1)

113 Notes
for Weds 29 Aug 2011

Continued from Monday:

Spivak has a discussion of vectors in Appendix I, p75-79, cf also p58.

I'm going to use this as an excuse to review some background ALGEBRA, in particular INEQUALITIES.

Recall that if $a \in \mathbb{R}$ then $a^2 \geq 0$! Let $|a| = \max(a, -a)$
(or something...)

Notation: If $\underline{x} \in \mathbb{R}^n$

If $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then

$$|\underline{x}| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0 \quad \text{(non-negative square root)}$$

Ex if $n = 1$ then $|x| = |\underline{x}|$

If $\underline{v}, \underline{w} \in \mathbb{R}^n$ then $\underline{v} - \underline{w} = (v_1 - w_1, \ldots, v_n - w_n)$.

We showed (or decided) on Monday that if $\underline{v}, \underline{w} \in \mathbb{R}^n$, then

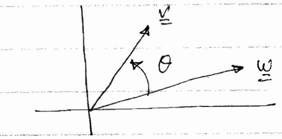
$$\operatorname{dist}(\underline{v}, \underline{w}) = |\underline{v} - \underline{w}| = \sqrt{(v_1 - w_1)^2 + \cdots + (v_n - w_n)^2} \geq 0$$

$(= 0 \iff \underline{v} = \underline{w})$.

2)

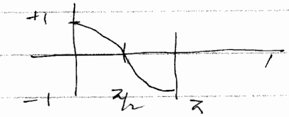
On p ii) of the notes from Monday, I claimed that

if $\underline{v}, \underline{w} \in \mathbb{R}^2$, then




$$\cos\theta = \frac{v_1 w_1 + v_2 w_2}{|v| \cdot |w|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2} \cdot \sqrt{w_1^2 + w_2^2}}$$

Since $|\cos\theta| \leq 1$




This doesn't make sense unless

$$|v_1 w_1 + v_2 w_2| \leq |v| \cdot |w|.$$

Problem 1.19 (p17) asks you to prove this.
Here is 1.19c:

Claim $(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2$

expand, using distributive axiom

$$x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2$$

$$x_1^2 y_1^2 + 2 x_1 y_1 x_2 y_2 + x_2^2 y_2^2$$

Cancel

$$x_1^2 y_2^2 - 2 x_1 y_2 x_2 y_1 + x_2^2 y_1^2$$

3)

Corollary:

$$(x_1^2+x_2^2)\cdot(y_1^2+y_2^2) = (x_1y_1+x_2y_2)^2 + \underbrace{(x_1y_2-x_2y_1)^2}_{\geq 0,\ \text{since it's a square!}}$$

$$\geq (x_1y_1+x_2y_2)^2.$$

Now take square roots:

(claim, if $a > b \geq 0$ then $\sqrt{a} > \sqrt{b}$.)

$$|\underline{x}|\cdot|\underline{y}| \geq |x_1y_1+x_2y_2|$$

$$\Rightarrow \quad 1 \geq \frac{|x_1y_1+x_2y_2|}{|\underline{x}|\cdot|\underline{y}|}$$

(I guess we should assume that $\underline{x}, \underline{y} \neq 0$!)

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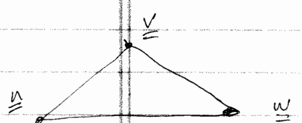
We can also use this to prove the triangle inequality (in $\mathbb{R}^2$: but there is a very similar proof in $\mathbb{R}^n$):




$$\text{dist}(\underline{u},\underline{v}) + \text{dist}(\underline{v},\underline{w}) \geq \text{dist}(\underline{u},\underline{w})$$

$$\Leftrightarrow$$

$$|\underline{u}-\underline{v}| + |\underline{v}-\underline{w}| \geq |\underline{u}-\underline{w}|$$

Simplify notation: let $\underline{x} = \underline{u} - \underline{v}$

$\underline{y} = \underline{v} - \underline{w}$ :

Then $\underline{x} + \underline{y} = \underline{u} - \underline{w}$,

and the $\Delta$ inequality becomes

$$|\underline{x}| + |\underline{y}| \geq |\underline{x}+\underline{y}|.$$

4)

It therefore suffices to show that

$$(|\underline{x}| + |\underline{y}|)^2 \geq |\underline{x}+\underline{y}|^2$$

$$\|$$

$$|\underline{x}|^2 + 2|\underline{x}|\cdot|\underline{y}| + |\underline{y}|^2 \geq (x_1+y_1)^2 + (x_2+y_2)^2$$

$$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2$$

$$= (x_1^2+x_2^2)^2 + 2(x_1y_1+x_2y_2) + (y_1^2+y_2^2)$$

or in other words that

$$2|\underline{x}|\cdot|\underline{y}| \geq 2\cdot(x_1y_1+x_2y_2)$$

which is what we've just done, on the previous page. //

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Homework for FRIDAY NEXT. (9 Sept)

§1 #22
§2 #1, 17d
§3 #8, 14
§4 #2, 4, 9

choose 5

— HAVE A ☺ LABORDAY WKEND! —