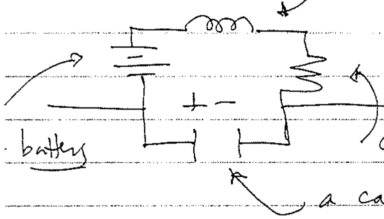
i)

Starting or initial transients:

RLC circuit

inductance (a coil)



a node

battery

a resistor

a capacitor

cf. Thomas Pynchon, V

"Kilroy was here"

$I = I(t)$ = current (in amperes)

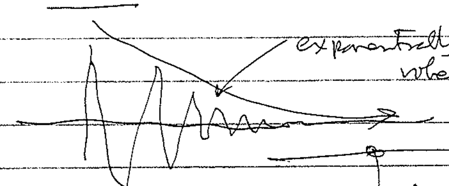
through the circuit at time t

satisfies a 2nd order differential equation

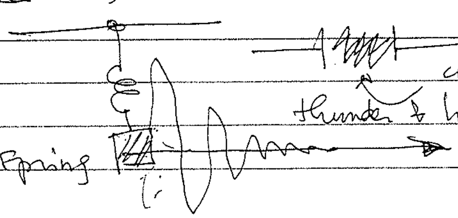
$$\boxed{L\cdot\frac{d^2I}{dt^2} + R\cdot\frac{dI}{dt} + C\cdot I = 0}$$

L = inductance
C = capacitance
R = resistance ≥ 0
constants

with a solution $I(t)$



exponentially decaying oscillation when a spark jumps across the capacitor



cf. eg thunder & lightning

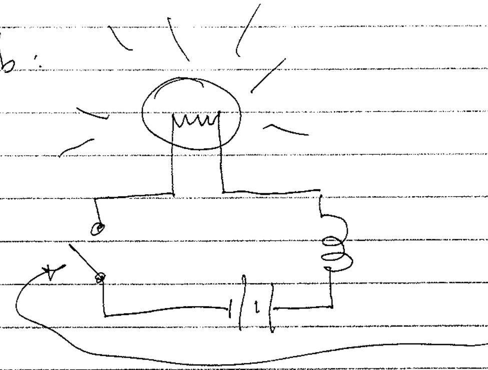
Formally the same eqn as the hanging spring

ii)

This is pretty complicated, so let's simplify

to the case of an oldfashioned incandescent

lightbulb:



No capacitor, but instead we have a switch

which we turn on at $t=0$: thus we

have the system of equations

$$L\,\frac{d^2I}{dt^2} + R\,\frac{dI}{dt} + \cancel{C\cdot I} = 0$$

and initial conditions $I(t) = 0$ at $t = 0$

To simplify, let $J = \frac{dI}{dt}$: then we have

$$L\,\frac{dJ}{dt} + RJ = 0 \iff L\,dJ + RJ\,dt = 0$$

iii)

Divide by I, and integrate:

$$\frac{dI}{I} = -\underbrace{\frac{R}{L}}_{\text{constant}}\,dt \quad \Rightarrow$$

$$\int \frac{dI}{I} = \boxed{\log I = -\frac{R}{L}t + \text{Constant of integration}}$$

Exponentiate to get rid of the logarithm

$\Rightarrow$

$$\frac{dI}{dt} = I = \exp\left(-\frac{R}{L}t + \text{Const}\right)$$

$$= A e^{-R/L\,t} \quad \text{for some } A = e^{\text{const.}}$$

Integrate again to get

another const of integration

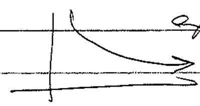
$$I(t) = \int A e^{-R/L\,t}\,dt = -\frac{AL}{R}e^{-R/L\,t} + \text{Const.}$$

rearrange

some constant

$$= B(1 - e^{-R/L\,t}) = 0 \quad \text{at } t = 0$$

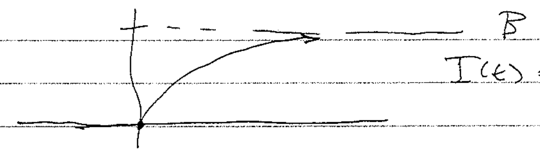
$$\longrightarrow B \text{ as } t \to \infty, \text{ because } e^{-R/L\,t} \to 0 \text{ as } t \to \infty$$



exponential decay.

