Math 201: Linear Algebra
Spring 2019
Exam 1
3/4/19
Time Limit: 50 Minutes

Name (Print): Solutions

JHU-ID: ____________________

Teaching Assistant ____________________

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	30	
2	20	
3	30	
4	7	
Total:	87	

Median: 75/87 (86%)

Mean: 72.45/87 (83%)

StDev: 10 (out of 87)

1. Consider the system of Linear equations:

$$\begin{aligned} 3x + 2y + z &= -5, \\ 2x + y &= -1, \\ x &= -5. \end{aligned}$$

(a) (5 points) Express this system of linear equations as a single matrix equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -5 \end{bmatrix}.$$

$$A \cdot \vec{x} \qquad \vec{b}$$

(b) (15 points) Is the matrix A invertible? If so, calculate A^{-1}.

Form augmented matrix.

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{I} \leftrightarrow \text{III}]{\text{row reduce}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\text{III}-3\text{I}]{\text{II}-2\text{I}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 2 & 1 & 1 & 0 & -3 \end{array}\right] \xrightarrow{\text{III}-2\text{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

rref(A) = I so A is invertible

1

A^{-1}!

$\boxed{\text{Since } \operatorname{rref}(A) = I, \ A \text{ is invertible.} \quad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}.}$

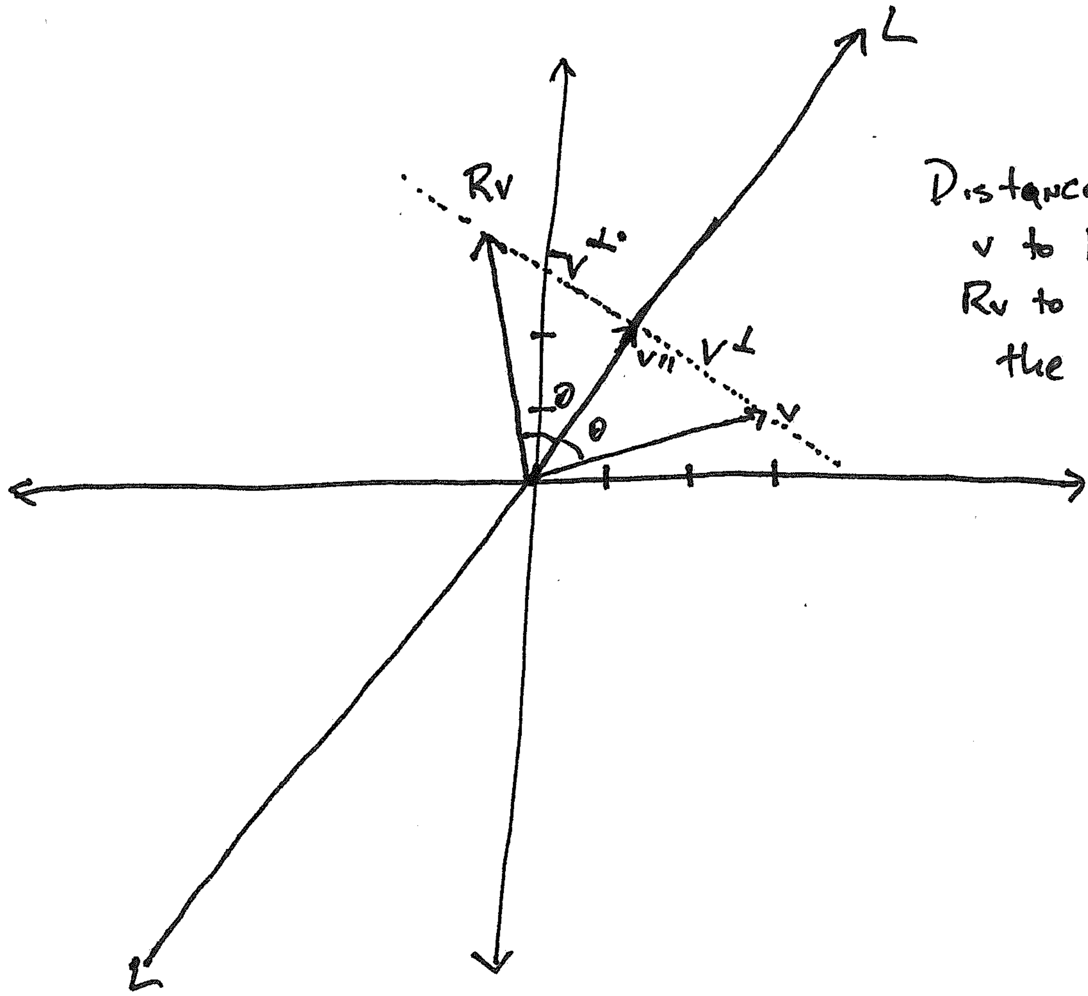
(c) (10 points) Find all solutions to this system of linear equations (Hint: use A^{-1} if A is invertible).

