $A = A^T$, then A is diagonalizable.

True False The span of k vectors $v_1, \ldots, v_k \in \mathbf{R}^n$ is a k-dimensional subspace.

True False If A is 3×3 matrix, then there is a line $L \subset \mathbf{R}^3$ through the origin such that if $v \in L$ then $Av \in L$.

True False If $a, b, c \in \mathbf{R}$ are positive numbers, then solution set to the equation $ax^2 + bxy + cy^2 = 1$ is an ellipse.

True False The matrices $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are similar.

True False If A is a square matrix, then $\det(A^T A) \geq 0$.

True False There exists an invertible 10×10 matrix that has 92 ones among its entries.

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