# Some Common Tor and Ext Groups

**Abstract**

We compute all the groups $G \otimes H$, $\mathrm{Tor}(G,H)$, $\mathrm{Hom}(G,H)$, and $\mathrm{Ext}(G,H)$, where $G$ and $H$ can be any of the groups $\mathbb{Z}$ (the integers), $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$ (the integers mod $n$), or $\mathbb{Q}$ (the rationals). All but one are reasonably accessible. Because all these functors are biadditive, these cases suffice to handle any finitely generated groups $G$ and $H$.

The emphasis here is on computation, not on the abstract definitions (which we don't give).

From the symmetry of the definition, we obviously have

$$G \otimes H \cong H \otimes G\ . \tag{1}$$

We write $\theta: G \times H \to G \otimes H$ for the universal bilinear pairing. Then by the definition of $G \otimes H$, any bilinear pairing $f: G \times H \to K$ of abelian groups factors uniquely through a homomorphism $\bar{f}: G \otimes H \to K$. For any integer $n$, we have

$$f(ng,h) = nf(g,h) = f(g,nh) \tag{2}$$

and hence, in $G \otimes H$,

$$(ng) \otimes h = n(g \otimes h) = g \otimes (nh). \tag{3}$$

Although the definition of $\mathrm{Tor}(G,H)$ is *not* symmetric, it is true that $\mathrm{Tor}(G,H) \cong \mathrm{Tor}(H,G)$. This requires some proof; we shall refrain from using this fact, choosing always to compute $\mathrm{Tor}(G,H)$ as the derived functor of $-\otimes H$, by using a free resolution of $G$.

**Computing $G \otimes H$ and $\mathrm{Tor}(G,H)$** We begin with a triviality.

PROPOSITION 4 *For any group $H$, $0 \otimes H = 0$.*

*Proof* From equation (3) with $n = 0$, we have

$$0 \otimes h = (00) \otimes h = 0(0 \otimes h) = 0. \quad \square$$

PROPOSITION 5 *For any $H$, we have $\mathbb{Z} \otimes H \cong H$, with the universal pairing $\theta: \mathbb{Z} \times H \to H$ given by $n \otimes h \mapsto nh$.*

*Proof* This proof we give in full detail, as a pattern for other proofs.

Given a bilinear pairing $f: \mathbb{Z} \times H \to K$, we need to show there is a unique homomorphism $\bar{f}: H \to K$ such that $f = \bar{f} \circ \theta$. This forces $\bar{f}h = \bar{f}\theta(1,h) = f(1,h)$; we therefore *define* $\bar{f}$ by $\bar{f}h = f(1,h)$ for all $h$. This *is* a homomorphism, because by bilinearity,

$$\bar{f}(h+h') = f(1,h+h') = f(1,h) + f(1,h') = \bar{f}h + \bar{f}h'.$$

It satisfies $\bar{f} \circ \theta = f$, because for any integer $n$ we have

$$\bar{f}\theta(n,h) = \bar{f}(nh) = n\bar{f}h = nf(1,h) = f(n,h). \quad \square$$

PROPOSITION 6 *We have $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$, with $\theta: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by $\theta(a \times b) = ab$.*

*Proof* Given a bilinear pairing $f\colon \mathbb{Q}\times\mathbb{Q}\to K$, we are forced to define $\bar f\colon \mathbb{Q}\to K$ by $\bar f a = f(1,a)$, and this is a homomorphism. We have to check that it satisfies $\bar f\circ\theta = f$. Now $\bar f\theta(a,b)=\bar f(ab)=f(1,ab)$. To see that this agrees with $f(a,b)$, we write $a$ as a fraction $m/n$, with $m$ and $n$ integers; then by equation (2),

$$f(1,ab)=f\left(n\frac{1}{n},ab\right)=f\left(\frac{1}{n},nab\right)=f\left(\frac{1}{n},mb\right)=f\left(\frac{m}{n},b\right)=f(a,b).$$

(We warn that we may not be able to divide by $n$ in the group $K$.) □

This leaves $\mathbb{Z}/n\otimes H$. For this we use the usual free resolution

$$0\longrightarrow\mathbb{Z}\xrightarrow{\ n\ }\mathbb{Z}\longrightarrow\mathbb{Z}/n\longrightarrow 0, \tag{7}$$

in which the $n$ denotes multiplication by $n$. We might as well compute $\operatorname{Tor}(\mathbb{Z}/n,H)$ at the same time.

