TAYLOR SERIES

CONCEPTS

GIVEN $f(x)$,

- n^{th} Taylor POLYNOMIAL:

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n.$$

- TAYLOR SERIES (IN X):

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

$$= \underbrace{f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n}_{} + \frac{f^{(n+1)}(0)}{(n+1)!}x^{n+1} + \dots\dots$$

$P_n(x)$ IS THE n^{th} PARTIAL SUM OF THE TAYLOR SERIES (IN X).

Basic Relation :

$$f(x) = P_n(x) + R_n(x)$$

$$f(x) = \left\{ f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n \right\} + R_n(x)$$

(A) $\boxed{\text{IF}}$ $|R_n(\tfrac{1}{2})| < 10^{-5}$

$\boxed{\text{THEN}}$ $f(\tfrac{1}{2}) \cong P_n(\tfrac{1}{2})$ to 4 DIGITS

$\downarrow$

GOOD FOR COMPUTING $f(\tfrac{1}{2})$ APPROXIMATELY

[SPECIFIC EXAMPLES]

(B) $\boxed{\text{IF}}$ $R_n(x) \to 0,\ n \to \infty$

$\boxed{\text{THEN}}$ GET NICE IDENTITY

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(n)}(0)}{n!} + \overset{R_n}{\dots}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

[TAYLOR SERIES EXPANSION]

Recall (Definition of SERIES):

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} (a_0 + a_1 + \dots + a_n)$$

[provided the latter limit exists]

$\boxed{\text{IF}}$ $R_n(x) \longrightarrow 0 \quad n \to \infty$

$\boxed{\text{THEN}}$ $$\lim_{n\to\infty} \Big(\underbrace{f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(n)}(0)}{n!}x^n}_{P_n(x)} \Big) = f(x)$$

(SIMPLY BECAUSE $f(x) = P_n(x) + R_n(x)$)

$\boxed{\text{THIS ACTUALLY MEANS:}}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(x)$$

THE TAYLOR EXPANSION OF $f(x)$ IN POWERS OF ~~[illegible]~~ X.

KEY (IN BOTH A & B):

CONTROL $R_n(x)$!

INPUT:

LAGRANGE FORMULA:

$$\boxed{R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}\, x^{n+1}}$$

c BETWEEN 0 AND x.

$$\lim R_n(x) = \lim \cancel{\frac{f^{(n+1)}(c_n)}{(n+1)!}}\, x^{n+1} =$$

$\downarrow$

ESTIMATE: $|R_n(x)| \leq \ldots\ldots$ hopefully small

(GET RID OF C)

STUDY: $\lim_{n\to\infty} R_n(x) \xrightarrow{\text{Hopefully}} 0$

BASIC EXAMPLES.

FOR $\forall x \in \mathbb{R}$ ARBITRARY,

Ex. 1) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Ex. 2) $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Ex. 3) $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin(10) = \sum_{n=0}^{\infty} (-1)^n \frac{10^{2n+1}}{(2n+1)!}$$

QUESTION

$$\left|\sum a_n\right| \neq \sum |a_n|$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \quad ?$$

$$\cancel{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}} \quad \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \quad ?$$

Examples.

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} = \frac{1}{e}$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2\pi)^{2n}}{(2n)!} = \cos(2\pi) = 1$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(12\pi)^{2n}}{(2n)!} = \cos(12\pi) = 1$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!} = \sin(3).$$

Example ~~5~~ 4 LOGARITHM

$f(x) = \ln(1+x)$

$$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}}_{\text{TAYLOR SERIES OF } f(x)=\ln(1+x)} \begin{cases} = \ln(1+x), & \cancel{0 \le x \le 1} \quad -1 < x \le 1 \\ \text{is DIVERGENT } (\neq \ln(1+x)), & \cancel{x > 1} \quad \text{otherwise} \end{cases}$$

CC

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + \sqrt{n}}$$

$\rightarrow$ Example : $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots = \ln(2)$

NOT ALL FUNCTIONS EQUAL THEIR TAYLOR SERIES! (HAVE TAYLOR SERIES EXPANSIONS).