If A is invertible unique solution to $Ax = b$ is $x = A^{-1}b$.

$$A^{-1}\begin{bmatrix} -5 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -1 \\ -5 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ 9 \\ -8 \end{bmatrix}}.$$

2. Let $L \subseteq \mathbf{R}^2$ be the line of slope 2 through the origin. Let $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Let Rv be the reflection of the vector v over the line L.

   (a) (4 points) Draw the line L and the vector v on a coordinate plane. Using geometric reasoning, draw the (approximate) location of Rv.



Distances from v to L and Rv to L are the same

(b) (16 points) Determine the location of Rv algebraically. (Check that your geometric and algebraic solutions are consistent.)

$$L = \mathrm{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right).$$

$$v = v^{\parallel} + v^{\perp}, \quad \text{where } v^{\parallel} = c\begin{bmatrix}1\\2\end{bmatrix} \text{ and } v^{\parallel}\cdot v^{\perp} =$$

$$Rv = v^{\parallel} - v^{\perp}.$$

$$c = \frac{v \cdot \begin{bmatrix}1\\2\end{bmatrix}}{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}} = \frac{3+2}{1+4} = 1$$

So $v^{\parallel} = \begin{bmatrix}1\\2\end{bmatrix}.$

$$v^{\perp} = v - v^{\parallel} = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}.$$

$$Rv = \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}.$$

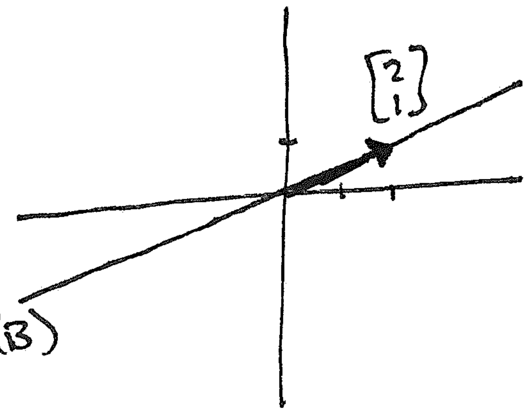
3. Let $k \in \mathbf{R}$ and $B = \begin{bmatrix} 2 & 2k \\ 1 & k \end{bmatrix}$.

(a) (8 points) Find a basis for the image of B and describe the image of B geometrically (your geometric description should be precise and a high school student should be capable of understanding it; both your geometric and algebraic solutions may depend on k).

$$\mathrm{Rref}(B) = \begin{bmatrix} 1 & K \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is a basis for } \mathrm{Im}(B).$$

Pivot column

The image of B is the line through $\vec{0}$ of slope $1/2$.



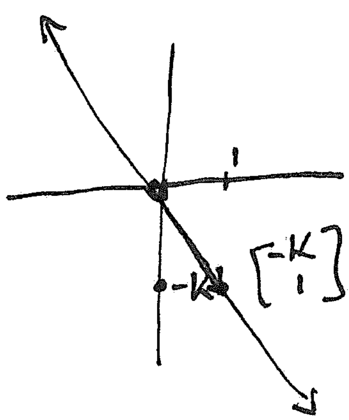
(b) (8 points) Find a basis for kernel of B and describe the kernel of B geometrically (your geometric description should be precise and a high school student should be capable of understanding it; both your geometric and algebraic solutions may depend on k).

$\mathrm{Rref}(B) = \begin{bmatrix} 1 & K \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an element of the kernel if and only if $x_1 + Kx_2 = 0 \Rightarrow x_1 = -Kx_2$.

$$\Rightarrow \mathrm{Ker}(B) = \left\{ \begin{bmatrix} -K \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\}.$$

$$\Rightarrow \begin{bmatrix} -K \\ 1 \end{bmatrix} \text{ is a basis for } \mathrm{Ker}(B).$$

Ker(B)



The kernel of B is the line of slope $-1/K$ through the origin.

