

**WEEK 11- LECTURE OF 11/17/04**  
**SECTION 11.4: NON-POSITIVE SERIES**

1. ABSOLUTELY CONVERGENT SERIES

**Definition.** The "absolute series" of  $\sum a_n$  is the positive series  $\sum |a_n|$ .

**Theorem 1.1** (Theorem AC). *If  $\sum |a_n|$  is convergent, then the original series  $\sum a_n$  must be convergent as well.*

This is Theorem 11.4.1. of the textbook, the proof is on page 657.

**Definition**

$\sum a_n$  is absolutely convergent (**AC**) iff  $\sum |a_n|$  is convergent.

$\sum a_n$  is conditionally convergent (**CC**) iff  $\sum a_n$  is convergent but  $\sum |a_n|$  is not.

In other words,

**AC:** both  $\sum a_n$  and  $\sum |a_n|$  are convergent.

**CC:**  $\sum a_n$  convergent, but  $\sum |a_n|$  divergent.

**Proposition 1.2** (Basic Comparison). *If:*

- $|a_n| \leq b_n \quad n \geq n_0$
- $\sum b_n$  is convergent

*Then  $\sum a_n$  is absolutely convergent (and hence convergent).*

*Example 1.* The series  $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$  is convergent, since  $\left| \frac{\sin(k)}{k^2} \right| \leq \frac{1}{k^2}$  and  $\sum \frac{1}{k^2}$  is convergent.

2. ALTERNATING SERIES

*Example 2.* The series  $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$  is clearly NOT AC, since its *absolute series* is the harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \dots$  (divergent).

But is this series convergent? The answer is **yes**, and the reason is the following theorem. [We say that such a series is conditionally convergent (**CC**).]

**Theorem 2.1** (Alt). *If  $a_1 \geq a_2 \geq a_3 \geq \dots$  and  $\lim a_n = 0$ , then the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

*is convergent.*

**Remark:** this theorem helps us to determine the nature of a certain series without saying anything about the actual sum of the series.

## 3. THE RATE OF CONVERGENCE OF A GIVEN SERIES

**3.1. Tuesday Lecture.** On Tuesday we determined the rate of growth for the partial sums of the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$ . We saw that  $s_n \simeq \ln(n)$  (for this we used the picture from the integral test), hence  $s_n \geq M$  for  $n \gg e^M$ .

*Example.* If we want the partial sum to be larger say than 1000 (and we know that this eventually will happen since  $\sum \frac{1}{n}$  is divergent) we have to sum the first roughly  $N = e^{1000}$  terms, which is a fairly large number.

In contrast, notice that  $s_{100} \approx \ln(100) = 2 \ln(10)$  which is roughly a number between 4 and 6, i.e. quite small.

Similarly,  $s_{1000} \approx \ln(1000) = 3 \ln(10)$  which is also a small number, certainly  $< 10$ . This shows that  $\sum \frac{1}{n}$  diverges very slowly.

*Remark.* If we want to be more precise, recall that

$$y_n = 1 + \frac{1}{2} + \cdots + \frac{1}{1/n} - \ln(n) \rightarrow \gamma, \quad n \rightarrow \infty$$

Here  $\gamma$  is Euler's constant, and the numerical value is  $\gamma = 0.577..$

[Reference: <http://mathworld.wolfram.com/Euler-MascheroniConstant.html>]

Moreover, we proved in class that  $\gamma < y_n < \gamma + \frac{1}{n}$ , which can be written in short-hand notation as  $y_n = \ln(n) + \gamma + O(\frac{1}{n})$ . Hence

$$s_n = \sum_{k=1}^n \frac{1}{k} = \ln(n) + \gamma + O(\frac{1}{n})$$

We see now that for example  $s_{1000} \approx \ln(1000) + 0.577$  roughly up to 2 decimal places. Since  $\ln(1000) = 3 \ln(10)$  (and knowing  $\ln(10) = 2.30..$ ) we have the rough estimate

$$s_{1000} = \sum_{k=1}^{1000} \frac{1}{k} \approx 3 \times 2.30 + 0.57 = 6.90 + 0.57 = 7.47$$

The calculator actually gives  $s_{1000} = 7.4855...$ , so we were not too far in our estimate.

### 3.2. Convergence rate for alternating series.

In the case of a convergent alternating series of sum  $S$ ,  $\sum_{k=1}^{\infty} \frac{(-1)^k}{a^k} = S$ , we have

$$|s_n - L| \leq a_n$$

and this is enough to determine the convergence rate of the series. We arrive at this fact in the proof of the theorem on alternating series (Theorem Alt).

We will then determine the convergence rate in the case of the alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \text{ and compare it to the case of } \sum_{k=1}^{\infty} \frac{1}{k^2}.$$