...... ALTHOUGH WE WISH THEY DID

PROOFS

Example 1 $f(x) = e^x$

$$e^x = P_n(x) + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c \cdot x^{n+1}}{(n+1)!} \quad \left[\begin{array}{l} c \text{ between} \\ 0 \text{ and } x \end{array}\right]$$

Estimate
$$|R_n(x)| = \frac{|x|^{n+1}}{(n+1)!} \cdot e^c \begin{cases} \leq \dfrac{|x|^{n+1}}{(n+1)!} \cdot e^x, & 0<c<x \\ \leq \dfrac{|x|^{n+1}}{(n+1)!} \cdot 1, & x<c<0 \end{cases}$$

In any case:

$$|R_n(x)| \leq e^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim\limits_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$ (special limit)

$\Downarrow$

$$\lim_{n\to\infty} \boxed{R_n(x) = 0. \Rightarrow e^x = \text{Taylor}(x)}$$

$$\sum_{n=0}^{\infty} \frac{(e^x)^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 2. $f(x) = \sin x$.

$$\boxed{\begin{aligned} \sin(x) &= P_n(x) + R_n(x) \\ R_n(x) &= \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}, \end{aligned}}$$ c btw. 0 and x

Estimate: $|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \to 0$

$$\Downarrow$$

$$\lim_{n\to\infty} R_n(x) = 0.$$

Conclusion: $\sin(x) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} x^n$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cancel{\frac{x^9}{9!}} \cancel{- \cdots}$$

deg $P_n \le n$

$$\boxed{x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}} + \cdots$$

$$P_3(x) = \cancel{\sin(0)} + \frac{\sin'(0)}{1!} x + \cancel{\frac{\sin''(0)}{2!} x^2} + \frac{\sin^{(3)}(0)}{3!} x^3 = \boxed{x - \frac{x^3}{6}}$$

Example 4 $f(x) = \ln(1+x)$.

$$f^{(n)} = (-1)^{n+1}(n-1)!\,(1+x)^{-n-1}$$

$$\boxed{f^{(n)}(0) = (-1)^{n+1}(n-1)!}$$

Taylor series: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ → Convergent for $-1<x\le 1$

ASSUME FOR SIMPLICITY $\underline{x \geq 0}$

Start with: $-1<x\le 1$

$$\ln(1+x) = P_n(x) + R_n(x),$$

$$R_n(x) = \frac{(-1)^n\, n!}{(n+1)!}(1+c)^{-n-1}\cdot x^{n+1}, \quad 0<c<x$$

$$\rightarrow |R_n(x)| \le \left(\frac{x}{1+c}\right)^{n+1}\cdot\frac{1}{n+1} \le \frac{1}{n+1} \to 0$$

$$0<c<x \Rightarrow 1<1+c<1+x \Rightarrow 0<\frac{1}{1+x}<\frac{1}{1+c}<1$$

$$\frac{x}{1+x}<\frac{x}{1+c}<x$$

$$\Rightarrow |R_n(x)| \le \frac{x^{n+1}}{n+1}.$$

$\Rightarrow$ For $0\le x\le 1$, $\displaystyle\lim_{n\to\infty} R_n(x) = 0$.

"Similarly", for $-1<x\le 0$ (although harder)

Conclusion

$$\underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}_{\text{Taylor series of } \ln(1+x)} \begin{cases} = \ln(1+x), & -1 < x \le 1 \\ \text{is DIVERGENT}, & \text{OTHERWISE} \end{cases}$$

Examples

$x = 1$:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$$

$x = \frac{1}{2}$:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 2^n} = \ln\left(\frac{3}{2}\right)$$

$x = -\frac{1}{2}$
$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \ln(2)$$

$x = -\frac{1}{3}$
$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} = \ln\left(\frac{3}{2}\right)$$

$x = 2$
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} \quad \text{is DIVERGENT}$$

Recall:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Question How about $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = ?$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = ?$$

Answer

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\boxed{\begin{aligned} \frac{e^x + e^{-x}}{2} &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\ \frac{e^x - e^{-x}}{2} &= \frac{x}{1!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \end{aligned}}$$

This is one of the reason why we use notations:

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ (hyperbolic cosine)

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ (hyperbolic sine)