SECTION 11.8

1. DIFFERENTIATION AND INTEGRATION OF POWER SERIES

Start with a power series $\sum_{n=0}^{\infty} a_n x^n$ of radius R, and assume we know the sum of this series for $x \in (-R, R)$,

$$\sum_{n=0}^{\infty} a_n x^n = f(x), \quad x \in (-R, R)$$

The "Nice Theorem" says that we can differentiate or integrate this identity *term-by-term*, as long as we keep $x \in (-R, R)$:

$$\sum_{n=1}^{\infty} n a_n \, x^{n-1} = f'(x) \tag{1}$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{n+1} \, x^{n+1} = \int_0^x f(t) dt \tag{2}$$

The above identities are valid for $x \in (-R, R)$.
Moreover, the new power series obtained at (1) and (2) have the same radius of convergence R. as the original series.

Example 1. Starting with the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad R = 1$$

Integrate:

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \, x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad x \in (-1, 1) \tag{3}$$

Example 2. Start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad R = 1$$

Differentiate:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \quad x \in (-1, 1)$$

Multiply both sides of this identity by x to obtain

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n, \quad x \in (-1, 1)$$

Particular values: for $x = \frac{1}{2}$ we obtain

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

Example 3. Start with the identity from the previous example

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n, \quad R = 1$$

Differentiate:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{1+x}{(1-x)^3} \quad \text{(check!)}$$

Multiply both sides by x to obtain

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \quad \text{valid for } x \in (-1,1)$$

Particular values: set $x = -\frac{1}{3} \in (-1,1)$ to obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n} = \frac{-\frac{1}{3} \cdot \frac{2}{3}}{(\frac{4}{3})^2} = -\frac{1}{8}$$

Set $x = \frac{1}{3}$ to obtain

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{\frac{1}{3} \cdot \frac{4}{3}}{(\frac{2}{3})^2} = 1$$

Example 4. Replace x by $-x^2$ in the geometric series to obtain

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad x \in (-1,1)$$

Integrate:

$$\tan^{-1} x = \int_0^x \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad x \in (-1,1) \tag{4}$$

2. Abel's Theorem

The nice theorem allows us to obtain "formulas" (or power series expansions) and it guarantees for their validity in the open interval $x \in (-R, R)$.
However some interesting identities can be obtained by pushing x to the endpoints $\pm R$ of the interval of convergence whenever that's possible. We can do that using Abel's theorem which says that, **if**:

- $R \in S$, i.e. the series $\sum_{n=0}^{\infty} a_n R^n$ is convergent
- $f(x) = \sum_{n=0}^{\infty} a_n x^n$ (the sum of the series) is continuous at $x = R$

then:

$$\sum_{n=0}^{\infty} a_n R^n = f(R)$$

Example 5. In equation (3), the sum of the power series, namely $f(x) = \ln(1+x)$ is continuous at $x = 1$.
On the other hand, $1 \in S$ since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent (alternating series test).

Therefore the identity (3), previously guaranteed by the nice theorem only in the range $x \in (-1,1)$, extends now to $x = 1$ as well and we obtain

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Example 6. Similarly in the case of equation (4), the sum of the power series, namely $f(x) = \tan^{-1}(x)$, is continuous everywhere, so no problems at the endpoint $x = 1$.
On the other hand while $1 \in S$ since indeed the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ is convergent (alternating series test).
Therefore the identity (4) can be extended to $x = 1$ (thus valid now in $(-1,1]$) to obtain

$$\tan^{-1}(1) = \frac{\pi}{4} = \sum_{n=}^{\infty} \frac{(-1)^n}{2n+1}$$

Example 7. Say we want to compute the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$$

We determine the nature of this series by the alternating series test, since $\frac{1}{3n+1} \searrow 0$ as $n \to \infty$. So this is a convergent series. The question is what is its actual sum.
For that, we start the series (simple consequence of the geometric series)

$$\sum_{n=0}^{\infty} x^{3n} = \frac{1}{1-x^3} \quad \text{valid for } x \in (-1,1)$$

Integrate term by term:

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{3n+1} = \int_0^x \frac{dx}{1-x^3}, \quad x \in (-1,1)$$

Push the identity to the endpoint $x = -1$ (ok in view of Abel's theorem)

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} &= \int_0^{-1} \frac{dx}{1-x^3} \quad \text{(substitution } x = -u\text{)} \\ &= \int_0^{1} \frac{dx}{1+x^3} \end{aligned}$$

The computation of the last integral is the stuff of Chapter 8 (more specifically, 8.5):

$$\frac{1}{1+x^3} = \frac{\frac{1}{3}}{1+x} + \frac{\frac{2}{3} - \frac{1}{3}x}{1-x+x^2}$$

Hence our integral equals:

$$\frac{1}{3}\int_0^1 \frac{dx}{1+x} + \frac{1}{3}\int_0^1 \frac{2-x}{1-x+x^2}dx = \frac{1}{3}\ln(2) + \frac{1}{3}\int_0^1 \frac{2-x}{(x-1/2)^2+3/4}dx$$
$$= \frac{1}{3}\ln(2) + \frac{1}{3}\int_{-1/2}^{1/2} \frac{3/2-u}{u^2+3/4}du = \frac{1}{3}\ln(2) + \frac{1}{2}\int_{-1/2}^{1/2} \frac{du}{u^2+3/4}$$
$$= \frac{1}{3}\ln(2) + \frac{\sqrt{3}}{3}\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{du}{u^2+1}$$
$$= \frac{1}{3}\ln(2) + \frac{\sqrt{3}}{3}\tan^{-1} u\Big|_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} = \frac{1}{3}\ln(2) + \frac{\pi}{3\sqrt{3}}$$

and this is the actual sum of the series that we started with in this example.

3. Relation with the Taylor series

Assume we arrive at an identity of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (R, -R)$$

with R the radius of convergence of the power series on the left. Then

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{is the Taylor series of } f(x)$$

In other words,

$$a_n = \frac{f^{(n)}(0)}{n!}$$

This is easily obtained by differentiating repeatedly the above formula and evaluating at $x = 0$.

Example 8. Say we need the Taylor series expansion of $f(x) = e^{-x^2}$.
Start with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = +\infty$$

Replacing x by $-x^2$ we get the identity

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Then this **must** be the Taylor expansion of the function $f(x) = e^{-x^2}$!
In particular, if we need to compute $f^{(8)}(0)$ we can just identify the coefficient of x^8:

$$\frac{f^{(8)}(0)}{8!}x^8 = \frac{(-1)^n x^{2n}}{n!} \text{ (for some } n) \Rightarrow 2n = 8, n = 4$$

Therefore

$$\frac{f^{(8)}(0)}{8!} = \frac{(-1)^4}{4!}$$

which leads to $f^{(8)}(0) = \frac{8!}{4!}$, a computation which might be somewhat difficult by other means.

Example 7. Determine the Taylor expansion of the function $f(x) = x^3 \sin(x^2)$.
[We don't even know a priori whether such a function should have a Taylor series expansion

or not.]
Start with

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

Replace x by x^2 in this formula to obtain

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

Multiply both sides by x^3:

$$x^3 \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(2n+1)!}$$

According to our previous observation, the power series on the right-hand side can only be the Taylor series of the function $f(x) = x^3 \sin(x^2)$.
It means that what we have here is simply the Taylor series expansion of $f(x)$, and we even know that it holds true for any $x \in \mathbb{R}$, simply because the identity that we started with (the Taylor series expansion of $\sin(x)$) was valid $\forall x \in \mathbb{R}$.