THE REDUCED EQUATION OF ORDER TWO

1. General formulas

Consider the reduced equation of order two:

$$y'' + ay' + b = 0 \tag{1}$$

where a, b are given real numbers.

Characteristic equation. This is the algebraic equation

$$r^2 + ar + b = 0$$

It has discriminant $\Delta = a^2 - 4b$, and the roots of the characteristic equation are given by the formulas

$$r_1 = \frac{-a + \sqrt{\Delta}}{2}$$
$$r_2 = \frac{-a - \sqrt{\Delta}}{2}$$

Fundamental solutions. The fundamental solutions $u_1(x), u_2(x)$ of the reduced equation (1) are determined as follows:

If $\Delta > 0$ then $r_1 \neq r_2$ are real numbers and

$$\begin{cases} u_1(x) = e^{r_1 x} \\ u_2(x) = e^{r_2 x} \end{cases}$$

If $\Delta = 0$ then $r_1 = r_2 = r$ and

$$\begin{cases} u_1(x) = e^{rx} \\ u_2(x) = xe^{rx} \end{cases}$$

If $\Delta < 0$, then $r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta$ and

$$\begin{cases} u_1(x) = e^{\alpha x} \cos(\beta x) \\ u_2(x) = e^{\alpha x} \sin(\beta x) \end{cases}$$

General solution. The general solution of the reduced equation (1) is given by

$$y(x) = C_1 u_1(x) + C_2 u_2(x)$$

where C_1, C_2 are arbitrary coefficients.

2. Example

For the reduced equation $y''-2y'+1=0$, the characteristic equation is $r^2-2r+1=0$ which has roots $r_1=r_2=1$, therefore we are in case two (above) so the fundamental solutions are

$$\begin{cases} u_1(x) = e^x \\ u_2(x) = xe^x \end{cases}$$

The general solution to the equation $y''-2y'+y=0$ is thus of the form

$$y(x) = C_1e^x + C_2xe^x$$

where C_1, C_2 are some constants.

3. Initial-value problems

If we need to solve a differential equation subject to some initial conditions, we first determine the general form of the solution to the diff. equation (up to some constants C_1 and C_2) and then input the initial conditions to determine the coefficients C_1 and C_2, as in the next example.

Problem. Solve the differential equation $y''-2y'+1=0$ subject to the initial conditions $y(0)=1, y'(0)=-2$.

Answer. First we determine the general form of the solution to the diff. equation. We saw in the previous example that this is

$$y(x) = C_1e^x + C_2xe^x$$

The next step is to input the initial conditions to determine the constants C_1, C_2:

$$y(0) = C_1 = 1$$

$$y'(0) = C_1 + C_2 = -2$$

It follows that $C_1=1, C_2=-3$ so the solution is

$$y(x) = e^x - 3xe^x$$