Chapter 8: Integration

- antiderivative, definite/indefinite integrals
- Substitution
- Integration by parts
- Special substitutions
  $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$
  [website]
- $\displaystyle\int_0^1 \frac{dx}{1+x^3} = ?$
- $\displaystyle\int \frac{dx}{x(\ln x)^2} = ?$

Chapter 9 : Curves, etc. ...

A) Parametric curves $x = x(t),\ y = y(t)$.

- Tangent line @ t_0: $[X_0 = X(t_0),\ Y_0 = Y(t_0)]$

$$Y - Y_0 = \cancel{\frac{X'(t_0)}{Y'(t_0)}}\ \frac{Y'(t_0)}{X'(t_0)} \cdot (X - x_0).$$

Slope @ $t = t_0$: $m = \dfrac{Y'(t_0)}{X'(t_0)}$.

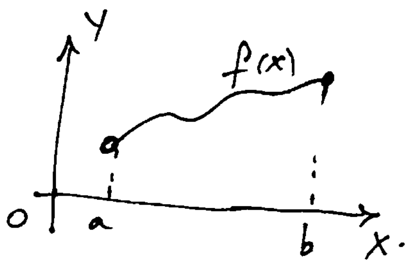
- Graph of a function $y = f(x)$.
  Standard parametrization:
  $x(t) = t$, $y(t) = f(t)$

- Length of a parametric curve: , $a \le t \le b$.

$$\ell = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

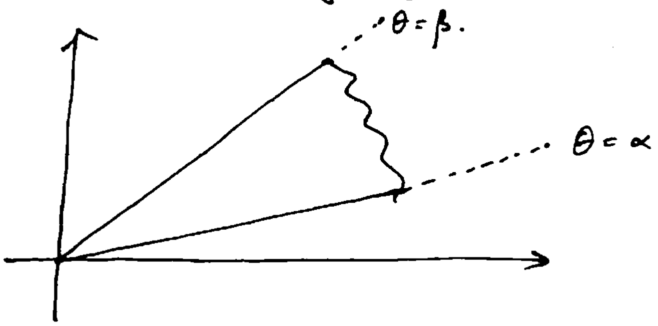
- Length of the graph of $y = f(x)$, $a \le x \le b$

$$\ell = \int_a^b \sqrt{1 + (f'(t))^2}\ dt$$



B) Polar Coordinates

Polar Curve $\rho = \rho(\theta), \quad \alpha \leq \theta \leq \beta.$



$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} [\rho(\theta)]^2 \, d\theta$$

$$\text{Length} = \int_{\alpha}^{\beta} \sqrt{(\rho(\theta))^2 + (\rho'(\theta))^2} \, d\theta$$

Standard parametrization – by angle

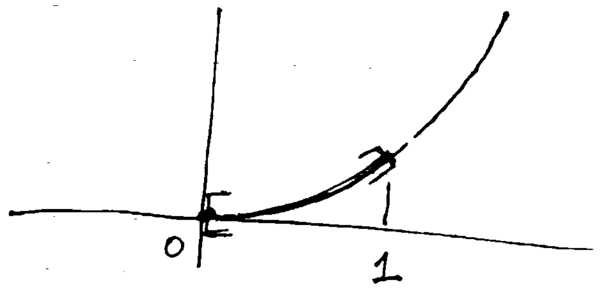
$$x(\theta) = \rho(\theta)\cos\theta, \quad y(\theta) = \rho(\theta)\sin\theta$$

<u>Example</u>: Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

using polar ~~coordinates~~ parametrization.

Example Find the length of the parabola $y = \frac{1}{2}x^2$ from $x=0$ to $x=1$.



$$\ell = \int_0^1 \sqrt{1+(f'(x))^2}\, dx$$

$$= \int_0^1 \sqrt{1+x^2}\, dx$$

$$= \left.\frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|\right|_{x=0}^{x=1} = ?$$

$$\int \sqrt{1+x^2}\, dx \overset{x=\tan u}{=\!=} \int \sec^2(u)\cdot \sec(u)\, du$$

$$= \int \sec^3(u)\, du = \frac{1}{2}\sec(u)\tan u + \frac{1}{2}\ln|\sec + \tan| + C$$

$$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right| + C.$$

Chapter 10: SEQUENCES

A) Convergent, monotone, bounded sequences.

Monotone & bounded $\Rightarrow$ Convergent.

Definitions!

