Math 201: Linear Algebra
Spring 2019
Exam 1
4/15/19
Time Limit: 50 Minutes

Name (Print): Solutions

JHU-ID:

TA name & Section No.

This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	23	
2	23	
3	30	
4	6	
Total:	82	

Do not write in the table to the right.

Median 70
(85.3 %)

Average 67.8
(82.7 %)

STDEV 9.967
(12.2 %)

1. (a) (20 points) Complete one (and only one) of the calculations below. Indicate which problem you have selected by marking the relevant box.

☒ Find an orthonormal basis u_1, u_2 for the subspace **span** $\left(\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right)$.

OR

☐ Find the least-squares (i.e. approximate) solution x^* to the matrix equation $Ax = b$, where:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\|v_1\| = \sqrt{2^2+2^2+1} = \sqrt{9} = 3$

$$\boxed{u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.}$$

$v_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $v_2 \cdot u_1 = 1$

$$v_2^{\perp} = v_2 - (v_2 \cdot u_1) u_1$$

$$= \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -5/3 \\ 8/3 \end{bmatrix}.$$

$$\boxed{u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\cancel{3}\sqrt{90}}\begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}}$$

1. (a) (20 points) Complete one (and only one) of the calculations below. Indicate which problem you have selected by marking the relevant box.

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$$A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\boxed{A^T A x^* = A^T b}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 \\ 3 & 11 \end{bmatrix}. \qquad (A^T A)^{-1} = \frac{1}{90} \begin{bmatrix} 11 & -3 \\ -3 & 9 \end{bmatrix}.$$

$$A^T b = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\boxed{x^* = (A^T A)^{-1} (A^T b) = \frac{1}{90} \begin{bmatrix} 11 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{90} \begin{bmatrix} 19 \\ 3 \end{bmatrix}}$$

(b) (3 points) In part (a) above, u_1, u_2, A, b, x^* are related by the formula:

$$Ax^* = (u_1 \cdot b)u_1 + (u_2 \cdot b)u_2. \quad (1)$$

Briefly explain (in 1 to 3 sentences) why you ~~would expect this equality to hold.~~

$\boxed{\text{Both the left and right hand side compute } \mathrm{Proj}_{\mathrm{Im}(A)}(b).}$

The vector Ax^* is the closest vector in $\mathrm{Im}(A)$ to b. This is $b^{\parallel}$, the parallel part of b to the image of A, or equivalently

$$(1) \quad Ax^* = b^{\parallel} = \mathrm{Proj}_{\mathrm{Im}(A)}(b).$$

Because u_1, u_2 are an orthonormal basis for the image of A, projection is given by the expression

$$(2) \quad \mathrm{Proj}_{\mathrm{Im}(A)}(b) = (u_1 \cdot b)u_1 + (u_2 \cdot b)u_2.$$

Combining (1) + (2) yields the above formula.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 1 \\ 2 & 1 & 9 & 9 & 9 & 9 \\ 0 & 0 & 1 & 1 & 9 & 4 \\ 0 & 0 & 2 & 2 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 4 & 3 \end{bmatrix}.$$

(a) (20 points) Compute $\det(A)$. (Choose an efficient method. Don't do more work than the problem requires.)

Compute determinant via row reduction

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 1 \\ 2 & 1 & 9 & 9 & 9 & 9 \\ 0 & 0 & 1 & 1 & 9 & 4 \\ 0 & 0 & 2 & 2 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow[\substack{II-2I \\ IV-2III \\ VI-4V}]{} \begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 1 \\ 0 & -3 & 1 & 7 & 3 & 7 \\ 0 & 0 & 1 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & -17 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

(determinant stays the same)

Upper triangular.

$$\boxed{\mathrm{Det}(A) = 1 \cdot (-3) \cdot 1 \cdot 0 \cdot 1 \cdot (-5) = 0.}$$

(b) (3 points) Does the parallelepiped generated by the columns of A contain the unit cube

$$\{c_1e_1 + c_2e_2 + \cdots + c_6e_6 : 0 \leq c_i \leq 1\}$$

in $\mathbf{R}^6$? Think geometrically. Explain your answer.

No! Here are two answers:

Geometric Answer: Since $|\det(A)| = 0$, the volume of the parallelepiped generated by the columns of A has volume 0; ~~so~~ therefore it cannot contain the unit cube which has volume **1**.

Algebraic Answer: Since $\det(A) = 0$, the column space spanned by the columns of A (i.e. the image of A) has dimension less than 6. It follows that the image of A cannot contain the 6 linearly independent vectors $e_1, \ldots, e_6$.

3. Let $k \in \mathbf{R}$ be a real number. Consider $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$.

   (a) (15 points) Compute the characteristic polynomial of A. Determine all values $k \in \mathbf{R}$ such that the matrix A has 2 real eigenvalues (counted with their algebraic multiplicity).

Char poly:

$$P_A(t) = \begin{vmatrix} 1-t & k \\ 1 & 1-t \end{vmatrix}$$

$$= (1-t)^2 - k$$

$$\boxed{= t^2 - 2t + (-k+1).}$$

Eigenvalues:

$$\lambda = \frac{2 \pm \sqrt{4 - 4(-k+1)}}{2}$$

$$\boxed{= 1 \pm \sqrt{k}.}$$

When $\boxed{k \geq 0}$, A has 2 real eigenvalues counted with algebraic multiplicity

(when $k > 0$, $P_A(t)$ has 2 distinct real roots.
when $k = 0$, $P_A(t)$ has a double root at 0.)