Proposition 8 *For any $H$, we have:*

(a) $\mathbb{Z}/n\otimes H\cong H/nH$, *where $nH$ denotes the subgroup of all elements of $H$ that are divisible by $n$;*

(b) $\operatorname{Tor}(\mathbb{Z}/n,H)\cong {}_nH$, *the subgroup $\{h\in H : nh=0\}$ of elements of $H$ of order $n$ (or some divisor of $n$).*

*Proof* When we tensor diagram (7) with $H$, the resulting long exact sequence simplifies by Proposition 5 to

$$0\longrightarrow\operatorname{Tor}(\mathbb{Z}/n,H)\longrightarrow H\xrightarrow{\ n\ }H\longrightarrow\mathbb{Z}/n\otimes H\longrightarrow 0\ . \tag{9}$$

We read off the kernel and cokernel of $n\colon H\to H$. □

Corollary 10 $\mathbb{Z}/n\otimes\mathbb{Q}=0$ *and* $\operatorname{Tor}(\mathbb{Z}/n,\mathbb{Q})=0$. □

In view of the symmetry (1), the only tensor product left is $\mathbb{Z}/n\otimes\mathbb{Z}/m$.

Lemma 11 *As subgroups of $\mathbb{Z}$, we have $n\mathbb{Z}+m\mathbb{Z}=d\mathbb{Z}$, where $d$ denotes the greatest common divisor* $\gcd(n,m)$ *of $n$ and $m$.*

*Proof* This is almost the definition of $\gcd(n,m)$. □

Proposition 12 *Let* $d=\gcd(n,m)$. *Then*

(a) $\mathbb{Z}/n\otimes\mathbb{Z}/m\cong\mathbb{Z}/d$;

(b) $\operatorname{Tor}(\mathbb{Z}/n,\mathbb{Z}/m)\cong\mathbb{Z}/d$.

*Proof* We apply Proposition 8. For (a) we get $\mathbb{Z}/(n\mathbb{Z}+m\mathbb{Z})$, which Lemma 11 identifies. For (b) we need $\{i\in\mathbb{Z}/m : ni=0\}$. Write $n=n'd$ and $m=m'd$, so that $m'$ and $n'$ are coprime; we need $ni$ to be divisible by $m$, which reduces to having $i$ divisible by $m'$. Thus the answer is $m'\mathbb{Z}/m\mathbb{Z}\cong\mathbb{Z}/d$. □

Now to finish off the remaining Tor groups. Because $\mathbb{Z}$ is projective, we have immediately

$$\operatorname{Tor}(\mathbb{Z},H)=0. \tag{13}$$

On the other hand, because $-\otimes\mathbb{Z}$ is essentially the identity functor by Proposition 5 and therefore exact, we have

$$\mathrm{Tor}(G,\mathbb{Z}) = 0. \tag{14}$$

As a companion to Corollary 10 we have

$$\mathrm{Tor}(\mathbb{Q},\mathbb{Z}/n) = 0. \tag{15}$$

This can be done by a simple trick, by writing multiplication by $n$ in $\mathrm{Tor}(\mathbb{Q},\mathbb{Z}/n)$ in two different ways. First, we write it as $\mathrm{Tor}(n,\mathbb{Z}/n)$, which is invertible with obvious inverse $\mathrm{Tor}(1/n,\mathbb{Z}/n)$. Second, we write it as $\mathrm{Tor}(\mathbb{Q},n)$, which is plainly zero. (More generally, $F(\mathbb{Q},\mathbb{Z}/n)=0$ for any biadditive functor $F$, as in Corollary 10.)

This leaves only $\mathrm{Tor}(\mathbb{Q},\mathbb{Q})$. To handle this properly, we need a better description of $G\otimes\mathbb{Q}$.