B) Operations with limits

i) $a_n \to A, \quad b_n \to B$

$\Downarrow$

$a_n \pm b_n \to A \pm B.$

$a_n b_n \to AB.$

$\dfrac{a_n}{b_n} \to \dfrac{A}{B}$ (provided $B \neq 0$).

ii) $f(x)$ continuous, $a_n \to A$,

then:

$$f(a_n) \longrightarrow f(A)$$

<u>Example</u>: $\left(1+\frac{1}{n}\right)^{3n} \longrightarrow e^3.$

$\sin\left(\left(1+\frac{1}{n}\right)^n\right) \longrightarrow \sin(e) = ?$

iii) Divide by highest power!

$$\lim_{n\to\infty} \frac{n^2+3}{3n^2+100\sin(\sqrt{n})+1} =$$

$$= \lim_{n\to\infty} \frac{1+3/n^2}{3+\frac{100\sin(\sqrt{n})}{n^2}+\frac{1}{n^2}}$$

$$= \frac{1}{3}.$$

~~iv)~~

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{\sqrt{n}}} = 2$$

$$\lim_{n\to\infty} \frac{n^2}{n^3+n\sin(n)} = \lim_{n\to\infty} \frac{1/n}{1+\frac{\sin(n)}{n^2}} = \frac{0}{1} = 0$$

c) L'Hopital Rule

$$\lim_{n\to\infty} \frac{\ln(n)}{n} \overset{n=x}{=\!=} \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\frac{\infty}{\infty}}{=\!=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\lim_{n\to\infty} n\left(2^{1/n}-1\right) = \lim_{n\to\infty} \frac{2^{1/n}-1}{1/n}$$

$$\overset{1/n=x}{=\!=} \lim_{x\to 0} \frac{2^x-1}{x}$$

$$\overset{\frac{0}{0}}{=\!=} \lim_{x\to 0} \frac{2^x \ln(2)}{1}$$

$$= 2^0 \ln(2)$$

$$= \ln(2)$$

$$\lim_{n\to\infty} \frac{\sin(\pi\sqrt{n})}{n^2}$$

D) Special limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1, \qquad \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{n \to \infty} n^{1/n} = 1$$

$$\lim_{n \to \infty} \frac{\ln(n)}{n} = 0$$

$$\lim_{n \to \infty} c^n = \begin{cases} +\infty, & c > 1 \\ 1, & c = 1 \\ 0, & 0 \le c < 1 \end{cases}$$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad (\text{any fixed } x \in \mathbb{R}).$$

$$\lim_{n \to \infty} (n!)^{1/n} = +\infty$$

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Chapter ~~8~~ 11: SERIES

$\sum_{n=0}^{\infty} a_n$

- main term: a_n
- partial sum: $S_n = a_0 + \dots + a_n$
- Sum: $S = \lim_{n\to\infty} S_n$.
  (if limit exists)
- Convergence/divergence

A) BDT.

1. $a_n \not\to 0 \implies \sum_{n=1}^{\infty} a_n$ (D)
2. $\sum_{n=1}^{\infty} a_n$ (C) $\implies a_n \to 0$

Example $\sum_{n=1}^{\infty} \frac{1}{1+2^{-n}}$ is DIVERGENT

B) Geometric Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad -1 < x < 1.$$

Ex Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{n+2}}$$

C) <u>P- series</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \begin{array}{c} p > 1 \\ 0 \leq p \leq 1 \end{array}$$

D) <u>POSITIVE SERIES</u>

CONVERGENCE TESTS:

- Comparison [Basic / Limit]
- Integral test
- Root $\rho = \lim_{n\to\infty} |a_n|^{1/n} \quad <> 1$
- Ratio $\lambda = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad <> 1$

Examples

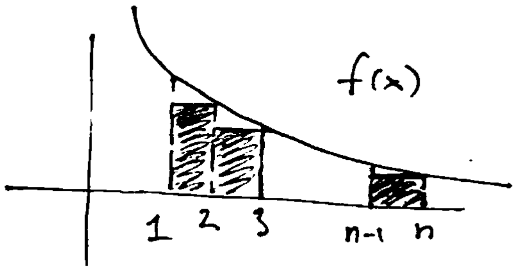
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3+\sqrt{n}+1}$$

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{(n!)^2}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{n^n}$$

Note on Integral test:



$$f(2)+\dots+f(n) \le \int_1^n f(x)\,dx$$

$$\boxed{S_n \le f(1) + \int_1^n f(x)\,dx}$$

f : monotone decreasing
$f \geq 0$

$$\sum_{n=1}^{\infty} f(n) \sim \int_1^{\infty} f(x)\,dx$$

Moreover,

$$\sum_{n=1}^{\infty} f(n) \le f(1) + \int_1^{\infty} f(x)\,dx.$$

E) Non-Positive Series

- Absolute Convergence
- Conditional Convergence

<u>Non-Positive series $\sum_{n=1}^{\infty} a_n$</u>

- Check BDT
- Check Absolute Series
- Check Alternating Series test.
- ?