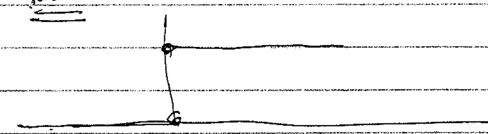
iv)

The MORAL of the story is that the lightbulb doesn't light up instantaneously, or discontinuously:



$I(t) = B(1 - e^{-R/L\,t})$

NOT



like this...

//

v)

§Limits

Ch 8 B3–B5

Ch 28 588–600

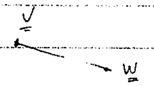
Previously $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$

line, plane, 3D

sufficient for geometry, physics, real world:

but: where does $\mathbb{R}$ come from?

In particular:



$\text{dist}(\underline{v}, \underline{w}) = |\underline{v} - \underline{w}|$

$= \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2}$

How do we know square roots exist?

Renaissance, decimal #'s:

```
       1 . 4  1  4 ...
    √ 2  00 00 00
          1²
  4
 20 ) 1 00
        96
   1
 280 ) 4 00
       2 81
    4
 2820 ) 1 19 00
        1 12 96
        ) 6 04 00
```

$d = d_0.d_1d_2\ldots$ infinite sequence

$d_i \in \{0, \ldots, 9\}$

§22 #14.e
p. 464

$d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n}$

$\in$ rational numbers: $\mathbb{Q}$

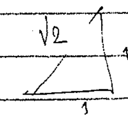
$\frac{10^n d_0 + 10^{n-1} d_1 + \cdots + d_n}{10^n}$

= ratio of integers.

Claim: a real number is the limit of (an increasing) sequence of rational numbers

vi)

Claim



Rational #'s are inadequate for geometry.

[Big problem for the Greeks]

There is no rational number whose square is 2

Proof: Fund Thm of arithmetic:

any integer ≥ 1 is the product of a unique set (up to order) of prime integers ≥ 1.

n is prime, or not. If it's prime, we're finished. If it's not prime, then it's divisible by something

$n = p \cdot q$ with $p, q < n$.

Induction: Suppose we've proved this for all #'s smaller than n. Then p, q are products of primes, etc.

vii)

viii)

Claim: no prime # can have a rational square root.

$p = r^2$, $r = \frac{n_1 \cdots n_k}{m_1 \cdots m_\ell}$ n's, m's prime, w/ repeats.

Then $p m_1^2 \cdots m_\ell^2 = n_1^2 \cdots n_k^2$

odd # of primes = even # of primes. — $\boxed{\text{impos}}$

A 'defect of beauty': decimal representations of real #'s are not unique:

$\boxed{.999\ldots = 1.000\ldots}$?

p72, 4Q2, §23 #7

$\frac{9}{10} + \frac{9}{10^2} + \cdots = \sum_{k\geq 1} 9 \cdot 10^{-k} = \frac{9}{10} \sum_{k\geq 0} 10^{-k}$

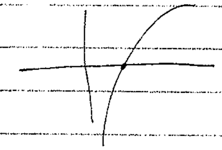
recall $(1+x+\cdots+x^n)(1-x) = 1-x^{n+1}$

$$\begin{array}{r} 1-x \\ \hline 1+x+\cdots+x^n \\ -x-x^2-\cdots-x^n-x^{n+1} \\ \hline 1-x^{n+1} \end{array}$$ "Telescoping sum"

$= 1+x+\cdots+x^n = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{1}{1-x} \cdot x^{n+1}$

