

## THE METHOD OF THE VARIATION OF PARAMETERS

### 1. THE FORMULAS

Assume we are trying to find all the solutions to the complete equation

$$y'' + ay' + by = \phi(x)$$

where  $a, b$  are real numbers (constants) and  $\phi(x)$  is a given function. We know that the general solution can be written as

$$y(x) = y_p(x) + C_1u_1(x) + C_2u_2(x)$$

where

- $u_1(x), u_2(x)$  are the fundamental solutions of the reduced equation  $y'' + ay' + by = 0$
- $C_1, C_2$  are arbitrary coefficients
- $y_p$  is a particular solution of the complete equation

So the most important part is to determine a particular solution  $y_p$ . In case the guessing game doesn't work (for example when  $\phi(x)$  is so complicated that you cannot guess the general form of a function "related" to it), a sure way of finding a particular solution is given by the following formula

$$y_p(x) = z_1(x)u_1(x) + z_2(x)u_2(x)$$

where

$$W(x) = \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} = u_1(x)u_2'(x) - u_1'(x)u_2(x)$$
$$z_1(x) = - \int \frac{u_2(x)\phi(x)}{W(x)} dx$$
$$z_2(x) = \int \frac{u_1(x)\phi(x)}{W(x)} dx$$

### 2. EXAMPLE

**Problem.** Find a particular solution to the equation

$$y'' + 2y' + y = e^{-x} \ln(x)$$

*Answer.* The characteristic equation

$$r^2 + 2r + 1 = 0$$

has roots  $r_1 = r_2 = -1$ , so the fundamental solutions of the reduced equation are

$$u_1(x) = e^{-x}, \quad u_2(x) = xe^{-x}$$

Compute the Wronskian.

$$\begin{aligned} W(x) &= \det \begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} \\ &= e^{-x}(e^{-x} - xe^{-x}) + e^{-x} \cdot xe^{-x} \\ &= e^{-2x} - xe^{-2x} + xe^{-2x} \\ &= e^{-2x} \end{aligned}$$

Compute  $z_1(x)$ .

$$\begin{aligned} z_1(x) &= - \int \frac{u_2(x)\phi(x)}{W(x)} dx \\ &= - \int \frac{xe^{-x} \cdot e^{-x} \ln(x)}{e^{-2x}} dx \\ &= - \int x \ln(x) dx = -\frac{x^2}{2} \ln(x) + \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= -\frac{x^2}{2} \ln(x) + \frac{x^2}{4} \end{aligned}$$

Compute  $z_2(x)$ .

$$\begin{aligned} z_2(x) &= \int \frac{u_1(x)\phi(x)}{W(x)} dx \\ &= \int \frac{e^{-x} \cdot e^{-x} \ln(x)}{e^{-2x}} dx \\ &= \int \ln(x) dx = x \ln(x) - \int x \cdot \frac{1}{x} dx \\ &= x \ln(x) - x \end{aligned}$$

*Note.* We used integration by parts to compute the integrals  $\int x \ln(x) dx$  and  $\int \ln(x)$ .

Final answer. The particular solution to our complete equation is

$$\begin{aligned} y_p(x) &= z_1(x)u_1(x) + z_2(x)u_2(x) \\ &= \left(-\frac{x^2}{2} \ln(x) + \frac{x^2}{4}\right) \cdot e^{-x} + (x \ln(x) - x) \cdot xe^{-x} \\ &= \frac{x^2}{2} \ln(x)e^{-x} - \frac{3x^2}{4}e^{-x} \\ &= \left(\frac{x^2}{2} \ln(x) - \frac{3x^2}{4}\right)e^{-x} \end{aligned}$$

*Note.* It would be hard to find this particular solution through the method of undetermined coefficients, since it is not clear (by looking at the answer) what should be a general form of  $y_p$  that we should try.