LEMMA 16 *Let $G$ be an abelian group.*

(a) *If $G$ is a torsion group, $G\otimes\mathbb{Q}=0$.*

(b) *If $G$ is a torsion-free group, every element of $G\otimes\mathbb{Q}$ has the form $g\otimes(1/n)$ for some $g\in G$ and integer $n\neq 0$, and $g\otimes(1/n)=g'\otimes(1/n')$ if and only if $n'g=ng'$ in $G$.*

*Proof* For (a), the only bilinear pairing $f\colon G\times\mathbb{Q}\to K$ is zero, because if $mg=0$,

$$f(g,b) = f\left(g, m\frac{b}{m}\right) = f\left(mg,\frac{b}{m}\right) = f\left(0,\frac{b}{m}\right) = 0.$$

For (b), we take the set of all formal symbols $g/n$, where $g\in G$ and $n\in\mathbb{Z}$ is nonzero, and impose the relation that $g/n = g'/n'$ if and only if $n'g=ng'$. This *is* an equivalence relation, because $n'g=ng'$ and $n''g'=n'g''$ imply $n'n''g=n'ng''$, and hence $n''g=ng''$ (this is where we use the hypothesis that $G$ is torsion-free).

We then define addition on the set $E$ of equivalence classes by the usual rule for fractions,

$$\frac{g}{n}+\frac{g'}{n'} = \frac{n'g+ng'}{nn'},$$

and show it is well defined and makes $E$ an abelian group. We define the bilinear pairing $\theta\colon G\times\mathbb{Q}\to E$ by $\theta(g,m/n)=(mg)/n$. Given a bilinear pairing $f\colon G\times\mathbb{Q}\to K$, we must define $\bar{f}\colon E\to K$ by $\bar{f}(g/n)=f(g,1/n)$. This is a homomorphism because

$$\begin{aligned}\bar{f}\left(\frac{n'g+ng'}{nn'}\right) &= f\left(n'g+ng',\frac{1}{nn'}\right)\\ &= f\left(n'g,\frac{1}{nn'}\right)+f\left(ng',\frac{1}{nn'}\right) = f\left(g,\frac{1}{n}\right)+f\left(g',\frac{1}{n'}\right).\end{aligned}$$

It satisfies $\bar{f}\circ\theta = f$ because

$$\bar{f}\theta\left(g,\frac{m}{n}\right) = \bar{f}\left(\frac{mg}{n}\right) = f\left(mg,\frac{1}{n}\right) = f\left(g,\frac{m}{n}\right). \qquad \square$$

Corollary 17 *For any group $G$, we have* $\operatorname{Tor}(G,\mathbb{Q})=0$.

*Proof* Take any free resolution

$$0 \longrightarrow F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} G \longrightarrow 0 \tag{18}$$

of $G$. The explicit description furnished by the Lemma shows that

$$\partial\otimes\mathbb{Q}\colon F_1\otimes\mathbb{Q} \xrightarrow{\partial\otimes\mathbb{Q}} F_0\otimes\mathbb{Q}$$

is a monomorphism, because if $x\in F_1$ is nonzero, so is $(\partial\otimes\mathbb{Q})x/n=\partial x/n$. □

**Computing** $\operatorname{Hom}(G,H)$ **and** $\operatorname{Ext}(G,H)$ We start with the analogues of Proposition 5 and equation (13).

Proposition 19 *For any group $H$ we have* $\operatorname{Hom}(\mathbb{Z},H)\cong H$ *and* $\operatorname{Ext}(\mathbb{Z},H)=0$.

*Proof* Homomorphisms $\omega\colon\mathbb{Z}\to H$ correspond 1–1 to elements $h\in H$ by $h=\omega 1$. Conversely, given $h$, we define $\omega n=nh$. (This is almost one *definition* of $\mathbb{Z}$.) The second statement is immediate because $\mathbb{Z}$ is projective. □

Next we deal with $G=\mathbb{Z}/n$, using the same free resolution (7) as before.

Proposition 20 *For any group $H$ we have:*

(a) $\operatorname{Hom}(\mathbb{Z}/n,H)\cong {}_nH$, *the subgroup of $H$ as in Proposition 8;*

(b) $\operatorname{Ext}(\mathbb{Z}/n,H)\cong H/nH$.