(c) (6 points) State the rank-nullity theorem and verify it for B.

Thm (Rank-nullity Theorem) Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, then

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(\text{Ker } T) + \dim(\text{Im}(T)) = \dim(\mathbb{R}^m) = m.$$

Since $Im(B)$ and $Ker(B)$ both have bases consisting of a single vector, the dimension of each space is 1. It follows

$$\dim(Ker(B)) + \dim(Im(B)) = 1 + 1 = 2 = \dim(\mathbb{R}^2).$$

(d) (8 points) Calculate $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}^2$ (i.e. B^2 when $k = -2$). Can you explain your answer geometrically in terms of the locations of $Im(B)$ and $Ker(B)$ when $k = -2$? Would you expect a similar result for other values of k?

$$B^2 = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 - 4 & -4\cdot 2 + (-2)\cdot(-4) \\ 2 - 2 & -4 + (-2)(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

When $k = -2$ the Kernel of B (which is the line of slope $-1/k$) coincides with the image of (which is the line of slope $1/2$). Hence, Bv is an element of $Ker(B)$ for all $v \in \mathbb{R}^2$. It follows

$$B^2(v) = B(Bv) = 0$$

for all $v \in \mathbb{R}^2$, i.e. B^2 is the zero transformation.

When $k \neq -2$, the Kernel and Image are distinct lines in $\mathbb{R}^2$, so if $Bv \in Im(B) \setminus \{0\}$ then $Bv \notin Ker(B)$. It follows $B(Bv) = B^2 v \neq 0$.

4. (7 points) Given a plane $P \subseteq \mathbf{R}^3$ through the origin, let $\text{Proj}_P : \mathbf{R}^3 \to \mathbf{R}^3$ denote the othogonal projection onto P. Are there a pair of planes $P_1, P_2 \subseteq \mathbf{R}^3$ through the origin such that the product $\text{Proj}_{P_1}\text{Proj}_{P_2}$ has rank 0? Explain your answer.

$\boxed{\text{No}}$ why not?

Solution 1: Let P_1 and P_2 be two planes (through the origin) in $\mathbb{R}^3$. Then P_1 and P_2 intersect in a line (if $P_1 \neq P_2$) or a plane (if $P_1 = P_2$). Let v be a non-zero vector in the intersection. Then

$$\text{Proj}_{P_1}(v) = v \quad \text{since } v \in P_1$$

and

$$\text{Proj}_{P_2}(v) = v \quad \text{since } v \in P_2,$$

so $\text{Proj}_{P_1}\text{Proj}_{P_2}(v) = v$ and thus

$$\dim(\text{Im}(\text{Proj}_{P_1}\text{Proj}_{P_2})) \geq \dim(\text{Span}(v)) = 1.$$

In summary:

Any non-zero vector in the intersection $P_1 \cap P_2$ is a non-zero vector in the image of $\text{Proj}_{P_1}\text{Proj}_{P_2}$.

Solution 2: A matrix has rank 0 if and only if it's the zero matrix. (As in problem 3(d)), for the product $\mathrm{Proj}_{P_1}\mathrm{Proj}_{P_2}$ to be the zero matrix it must be the case that every element in the <u>image</u> of Proj_{P_2} is <u>contained</u> in the <u>kernel</u> of Proj_{P_1}. The image of Proj_{P_2} <u>is the plane P_2</u>, whereas the kernel of Proj_{P_1} is <u>the line orthogonal to P_1</u>.

As P_2 is not contained in a <u>line</u> it must be the case that

$$\mathrm{Proj}_{P_1}\mathrm{Proj}_{P_2} \neq 0\text{-matrix}.$$

<u>In summary:</u>

$$\boxed{\text{The kernel of } \mathrm{Proj}_{P_1}\text{, the line othogonal to } P_1\text{, } \underline{\text{does not}} \text{ contain the image of } \mathrm{Proj}_{P_2}\text{, the plane } P_2\text{, so } \mathrm{Proj}_{P_1}\mathrm{Proj}_{P_2} \neq 0\text{-matrix.}}$$