Claim: If $0 < x < 1$ then $x^n \to 0$

More precisely: For any $\epsilon > 0$, $\exists n$ such that $x^n < \epsilon$

log  [increasing fn, $\log' = \frac{1}{x} > 0$ if $x > 0$]

$\epsilon > x^n \iff \log \epsilon > n \log x$

$-\log \epsilon < n(-\log x)$

$n > \frac{-\log \epsilon}{-\log x}$

$\boxed{n = \frac{\log \epsilon}{\log x}}$

if $|x| < 1$, $h > n$

$\sum_{k=0}^{n} x^k \to \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \to \frac{1}{1-x}$

Ex. $x = \frac{1}{10}$

$\lim_{n\to\infty} (a + b_n) = a + \lim_{n\to\infty} b_n = a$

$.999\ldots = \frac{9}{10} \frac{1}{1-1/10} = 1.00\ldots$

? More 'elementary' argument ?

$\inf \{x^n \mid n = 1, 2, \ldots\}$ = greatest lower bound $= B$,

$xB = x \cdot \inf\{x^n \mid n = 1, 2, \ldots\} \overset{?}{=} \inf\{x^n \mid n = 2, 3, \ldots\}$

$\geq \inf\{x^n \mid n = 1, 2, \ldots\} = B$

But $B > xB \geq B$ if $B \neq 0$!

iv)

$B = \inf\{x \mid x \in S\}$, $a>0$

$\Rightarrow$

$aB = \inf\{ax \mid x \in S\}$.

Proof

1) $x \in S \Rightarrow B \le x$

2) $(x \in S \Rightarrow B' \le x) \Rightarrow (B' \le B)$

$a>0$

1) $x \in S \Rightarrow C \le ax \Leftrightarrow a^{-1}C \le x$

2) $(x \in S \Rightarrow C' \le ax) \Rightarrow C' \le C$

$a^{-1}C \le x \Rightarrow a^{-1}C' \le a^{-1}C$

$\Rightarrow a^{-1}C = B$

//

Real numbers contain rational #'s,

S bdd below has least upper bound

Ex $\boxed{\sqrt{2} = \inf\{r \in \mathbb{Q} \mid r^2 < 2\}}$.

Dedekind cuts $r = \{S \subset \mathbb{Q} \mid$ if $s' \in \mathbb{Q}, s' > s$ then $s' \in S\}$.

x)

Blame Socrates! what is the love

$\mathbb{N} = 0, 1, \ldots$ = natural numbers

von Neumann: $0 = \{0\}$ successor.

$n+1 = \{n, \{n\}\}$: $1 = \{0, \{0\}\}$

A set is a kind of conceptual box, which may or may not have anything inside of it.

$S \notin S$, $S \subset S$

Subsets are boxes containing collections of things in the box containing S.

$\forall m+n \in \mathbb{N}$ $\exists m+n \in \mathbb{N}$,

$l + (m+n) = (l+m)+n$, etc.

$\exists\, \mathbb{Z} = \ldots -2, -1, \ldots, 1, 2, \ldots$

$\forall k \in \mathbb{Z}\ \exists -k \ni k + (-k) = 0$, etc.

$\mathbb{Q}$ = rational numbers = equivalence classes $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

$(n, m) \to n/m$

A, B sets

$(a,b) \to$

$A \times B = \{\{a, \{a,b\}\}\}$ $a \in A, b \in B$,

1§3 p54

$(a,b) = (c,d)$

$\Leftrightarrow$

$a, \{a,b\} \Leftrightarrow c, \{c,d\}$

$a \in \{a,b\}$, $c \in \{c,d\}$

if $\{a,b\} = c$, ... trouble

$\exists$ eq. relation $(n,m) \sim (a,b) \Leftrightarrow$

$\exists j, k \in \mathbb{Z} - 0$ such that

$kn = ja$

$km = jb$

$(6, 9) \sim (8, 12)$