i)

Week of 9/12 : $\boxed{\text{intro to limits \& continuity}}$

domain

Limits & Continuity

range

$f : \mathbb{R} \to \mathbb{R}$

$x \in \mathbb{R} \mapsto f(x) \in \mathbb{R}$

Ex

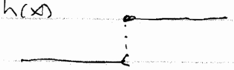
$f(x) = x + k$

$= kx$

$= x^2, \sqrt{x}, \ldots$

domain $= \mathbb{R}_+$

$|x|, h(x)$



$h(x) = 0 \quad x < 0$

$= 1 \quad x \geq 0$

Heaviside fn.

graph of f

$f : \mathbb{R} \longrightarrow \mathbb{R}^2$

$x \longmapsto (x, f(x))$

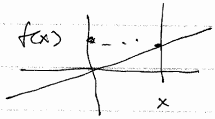
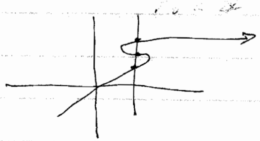


image of f $\subset \mathbb{R}^2$

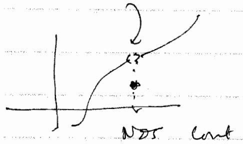
determines f

Subset G of $\mathbb{R}^2$ is the graph of a fn. def'd on $D \subset \mathbb{R}$ $\Leftrightarrow$

- $\forall x \in D \; \exists \, (x, y) \in G$
- $(x, y_0), (x, y_1) \in G \Rightarrow y_0 = y_1$



NOT the graph of a fn.

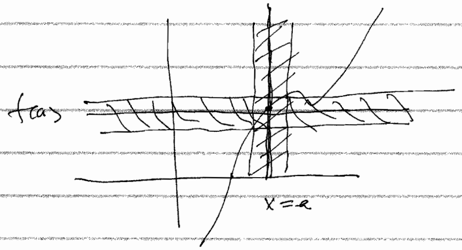


NOT cont!

Def'n of continuity at $a \in$ domain f

$$\lim_{x \to a} f(x) = f(\lim_{x \to a} x) \quad (= f(a))$$

ii)



WANT 1) x close to a $\Rightarrow$ f(x) close to f(a)

2) $|x - a|$ small $\Rightarrow$ $|f(x) - f(a)|$ small

? small = as small as we want

$= < \epsilon$, for any $\epsilon > 0$.

3) We can make $|f(x) - f(a)|$ small $(< \epsilon)$ by making $|x - a|$ small $(< \delta(\epsilon))$

$$\boxed{\begin{array}{l} \text{Given } \forall \epsilon > 0 \;\; \exists \, \delta(\epsilon) \;\; \text{such that} \\ \\ |x - a| < \delta(\epsilon) \Rightarrow |f(x) - f(a)| < \epsilon . \end{array}}$$

Ex $f(x) = x + k$

Claim If $|x - a| < \delta$ then $|f(x) - f(a)| < \delta$

$\|$

$|(x+k) - (a+k)| = |x - a|$

$\therefore \forall \epsilon > 0 \; \boxed{\exists \, \delta(\epsilon) := \epsilon}$ such that

$|x - a| < \delta(\epsilon) \Rightarrow |f(x) - f(a)| < \delta(\epsilon) = \epsilon$!

Ex. $g(x) = kx$, $(k \neq 0)$

$|g(x) - g(a)| = |kx - ka| = |k|.|x-a|$

$\therefore$ if $|x-a| < \delta$ then $|g(x)-g(a)| < |k| \cdot \delta$

want $|g(x)-g(a)| < \epsilon$?

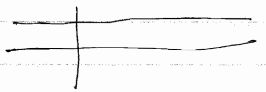
Take $\delta = \epsilon/|k|$:

$|x-a| < \epsilon/|k| \Rightarrow |g(x)-g(a)| < |k|. \epsilon/|k| = \epsilon$.

