WEEK 9- NOV 1: OUTLINE OF LECTURES

1. MONDAY.

1) Sequences. Theorem A: every sequence has a monotone subsequence. Theorem B: every bounded sequence has a convergent subsequence.
2) Improper integrals: integrals of type $\int_a^{+\infty} f(x)dx$. Definition.
The *p-integral* $\int_1^{+\infty} \frac{dx}{x^p}$, $p > 0$.
Comparison theorem: if $f, g > 0$ and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L \neq 0$, then $\int_a^{\infty} f(x)dx \sim \int_a^{\infty} g(x)dx$ [the two integrals have the same nature]
Steps in applying this theorem: guess the comparison; check that the comparison holds (compute the limit); draw the conclusion.
Example: $\int_{10}^{\infty} \frac{dx}{\sqrt{x^3+x}}$. Compare to $\frac{1}{x^{3/2}}$: $\lim\limits_{x\to\infty} \frac{\frac{1}{\sqrt{x^3+x}}}{\frac{1}{x^{3/2}}} = 1$ ($\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ exists and is not zero), hence the comparison holds. Conclusion: $\int_{10}^{\infty} \frac{dx}{\sqrt{x^3+x}}$ has the same nature as $\int_{10}^{\infty} \frac{dx}{x^{3/2}}$, i.e. convergent, since the latter is a p-integral with $p = 3/2 > 1$.

2. TUESDAY.

More examples of the comparison theorem: $\int_1^{\infty} \sin^2(\frac{1}{x})dx \sim \int_1^{+\infty} \frac{dx}{x^2}$, convergent.
The integral $\int_1^{\infty} \frac{dx}{x(x+1)}$, its nature and actual value.
Series. Definition. Partial sums. Terms of the series. Definition of convergence.
[Recall: a sequence is divergent if *either* it doesn't have a limit *or* it has a limit but the limit is infinite. So the sequence $(2^n)_{n\geq 1}$ is divergent since $\lim 2^n = +\infty$.]
Examples:

$$1)\sum_{n=1}^{\infty} 1 \quad 2)\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad 3)\sum_{n=1}^{\infty} (-1)^n \quad 4)\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$$

Theorem 1. (Basic Convergence Test)

- $\displaystyle\sum_{n=1}^{\infty} a_n$ convergent $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$.
- $\lim\limits_{n\to\infty} a_n \neq 0 \Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ divergent.

Application: in example 3), $\sum_{n=1}^{\infty}(-1)^n$ is divergent since $\lim\limits_{n\to\infty}(-1)^n \neq 0$ (in fact, the limit doesn't even exist).
The converse of this theorem is not true: in example 4) the series $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$ is divergent, yet its main term tends to zero, $\lim\limits_{n\to\infty} \ln(1+\dfrac{1}{n}) = 0$.

3. Wednesday

Given x a (fixed) real number, the *geometric series* of x is $\sum_{n=0}^{\infty} x^n$. The main term of this series is x^n, and its n^{th} partial sum is $s_n = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$ (if $x \neq 1$).

Theorem 2.

$$\sum_{n=0}^{\infty} x^n \text{ is } \begin{cases} \text{convergent}, & \text{if } x \in (-1,1) \\ \text{divergent}, & \text{if } x \notin (-1,1) \end{cases}$$

Moreover, $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, for $x \in (-1,1)$.

Examples:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}, \quad 0.252525\cdots = \frac{25}{99}$$

Derived formulas. Also, for $x \in (-1,1)$ we have

$$\sum_{n=0}^{\infty} x^{2n+1} = \frac{x}{1-x^2}, \quad \sum_{n=3}^{\infty} = \frac{x^3}{1-x}$$