WEEK 7: CONVERGENT SEQUENCES

A. Concepts introduced so far:

1. Bounded sequences. Mononotone sequences. How to decide whether a sequence is monotone.

2. The definition of a convergent sequence using $\epsilon, N(\epsilon)$. We discussed in detail the sequences $\frac{1}{n}$, $\frac{1}{\sqrt{n}}$, $\frac{2\sqrt{n}}{\sqrt{n}+1}, \dots$.

3. Main theorems:

- Any convergent sequence is bounded. The converse is of course not true (think of examples).
- Operation with sequences: if $a_n \to A$, $b_n \to B$, then $a_n \pm b_n \to A \pm B, a_n b_n \to AB$ etc...
- If $a_n \to L$ and $f(x)$ is continuous then $f(a_n) \to f(L)$. Example: $e^{\frac{n}{n+1}} \to e$.
- Pinching theorem. Application: $\frac{\sin(n^3)}{\sqrt{n}} \to 0$.
- Monotone & Bounded $\Rightarrow$ Convergent. The converse is of course not true [example: the sequence $\{\frac{(-1)^n}{n}\}_{n \geq 1}$ is convergent without being monotone.

B. The sequence $a_n = (1 + \frac{1}{n})^n$ and the number e.

Step 1. Inequality $\frac{x-1}{x} \leq \ln(x) < x - 1$ for $x > 1$.

Step 2. Inequality $\frac{1}{n+1} < ln(1 + \frac{1}{n}) < \frac{1}{n}$ for $n \geq 1$.

Step 3. Inequalities $e^{\frac{n}{n+1}} < (1 + \frac{1}{n})^n < e$ and $(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}$.

Step 4. Conclude that $\lim_n (1 + \frac{1}{n})^n = e$ by Pinching theorem.

Q: What is $N(\epsilon)$ for the sequence $a_n = (1 + \frac{1}{n})^n$?

To answer, let $b_n = (1 + \frac{1}{n+1})^{n+1}$. Since $b_n = (1 + \frac{1}{n})a_n$ and $a_n \to e$ it is clear that $b_n \to e$ as well (operations with limits). Moreover, we saw at step 3 that $a_n < e < b_n$ for any $n \geq 1$. In particular, for $n = 1$ we obtain the estimate $2 < e < 4$. If we want to see how fast a_n approaches e, we have to gauge the difference $|a_n - e|$. We do this by noticing that

$$|a_n - e| < |b_n - a_n| = \frac{1}{n} a_n < \frac{e}{n} < \frac{4}{n}$$

where we used the available information that $e < 4$. Hence

$$\boxed{|a_n - e| < \frac{4}{n}} \quad \text{which is the same as} \quad a_n < e < a_n + \frac{4}{n}$$

Hence if we want to make sure that $|a_n - e|$ is "small" (i.e. $< \epsilon$), it's enough to ensure that $\frac{4}{n}$ is "small" ($< \epsilon$). We conclude with:

$$\text{for } n \geq \frac{4}{\epsilon} \Rightarrow |a_n - e| < \epsilon \quad \text{i.e.} \quad N(\epsilon) = \frac{4}{\epsilon}$$

In particular for $\epsilon = 0.01$,

$$a_{401} < e < a_{401} + 0.01$$

Since $a_{401} = 2.7149..$ we see that $2.7149.. < e < 2.7249..$ hence $e = 2.7..$ to one decimal place.

Homework problem. Show that the sequence a_n is monotone increasing.

Remark. As we noticed, *wait time* to get within ϵ of the limit is proportional to $\frac{1}{\epsilon}$, which is pretty large. We will see that there are faster ways to approximate e, and one of them will be using the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ which will be studied towards the end of the semester.

C. The sequence $y_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)$ and Euler's constant γ.

i) The sequence y_n is *monotone* decreasing, as $y_{n+1} - y_n = \frac{1}{n+1} - \ln(1 + \frac{1}{n}) < 0$.

ii) Since y_n is monotone decreasing, we have $y_n \leq y_1 = 1$ for any $n \geq 1$. Moreover, since $1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \int_1^{n+1} \frac{dt}{t} = \ln(n+1)$ (draw the graph of $1/x$) it follows that $Y_n \geq \ln(n+1) - \ln(n) > 0$ for any $n \geq 1$. So we proved that $0 < y_n \geq 1$, for any $n \geq 1$, i.e. y_n is a *bounded* sequence.

Having proved that y_n is monotone & bounded we conclude that y_n is convergent. Let γ denote the limit of this sequence:

$$\gamma = \lim_{n \to \infty} y_n = \lim_{n \to \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n}) - \ln(n)$$

This special number is also called the Euler-Mascheroni constant. We can re-write what we proved so far as

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln(n) + \gamma + z_n$$

where $z_n = y_n - \gamma \to 0$, as $n \to \infty$.

Homework. Show that $\frac{1}{2(n+1)} < z_n < \frac{1}{2n}$ for $n \geq 1$.