Ex. $g(x) = $ const $(=0)$

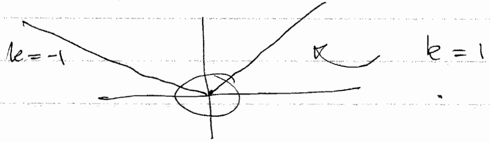
$|g(x) - g(a)| = |\text{const} - \text{const}| = 0 < \epsilon \quad \forall \epsilon$

$\therefore$



continuous

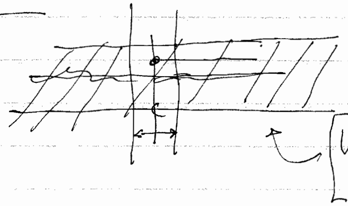
Ex.



? $a=0$ $|f(x)-f(0)| = ||x|-0| = |x| < \epsilon$?

$|x-a| = |x-0| = |x| < \delta$? Take $\delta = \epsilon$

Ex.



[large no matter how small.

§6 p116-7 Thm 1 & 2 ;

THEOREM 2

Given fns $f, g : \mathbb{R} \to \mathbb{R}$ & a, such that g is cont at a

& f is cont at $g(a)$

Then $(f \circ g)(x) := f(g(x))$ is cont at a:

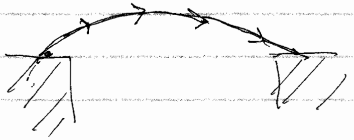
$$\begin{tikzcd}\mathbb{R} \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R}\end{tikzcd}$$

Ex $f(x) = x+k$

$g(x) = cx$

$(f \circ g)(x) = f(cx) = cx+k$

⍟ [The most efficient way to prove a fn continuous is to break it down as a composition of simple fns, that you already know to be continuous



[bridge-building; weakest link issue, etc.]

Proof

Want to show $|f(g(x)) - f(g(a))|$ small. $(<\epsilon)$

know that $|g(x) - g(a)| < \delta_f(\epsilon) \Rightarrow$ desired result.

know that if $|x-a| < \delta_g(\delta_f(\epsilon))$ then $|g(x)-g(a)| < \delta_f($

$\therefore \delta_{f\circ g}(\epsilon) = \delta_g(\delta_f(\epsilon))$ does the trick, QED v)

THM 1: If $f, g : \mathbb{R} \to \mathbb{R}$ continuous at a,

then $f+g$, $f g$ cont. at a.

Moreover, if $f(a) \neq 0$, then $x \longmapsto \frac{1}{f(x)}$ cont. at a.

Proof, deferred:

Note however that the last assertion follows from the claim: $x \mapsto 1/x : \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\} \to \mathbb{R}$ ($\mathbb{R}_+ :=$) is continuous: $x \mapsto f(x) \longmapsto 1/f(x)$:

(What about if $f(x) < 0$?)

More examples: $x \mapsto x^2$ is continuous

Need to show

$\forall \epsilon \exists \delta \quad |x-a| < \delta \Rightarrow |x^2 - a^2| < \epsilon$

$= |x-a| \cdot |x+a|$.

Now $|x+a| \leq |x| + |a|$, and if x is near a, then $|x| < 1+|a|$;

so $|x+a| \leq 1 + 2|a|$ for x near a.

$\therefore \; |x^2 - a^2| \leq |x-a| \cdot (1+2|a|)$

$\therefore$ if $|x-a| < \frac{\epsilon}{1+2|a|} := \delta$,

then $|x^2 - a^2| < \epsilon$.

$\Leftarrow$ easier: use thm 1, $f(x) = x$, $g(x) = x$, $f \cdot g = x^2$ is cont!

? $\mathbb{R}_+ \ni x \longmapsto \sqrt{x} \in \mathbb{R}_+$

$|\sqrt{x} - \sqrt{a}| \cdot |\sqrt{x} + \sqrt{a}| = |x-a|$.

We want to bound $|\sqrt{x} - \sqrt{a}|$ from above.

Note $|\sqrt{x} + \sqrt{a}| \geq |\sqrt{x}| + |\sqrt{a}| > \sqrt{a}$,

so $\frac{1}{\sqrt{a}} > \frac{1}{|\sqrt{x}+\sqrt{a}|}$.

$\therefore \; |\sqrt{x} - \sqrt{a}| < \frac{1}{\sqrt{a}} \cdot |x-a|$

$\therefore$ if $|x-a| < \sqrt{a}\,\epsilon := \delta$

then $|\sqrt{x} - \sqrt{a}| < \epsilon$.