<u>Examples</u>

$$\sum_{n=1}^{\infty} (-1)^n \frac{4^{n+1}}{e^n}.$$

$$\sum_{n=1}^{\infty} \frac{\sin(\pi\sqrt{n})}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+4}.$$

11.5 Taylor Series, Polynomials.

Given $f(x)$

- Definitions $\rightarrow$ $P_n(x) = f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(n)}(0)}{n!}x^n$

  n^{th} Taylor poly

  $\deg P_n \leq n$

  $\rightarrow$ Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Relation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \iff R_n(x) \xrightarrow[n\to\infty]{} 0$$

Starting Point:

$f(x) = P_n(x) + R_n(x)$

$R_n(x) = [\text{ Lagrange }]$

Estimate $|R_n(x)| \leq \dots\dots$ [get rid of $\underline{\underline{c}}$]

Examples

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\text{-----} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\cos(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

$$\text{-------} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

Power Series.

Last year exam:

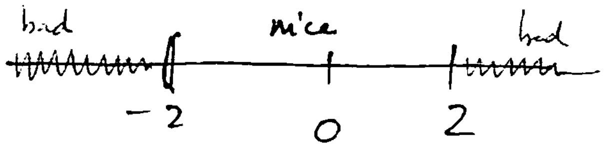
$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n \sqrt{n}}.$$

Coefficient:

$$a_n = \frac{(-1)^n}{2^n \cdot \sqrt{n}}$$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{2 \cdot n^{2/n}} = \frac{1}{2}$$

$R = 2$



__Endpoints__

$x = 2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$x = -2$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$

$$\boxed{S = (-2, 2]}$$

Obtain New Taylor Exp. from Old Ones

$$\boxed{\sum_{n=0}^{\infty} \frac{n}{4^n} = ?}$$

1) Compute Power Series

$$\sum_{n=0}^{\infty} n x^n$$

2) Set $x = \frac{1}{4}$.

1) Start : $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $x \in (-1,1)$

Diff. : $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

Mult. x : $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

2) Set $x = 1/4$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \frac{1/4}{(1-1/4)^2} = \frac{4}{3}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{n^3}{3^n} = \ ?}$$

1) Compute Power Series

$$\sum_{n=0}^{\infty} n^3 x^n$$

2) Set $x = 1/3$

1) Start: $\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)}$, $x \in (-1,1)$

Diff: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

$\cdot x$: $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

Diff: $\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$

$\cdot x$: $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$

Diff: $\sum_{n=1}^{\infty} n^3 x^{n-1} = \frac{1+8x-9x^3+x^4}{(1-x)^6}$

$\cdot x$: $\sum_{n=1}^{\infty} n^3 x^n = \frac{x(1+8x-9x^3+x^4)}{(1-x)^6}$.

2) Set $x = 1/3$

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} = \frac{813}{64}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!(n+2)} = ?}$$

1) Compute Power Series

$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!\,(n+2)}$$

2) Set $x = 1$

1) Start: $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

• x: $$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x e^x$$

• Integrate:
$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!\,(n+2)} = \int_0^x t e^t \, dt$$
$$= x e^x - e^x + 1$$

2) SET $x=1$
$$\sum_{n=0}^{\infty} \frac{1}{n!\,(n+2)} = 1$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = ?}$$

1) $$\sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+1)(n+2)} = ?$$

2) Set $x=1$

1) $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad x \in (-1,1)$$

Int. $$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x), \qquad x \in (-1,1)$$

i.e. $$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \qquad x \in (-1,1)$$

Int. $$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = \int_0^x -\ln(1-t)\,dt$$

$$= (1-x)\ln(1-x) - x, \qquad x \in (-1,1)$$

Int. $$\sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+1)(n+2)} = \int_0^x (t-1)\ln(1-t) - t \quad dt$$

$$= -\frac{1}{2}(x-1)^2 \ln(1-x) + \frac{3}{4}x^2 - \frac{x}{2}$$

2) SET $x = 1$ [OK by ABEL]

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{x \to 1} \left\{ \frac{1}{2}(x-1)^2 \ln(1-x) - \frac{3}{4}x^2 + \frac{x}{2} \right\}$$

$$= \frac{3}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}$$

L'H

$$\lim_{x \to 1^-} (x-1)^2 \ln(1-x)$$

$1-x=y$

$$= \lim_{y \to 0^+} y^2 \ln(y)$$

$$= - \lim_{z \to +\infty} \frac{\ln(z)}{z^2} = 0$$