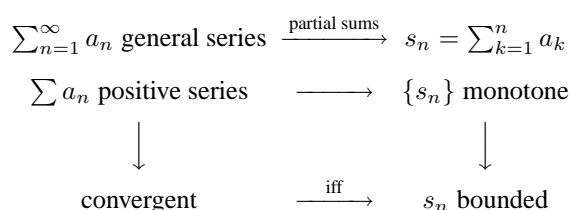


## WEEK 10- NOV 8: OUTLINE OF LECTURES

This week material: convergence tests for **positive series**.

**Definition:** a positive series is a series  $\sum a_n$  whose terms are positive,  $a_n \geq 0, \forall n \geq 1$ .

The convergence tests that we use for positive series are based on the following simple observation: the partial sums of a positive series form a monotone (increasing) sequence, therefore convergence is synonymous (in the case of positive series) with the boundedness of the sequence of partial sums [recall that a monotone sequence is convergent if and only if it is bounded].



- (1) **Integral test:** use it to determine the nature of the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ .  
 [Recall: the  $p$ -integral is convergent when  $p > 1$  and divergent otherwise.]  
 Other examples that can be settled with this test are  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ ,  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ .
- (2) **Comparison test:** use it to determine the nature of a variety of series based on our prior knowledge of the  $p$ -series and geometric series.  
*Example:* the series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$  is convergent by comparison to  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ .  
 To justify the comparison,  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^3+1}} : \frac{1}{n^{3/2}} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3+1}} = 1$ .  
*Remark.* There are two versions of the comparison theorem:
  - **a)**  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0 \Rightarrow \sum a_n \sim \sum b_n$ .
  - **b)**  $0 \leq a_n \leq b_n$  and  $\sum b_n$  is convergent  $\Rightarrow \sum a_n$  is convergent.
  - **b)**  $0 \leq a_n \leq b_n$  and  $\sum a_n$  is divergent  $\Rightarrow \sum b_n$  is divergent.
- (3) **Root test:** use it when the general term of the series contains exponentials.  
*Example:* the series  $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$  is convergent by applying the root test. We find in this particular case  $\rho = \frac{1}{2}$ .
- (4) **Ratio test:** use it when the general term contains factorials.  
*Example:* the series  $\sum_{n=1}^{\infty} \frac{x^n}{n!}$  (where  $x$  is a fixed positive real number) is convergent by applying the ratio test. We find in this case we find  $\lambda = 0$ .

### 1. REVIEW MATERIAL

Recall that a series  $\sum_{n=1}^{\infty} a_n$  is convergent provided the sequence  $(s_n)_{n \geq 1}$  of partial sums is convergent, i.e.

$$\lim_{n \rightarrow \infty} s_n = S \quad (S = \text{finite real number})$$

Divergent=not convergent. In other words, a series  $\sum a_n$  can be divergent for two reasons:

- **EITHER**  $\lim_{n \rightarrow \infty} s_n$  doesn't exist
- **OR**  $\lim_{n \rightarrow \infty} s_n = \pm \infty$

In any case, the **sum of the series** is always given by

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n$$

provided the limit exists. Otherwise, the series has no sum.

**Examples.**

- a) The series  $\sum_{n=1}^{\infty} (-1)^n$  is divergent because the sequence  $(s_n)$  has no limit.  
 b) On the other hand, the series  $\sum_{n=1}^{\infty} 1$  is divergent since in this case  $s_n = n \rightarrow +\infty$ .  
 Therefore we can say that  $\sum_{n=1}^{\infty} 1 = +\infty$ .

**The harmonic series.** We have discussed two ways in which to prove that the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

The first method is based on the integral test (this is how we settled all the  $p$ -series), as  $f(x) = \frac{1}{x}$  is positive decreasing on  $[1, \infty)$ , hence  $\sum_{n=1}^{\infty} \frac{1}{n} \sim \int_1^{\infty} \frac{dx}{x}$ . The latter improper integral can be evaluated explicitly to see that it is divergent. Therefore our series is divergent.

The second method is based on the sequence  $y_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n)$ . We proved in class that this is a convergent sequence (monotone and bounded), and the limit is denoted by  $\gamma$ , Euler's number:

$$\lim_{n \rightarrow \infty} y_n = \gamma$$

( $\gamma$  is a special number like  $e$  and  $\pi$ .)

Writing  $1 + \frac{1}{2} + \dots + \frac{1}{n} = y_n + \ln(n)$ , this simply means that  $s_n = y_n + \ln(n)$ , where  $s_n$  is the  $n^{\text{th}}$  partial sum of  $\sum_{n=1}^{\infty} \frac{1}{n}$ . But  $y_n \rightarrow \gamma$  and  $\ln(n) \rightarrow \infty$ , therefore  $s_n \rightarrow \infty$ . It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n \rightarrow \infty} s_n = +\infty$$

which shows that the series is divergent.

If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then:

- Main term  $a_n \rightarrow 0$  (basic divergence test)
- Partial sum  $s_n \rightarrow S$  (finite limit)
- $\sum_{n=1}^{\infty} a_n = S$

If the series  $\sum_{n=1}^{\infty} a_n$  is divergent, then:

- Main term  $a_n \rightarrow$  (don't know)
- Partial sums:  $(s_n)_{n \geq 1}$  is a divergent sequence, so either  $\lim s_n = \pm\infty$  or simply  $\lim s_n$  does not exist.
- $\sum_{n=1}^{\infty} a_n =$  either infinite, or no sum.

If the series  $\sum_{n=1}^{\infty} a_n$  is positive and divergent, then:

- Main term  $a_n \geq 0$  but  $a_n \rightarrow$  (don't know)
- Partial sums:  $s_n \rightarrow +\infty$  (monotone divergent sequence)
- $\sum_{n=1}^{\infty} a_n = +\infty$ .

[A monotone increasing sequence is either convergent or it tends to  $+\infty$ . In any case, it always has a limit, be it finite or infinite.]