SECTION 11.7: POWER SERIES

A series of the type

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

is called a power series. Here a_n are fixed real numbers, while x is a variable. a_n is the coefficient of x^n.

The nature of the series (1) is clearly influenced by the particular value of x.

The set of those x for which the power series converges is called the *domain of convergence*. We denote it by S. It is also called the *interval of convergence* (since it turns out to be an interval).

Example: the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is the well known geometric series which converges only for $-1 < x < 1$.
In other words, in this particular case, $S = (-1, 1)$.
We also know the sum of this series when $x \in S$, namely $\frac{1}{x-1}$. Example:
$\sum_{n=0}^{\infty}(-\frac{1}{2})^n = \frac{1}{1+1/2} = \frac{3}{2} \quad (x = -\frac{1}{2})$
$\sum_{n=0}^{\infty} 2^n$ is divergent $\quad (x = 2)$.

Theorem 1 (Fundamental Theorem- J. Hadamard)**.** *Given a power series* $\sum_{n=0}^{\infty} a_n x^n$*, there exists a number R (radius of convergence) such that the power series is:*

- *Absolutely convergent for* $-R < x < R$
- *Divergent for* $|x| > R$
- *The endpoints* $x = -R$ *and* $x = R$ *need further investigation.*

Moreover, this radius of convergence is given explicitly by the following formula

$$R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}} \quad \text{or} \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} \tag{2}$$

(provided the right-hand side exists).

Remarks:

Once we compute the radius of convergence R, we know that the series converges for sure when $x \in (-R, R)$ and possibly when $x = R$ and $x = -R$ (although the endpoints have to be checked separately). Therefore the domain of convergence S can only be one of the following intervals: $(-R, R), [-R, R), (-R, R], [-R, R]$.

R can very well be 0 or ∞ (depends on the particular series), and we will encounter even these extreme cases.

Example 1. Consider the power series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Radius: $\frac{1}{R} = \lim \frac{1}{n^{1/n}} = 1 \Rightarrow R = 1$.
Check the endpoints:

- For $x = -1$ we obtain the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which is (conditionally) convergent, as an alternating series.
- For $x = 1$ we obtain the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.

Conclusion: $S = [-1, 1)$.

Example 2. Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Radius: $R = \lim \frac{1}{(n!)^{1/n}} = 0 \Rightarrow R = +\infty$.
Endpoints to check: none.
Domain of convergence: $S = (-\infty, +\infty) = \mathbb{R}$. (The series converges for any $x \in \mathbb{R}$, but we already knew that.)

Example 3. Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$$

Radius of convergence: $\frac{1}{R} = \lim_{n\to\infty} |\frac{1}{n^2 2^n}|^{1/n} = \lim \frac{1}{2n^{2/n}} = \frac{1}{2} \Rightarrow R = 2$.
Check endpoints:

- For $x = -1$ we obtain $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which is an absolutely convergent series.
- For $x = 2$ we obtain the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent.

Domain of convergence: $S = [-2, 2]$.

Problem: find the actual sum of the power series in Example 3 (section 11.8).

Useful limits

A number of 'special' limits are useful for computing the radius R using the formula (2): as $n \to \infty$,

1. $n^{1/n} \to 1$ (this one a special limit obtained by L'Hopital).

2. $(\ln)^{1/n} \to 1$ (can be derived from the above using pinching theorem)

3. $(n^6 + 2n^5 + 3n)^{1/n} \to 1$
(Try to argue this statement by invoking the pinching theorem and using the first special limit of this list.)

4. $(n!)^{1/n} \to \infty$

5. $2^{1/n} \to 1$