Section 11.8
COMPUTING POWER SERIES

So FAR:

1) $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \qquad R=\infty, \quad S=\mathbb{R}$$

2) $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin(x), \qquad R=\infty, \quad S=\mathbb{R}$$

3) $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos(x), \qquad R=\infty, \quad S=\mathbb{R}$$

4) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x), \qquad R=1, \quad S=(-1,1]$$

0) $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad R=1, \quad S=(-1,1)$$

Nice THEOREM (DIFFERENTIATION and INTEGRATION of Power Series)

Assume • $\sum_{n=0}^{\infty} a_n x^n$ has radius $R > 0$

• $\sum_{n=0}^{\infty} a_n x^n = f(x), \qquad -R < x < R$

i.e. $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots = f(x)$

THEN

A) $f'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1} + \dots = \sum_{n=1}^{\infty} n a_n x^{n-1}$

B) $\int_0^x f(t)\,dt = a_0 x + \frac{a_1}{2} x^2 + \dots + \frac{a_n}{n+1} x^{n+1} + \dots = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

FOR $\boxed{-R < x < R}$

NOTE • The series obtained through differentiation and integration have same radius R.
[Recall $n^{1/n} \to 1$, $n \to \infty$]

• $x = \pm R$ (endpoints) not covered by Theorem.

Example 1

Geometric Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad -1<x<1, \quad R=1.$$

A) Differentiate

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots$$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \quad \boxed{-1<x<1}$$

$$\sum' \frac{x^n}{n}$$

Obtain Identity:

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n, \qquad -1<x<1$$

e.g.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2 \quad (x=1/2)$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{3^n} = \frac{-1/3}{(1+1/3)^2} = -\frac{3}{4}$$

$(x=-1/3)$

$$\frac{1}{1-x} = \sum x^n$$

Example 2

Modify Geometric Series $(x \mapsto (-x))$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}, \qquad -1<x<1, \qquad R=1$$

B) Integrate

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\int_0^x f(t)\,dt = \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} .$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad \boxed{-1<x<1}$$

Note : We saw before that the identity holds in the range

$$-1 < x \le 1$$

ABEL'S THEOREM

Assume $\longrightarrow$ **RADIUS R**

- $$\sum_{n=0}^{\infty} a_n x^n = f(x), \qquad x \in (-R, R) \qquad (*)$$
- $\sum_{n=0}^{\infty} a_n R^n$ is convergent $(R \in S)$
- $f(x)$ is continuous at $x = R$

THEN

$$f(R) = \sum_{n=0}^{\infty} a_n R^n .$$

[In other words, extend identity $(*)$ whenever possible]

Example 3

1) Start with modified geom. series.

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} \quad , \quad -1 < x < 1$$

2) Integrate

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x), \qquad -1 < x < 1$$

3) What happens at $x = 1$?

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ = convergent
- $f(x) = \ln(1+x)$ continuous at $x = 1$

ABEL'S Theorem: have identity:

$$\boxed{f(1) = \ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.}$$

Example 4

1) Start with geom. series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad -1<x<1.$$

2) Derive ~~identities~~ identity

$$\sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2},$$

OR

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}, \quad -1<x<1.$$

3) Integrate:

$$\int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}, \quad -1<x<1$$

i.e.

$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}, \quad -1<x<1.$$

4) What happens @ $x=1$?

- $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ = convergent.
- $\tan^{-1}(x)$ = continuous @ $x=1$

INPUT ABEL'S Theorem: $(x=1)$

$$\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

i.e.

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\cdots$$

[Leibniz]

Example 5

$$\sum_{n=1}^{\infty} n^2 x^n = \; ?$$

- Radius: $\frac{1}{R} = \lim_{n\to\infty} n^{2/n} = 1 \Rightarrow \boxed{R=1}$
- $\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)}$
- Differentiate once:

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

- Differentiate twice:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1-x^2}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}}, \; -1<x<1$$

e.g.

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^3} = 6$$

Example 6

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} = \qquad ?$$

1) $$\sum_{n=0}^{\infty} x^{3n} = \frac{1}{1-x^3}, \quad -1<x<1$$

2) $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1} = \int_0^x \frac{dt}{1+t^3}, \quad -1<x<1$$

3) Input ABEL'S Theorem @ $x=1$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} = \int_0^1 \frac{dt}{1+t^3}$$

$$= \text{. . . . .} \quad ?$$

(Compute integral)

Relation to Taylor Series

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{Radius } R$$

Put: $f(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad -R < x < R.$

$\Longrightarrow \sum_{n=0}^{\infty} a_n x^n$ <u>is the Taylor Series of f</u> !

<u>Proof</u>.

Differentiate :

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad -R<x<R$$

@ $x=0 \Rightarrow f'(0) = a_1$

Differentiate :

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \qquad -R<x<R$$

@ $x=0 \Rightarrow f''(0) = 2\cdot 1\cdot a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$

Differentiate $\underline{\underline{n}}$' times :

$$\Rightarrow \boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

Example 1

Let $f(x) = x^2 \ln(1+x^3)$.

a) Find the Taylor series of f

b) Compute $f^{(23)}(0)$

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a)

1) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, $-1<x<1$

2) $\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}$, $-1<x<1$

3) $\boxed{x^2 \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n+2}}{n}}$, $-1<x<1$

Taylor series of $f(x)$!

b) $3n+2 = 23 \Rightarrow \underline{n=7}$

$$\frac{f^{(23)}(0)}{23!} = \frac{(-1)^{7+1}}{7} = \frac{1}{7}$$

$$\Downarrow$$

$$\boxed{f^{(23)}(0) = \frac{23!}{7}}$$