*Proof* When we apply the functor $\operatorname{Hom}(-,H)$ to equation (7) and use Proposition 19, we obtain the exact sequence

$$0\longrightarrow \operatorname{Hom}(\mathbb{Z}/n,H)\longrightarrow H\xrightarrow{n} H\longrightarrow \operatorname{Ext}(\mathbb{Z}/n,H)\longrightarrow 0,$$

which has to be isomorphic to diagram (9). (Part (a) was obvious directly.) □

This result allows us to read off all the groups $\operatorname{Hom}(\mathbb{Z}/n,H)$ and $\operatorname{Ext}(\mathbb{Z}/n,H)$ as in Proposition 12 etc.; several of them are obvious anyway.

Corollary 21 *We have the following groups:*

(a) $\operatorname{Hom}(\mathbb{Z}/n,\mathbb{Z})=0$ *and* $\operatorname{Ext}(\mathbb{Z}/n,\mathbb{Z})\cong\mathbb{Z}/n$;

(b) $\operatorname{Hom}(\mathbb{Z}/n,\mathbb{Z}/m)\cong\mathbb{Z}/d$ *and* $\operatorname{Ext}(\mathbb{Z}/n,\mathbb{Z}/m)\cong\mathbb{Z}/d$, *where* $d=\gcd(n,m)$;

(c) $\operatorname{Hom}(\mathbb{Z}/n,\mathbb{Q})=0$ *and* $\operatorname{Ext}(\mathbb{Z}/n,\mathbb{Q})=0$. □

This leaves only the case $G=\mathbb{Q}$.

Proposition 22 *For the group $\mathbb{Q}$ we have:*

(a) $\operatorname{Hom}(\mathbb{Q},\mathbb{Q})\cong\mathbb{Q}$;

(b) $\operatorname{Ext}(G,\mathbb{Q})=0$ *for any group $G$.*

*Proof* Part (a) is easy enough: homomorphisms $\omega\colon \mathbb{Q} \to \mathbb{Q}$ correspond 1–1 to elements $h \in \mathbb{Q}$ by $h = \omega 1$ and $\omega a = ah$. These correspondences are inverse by the identities $1h = h$ and $\omega a = a(\omega 1)$. To see the second, we must write $a = m/n$ and use

$$n(\omega a) = n\omega\left(\frac{m}{n}\right) = \omega m = m(\omega 1) = na(\omega 1) .$$

Because $\mathbb{Q}$ is torsion-free, we deduce $\omega a = a(\omega 1)$.

Part (b) is equivalent to the injectivity of $\mathbb{Q}$. If we use the free resolution (18) of $G$, we have to show that $\mathrm{Hom}(F_0, \mathbb{Q}) \to \mathrm{Hom}(F_1, \mathbb{Q})$ is surjective. (The fact that any divisible group is injective is standard, but not trivial.) □

By the same trick as for equation (15), we have

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/n) = 0 \qquad \text{and} \qquad \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}/n) = 0, \tag{23}$$

except that this time, direct proof of the second equation from the definitions is not so easy.

It is obvious that

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0, \tag{24}$$

because no nonzero element of $\mathbb{Z}$ is divisible by $n$ for all $n$. This leaves only one group to determine.

**The group** $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ *This subsection is strictly optional.* The group $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is much more difficult to determine. It is easy to see that it is a rational vector space, simply from the presence of $\mathbb{Q}$, but harder to see what its dimension is. This group is not as mysterious as is sometimes claimed, but is related to adèle groups familiar to number theorists. [The result is surely not new, but I don't have a reference.]

Denote by $\mathbb{Q}_p$ the field of $p$-adic numbers, for each prime $p$, and by $\mathbb{Z}_p \subset \mathbb{Q}_p$ the subring of $p$-adic integers (not to be confused with $\mathbb{Z}/p$).

THEOREM 25 *We have* $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong A/\mathbb{Q}$, *where $A$ denotes the adèle group consisting of all sequences $(x_2, x_3, x_5, x_7, \ldots)$ of $p$-adic numbers $x_p \in \mathbb{Q}_p$ such that $x_p \in \mathbb{Z}_p$ for all except finitely many $p$, and $\mathbb{Q} \subset A$ denotes the subgroup of all sequences $(x, x, x, \ldots)$ with $x \in \mathbb{Q}$.* (Note to number theorists: $A$ has no coordinate indexed by the reals $\mathbb{R}$.) *It is thus an uncountable rational vector space.*

We begin by applying the functor $\mathrm{Hom}(\mathbb{Q}, -)$ to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

to obtain the long exact sequence

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \longrightarrow A \longrightarrow \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{Ext}(\mathbb{Q}, \mathbb{Q}), \tag{26}$$

where we write $A = \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ and both end groups vanish. Since $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$, we have only to identify $A$ to establish the Theorem.

