II. POWER SERIES

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots\dots$$

$a_0, a_1, \dots$ = COEFFICIENTS

X = VARIABLE

TWO QUESTIONS

(Q1) GIVEN $a_0, a_1, \dots\dots$,

DETERMINE THOSE $\underline{\underline{X}}$ FOR WHICH

THE SERIES $\sum_{n=0}^{\infty} a_n x^n$ CONVERGES.

[RANGE OF CONVERGENCE]

(Q2) FOR $x \in$ (RANGE OF CONVERGENCE),

$$\sum_{n=0}^{\infty} a_n x^n = ?$$

[DETERMINE ACTUAL SUM].

[RECOGNIZE TAYLOR SERIES!!]

YESTERDAY:

Example 4

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

1 : Range of Convergence : $(-1, 1]$

2 For $x \in (-1, 1]$,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x).$$

Example 1

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

1 Range of Convergence : $(-\infty, \infty) = \mathbb{R}$

2 For $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

By the way,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \overset{?}{=}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 2^n} \overset{?}{=}$$

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots$$

$$\overset{?}{=}$$

[How did we determine nature?]

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} \overset{?}{=}$$

ANSWER TO Q1 :

THEOREM (HADAMARD, 1892).

THERE EXISTS R $(0 \le R \le \infty)$

[RADIUS OF CONVERGENCE]

SUCH THAT THE POWER SERIES $\sum_{n=0}^{\infty} a_n x^n$

i) CONVERGES ABSOLUTELY FOR

$$-R < x < R$$

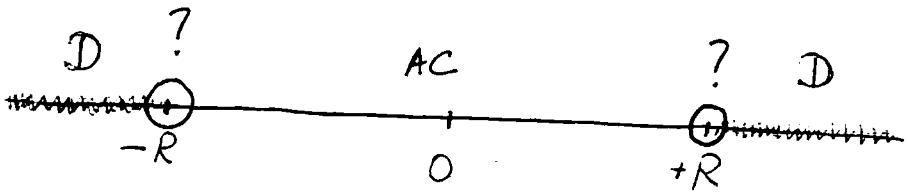
ii) DIVERGES FOR $|x| > R$

iii) ENDPOINTS $x = \pm R$ NEED

FURTHER INVESTIGATION.

MOREOVER,

$$\boxed{\frac{1}{R} = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}}$$



$$\sum_{n=0}^{\infty} a_n x^n$$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

$R =$ RADIUS OF CONVERGENCE

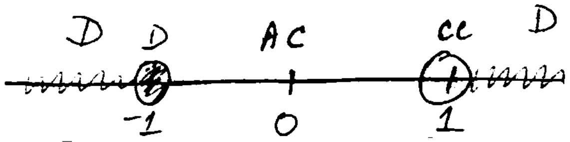
Example 4

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n} \qquad S = (-1, 1]$$

$$a_n = \frac{(-1)^{n+1}}{n}.$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

$$R = (-1, 1)$$



Check Endpoints:

$$X = 1 \rightsquigarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \qquad (cc)$$

$$X = -1 \rightsquigarrow \sum_{n=1}^{\infty} -\frac{1}{n} \qquad (D)$$

INTERVAL OF CONVERGENCE $= (-1, 1]$

Example 1

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad S = (\mathbb{R})$$

$$a_n = \frac{1}{n!}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{1}{(n!)^{1/n}} = 0$$

$$R = +\infty .$$

Interval of Convergence: $(-\infty, +\infty) = \mathbb{R}$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

$= $ RADIUS OF CONVERGENCE

Interval of Convergence is the set

$$S = \{ x \mid \sum_{n=0}^{\infty} a_n x^n \text{ is Convergent} \}.$$

Hadamard's Theorem:

S is one of the following:

$(-R, R)$, $(-R, R]$, $[-R, R)$

OR $[-R, R]$.

- $(-R, R)$ FOR SURE $\subseteq S$
- Check $R, -R$ separately.

Example 5

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n^n}$$

$$a_n = \frac{(-1)^n}{n^n}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{1}{n^n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$R = +\infty$, $\boxed{S = (-\infty, \infty) = \mathbb{R}}$.

[SERIES CONVERGES FOR ANY $x \in \mathbb{R}$].

Example 6.

$$\sum_{n=1}^{\infty} \frac{n^n x^n}{2^n}.$$

$a_n = \frac{n^n}{2^n}$. $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2} = +\infty$

$R = 0$, $S = \{0\}$.

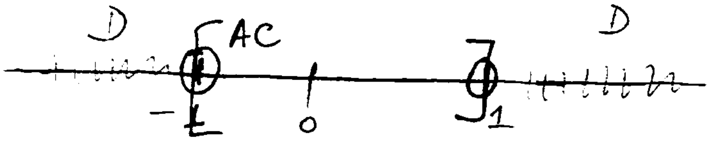
[The series does not converge for any $x \neq 0$].

<u>Example 7</u>

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$a_n = \frac{1}{n^2}$$

$$\frac{1}{R} = \lim_{n \to \infty} \frac{1}{n^{2/n}} = 1, \qquad \boxed{R = 1}.$$



Check Endpoints:

$x = 1 \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad C$

$x = -1 \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad AC$

$$S = [-1, 1].$$

<u>Q.2</u> $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2} = ?$ $\qquad \underline{-1 \leq x \leq 1}$

Example 8

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n \cdot n^3}$$

TRICK QUESTION

ASSUME $\sum_{n=0}^{\infty} a_n \cdot 2^n$ CONVERGENT

$\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ is . . . .

$\sum_{n=0}^{\infty} \frac{(-1)^n a_n}{3^n}$ is . . . .

$\sum_{n=0}^{\infty} (-1)^n 2^n \cdot a_n$ is . . . . . .