(b) (15 points) For which values of k is A diagonalizable (over $\mathbf{R}$)? For each such value, find an eigenbasis for A.

$\lambda = 1 + \sqrt{K}$

$$A - \lambda I = \begin{bmatrix} -\sqrt{K} & K \\ 1 & -\sqrt{K} \end{bmatrix} \xrightarrow{\text{II} \leftrightarrow \text{I}} \begin{bmatrix} 1 & -\sqrt{K} \\ -\sqrt{K} & K \end{bmatrix}$$

row reduce

$$\xrightarrow{\text{II} + \sqrt{K}\,\text{I}} \begin{bmatrix} 1 & -\sqrt{K} \\ 0 & 0 \end{bmatrix}$$

$\text{rref}(A - \lambda I)$

$$E_{1+\sqrt{K}} = \text{Ker}(A - \lambda I) = \text{Span}\left(\begin{bmatrix} \sqrt{K} \\ 1 \end{bmatrix}\right).$$

$\boxed{\underline{\text{Choose}}\ v_1 = \begin{bmatrix} \sqrt{K} \\ 1 \end{bmatrix}.}$

$\lambda = 1 - \sqrt{K}$

$$A - \lambda I = \begin{bmatrix} \sqrt{K} & K \\ 1 & \sqrt{K} \end{bmatrix} \xrightarrow{\text{I} \leftrightarrow \text{II}} \begin{bmatrix} 1 & \sqrt{K} \\ \sqrt{K} & K \end{bmatrix}$$

$$\xrightarrow{\text{II} - \sqrt{K}\,\text{I}} \begin{bmatrix} 1 & \sqrt{K} \\ 0 & 0 \end{bmatrix}$$

$$E_{1-\sqrt{K}} = \text{Ker}(A - \lambda I) = \text{Span}\left(\begin{bmatrix} -\sqrt{K} \\ 1 \end{bmatrix}\right).$$

$\boxed{\underline{\text{Choose}}\ v_2 = \begin{bmatrix} -\sqrt{K} \\ 1 \end{bmatrix}}$

<u>Answer</u>: v_1, v_2 is a eigenbasis if $K \neq 0$; hence for all $K > 0$ A is diagonalizable. If $K = 0$, then $\dim(E_0) = 1 < \text{almult}(0)$ so A is <u>not</u> diagonalizable.

4. **This problem does not require any computation.**

   (a) (2 points) Consider the orthogonal matrix

$$A = \frac{1}{4761}\begin{bmatrix} 433 & -448 & -1568 & -4452 \\ -448 & 4081 & -2380 & 384 \\ 1568 & 2380 & 3569 & -1344 \\ 4452 & -384 & -1344 & 945 \end{bmatrix}.$$

   Find $\det(A^6)$. No computation is required. Justify your answer.

Since A is orthogonal $\det(A) = \pm 1$.

Hence,

$\det(A^6) = \det(A)^6 = (\pm 1)^6 = 1.$

$$\boxed{\det(A^6) = 1.}$$

   (b) (2 points) Determine the characteristic polynomial of the third power of the projection matrix:

$$(\text{Proj}_V)^3 = \frac{1}{6619}\begin{bmatrix} 3869 & 1105 & 2625 & -1590 \\ 1105 & 1301 & 1653 & 1722 \\ 2625 & 1653 & 2609 & 916 \\ -1590 & 1722 & 916 & 5459 \end{bmatrix}, \text{ where } V = \text{span}\left(\begin{bmatrix} 7 \\ 11 \\ 13 \\ 17 \end{bmatrix}, \begin{bmatrix} 19 \\ 23 \\ 29 \\ 31 \end{bmatrix}\right).$$

   No computation is required. Justify your answer.

Proj_V is similar to $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$. Acts on V / Acts on $V^\perp$

Hence Proj_V^3 is similar to $\begin{bmatrix} 1^3 & & & \\ & 1^3 & & \\ & & 0^3 & \\ & & & 0^3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}.$

Hence $\boxed{P_{\text{Proj}_V^3}(t) = (1-t)^2 t^2.}$

(c) (2 points) Find

$$\det\left(\begin{bmatrix} 2001 & 42 & 1 \\ 1 & 18 & 97 \\ 1 & 311 & 7 \\ 1 & 2 & 1 \\ 9 & 7 & 1919 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 3 & 8 \\ 7 & 1 & 13 & 17 & 12 \\ 42 & 11 & 93 & 47 & 2019 \end{bmatrix}\right).$$

No computation is required. Justify your answer.

The determinant of this matrix is 0

Why?

A square matrix has determinant 0 if and only if it is not invertible.

The matrix under consideration is the composition of two linear transformations

$$\mathbb{R}^5 \xrightarrow{A} \mathbb{R}^3 \xrightarrow{B} \mathbb{R}^5.$$

By the rank nullity theorem, the image of B is at most 3 dimensional. Hence the image of BA (which is contained in Im(B)) is at most 3 dimensional. Therefore, BA is not invertible.