The torsion group $\mathbb{Q}/\mathbb{Z}$ decomposes as $\bigoplus_p \mathbb{Z}/p^\infty$, where $\mathbb{Z}/p^\infty$ is the well-known divisible group defined as $\mathbb{Z}[p^{-1}]/\mathbb{Z}$. Here, $\mathbb{Z}[p^{-1}] \subset \mathbb{Q}$ denotes the subring of all rationals of the form $m/p^n$, with integers $n \geq 0$ and $m$. We therefore study $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/p^\infty)$, which first requires $\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty)$.

LEMMA 27 *We have the following descriptions of the p-adic numbers:*

(a) *The endomorphism ring* $\operatorname{End}(\mathbb{Z}/p^\infty) = \operatorname{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty)$ *of* $\mathbb{Z}/p^\infty$ *may be identified with the ring of p-adic integers* $\mathbb{Z}_p$*;*

(b) *There are isomorphisms of groups*

$$\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}/p^\infty) \cong \operatorname{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}/p^\infty) \cong \mathbb{Q}_p,$$

*where* $\omega\colon \mathbb{Q} \to \mathbb{Z}/p^\infty$ *corresponds to an element of* $\mathbb{Z}_p \subset \mathbb{Q}_p$ *if and only if* $\omega 1 = 0$*.*

*Proof* In (a), the group $\mathbb{Z}/p^\infty$ is the union of the cyclic subgroups $\mathbb{Z}/p^n$ generated by $1/p^n$, for $n > 0$. Any homomorphism $\omega\colon \mathbb{Z}/p^\infty \to \mathbb{Z}/p^\infty$ must map $\mathbb{Z}/p^n$ into itself; thus the endomorphism ring $\operatorname{End}(\mathbb{Z}/p^\infty)$ is the limit $\lim_n \operatorname{End}(\mathbb{Z}/p^n)$ of the endomorphism rings $\operatorname{End}(\mathbb{Z}/p^n)$. By Corollary 21, $\operatorname{End}(\mathbb{Z}/p^n) \cong \mathbb{Z}/p^n$ as a ring, and we may therefore identify the limit with $\mathbb{Z}_p$.

Because $\mathbb{Z}/p^\infty$ has *unique* division by any integer $m$ prime to $p$, every homomorphism $\mathbb{Z}[p^{-1}] \to \mathbb{Z}/p^\infty$ extends uniquely to a homomorphism $\mathbb{Q} \to \mathbb{Z}/p^\infty$; hence the first isomorphism in (b). Any $\omega\colon \mathbb{Z}[p^{-1}] \to \mathbb{Z}/p^\infty$ must satisfy $p^n(\omega 1) = 0$ for some $n$, so that $(\omega p^n)\mathbb{Z} = 0$; then $\omega \circ p^n$ factors through $\mathbb{Z}/p^\infty = \mathbb{Z}[p^{-1}]/\mathbb{Z}$. This gives enough information to identify the inclusion $\operatorname{End}(\mathbb{Z}/p^\infty) \subset \operatorname{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}/p^\infty)$ with $\mathbb{Z}_p \subset \mathbb{Q}_p$. □

*Proof of Theorem 25* Instead of dealing with $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ directly, we first embed

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}/p^\infty \subset \prod_p \mathbb{Z}/p^\infty,$$

and write

$$A = \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \subset \operatorname{Hom}(\mathbb{Q}, \prod_p \mathbb{Z}/p^\infty) \cong \prod_p \mathbb{Q}_p,$$

with the help of Lemma 27. Then $A$ consists of those homomorphisms $\omega\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ whose coordinates $\omega_p\colon \mathbb{Q} \to \mathbb{Z}/p^\infty$ satisfy $\omega_p 1 = 0$ for all except finitely many $p$. Thus $A$ is as described. Finally, we note that the subgroup $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$ of $A$ given by diagram (26) is as indicated. □