THE COMPLETE EQUATION OF ORDER TWO

1. General principles

Consider the complete equation

$$y'' + ay' + by = \phi(x) \tag{1}$$

where a, b are real numbers and $\phi(x)$ is a given function. The general solution to this equation is of the form

$$y(x) = y_p(x) + y_0(x)$$

where y_p is a particular solution of the complete equation (1) and y_0 is a general solution to the reduced equation

$$y'' + ay' + by = 0$$

in other words

$$y(x) = y_p(x) + C_1 u_1(x) + C_2 u_2(x) \tag{2}$$

where u_1, u_2 are the fundamental solutions of the reduced equation, and C_1, C_2 are some constants. Therefore all that remains is to find a particular solution y_p to the complete equation (1).

2. Search of y_p: method of undetermined coefficients

This is the method of guessing y_p by starting with a general form which is "related" somehow to $\phi(x)$. If this method doesn't work, one should then try the formulas from the method of the variation of parameters.

Few examples:

a) $y'' + 2y' + 5y = 10e^{-2x}$. Try $y_p(x) = Ae^{-2x}$ and determine A.

b) $y'' - 3y' + 2y = e^x$. Here looking for y_p of the form Ae^x will not work, since $u_1(x) = e^x$ is a fundamental solution of the reduced equation. So in this case one should try $y_p = Axe^x$.

c) $y'' - 4y' + 4y = x^3 + x + 1$. Try $y_p(x) = Ax^3 + Bx^2 + Cx + D$ and determine the coefficients A, B, C, D.

d) Principle of superposition. To determine a particular solution to the equation, say,

$$y'' - 2y' + y = x^2 + e^{-x}\sin(x) \tag{3}$$

it is better to take the following steps:
i) Find a particular solution y_1 to the equation $y'' - 2y' + y = x^2$, and of course one should try the form $y_1(x) = Ax^2 + Bx + C$ and determine A, B, C.
ii) Find a particular solution y_2 to the equation $y'' - 2y' + y = e^{-x}\sin(x)$, and one should try the form $y_2(x) = Ae^{-x}\sin(x) + Be^{-x}\cos(x)$, and determine A, B.
iii) A particular solution of the original equation (3) is then given by $y_p(x) = y_1(x) + y_2(x)$.

3. Initial value problems

When solving a diff. equation subject to initial conditions, we first need to determine the general form of the solution to the diff. equation (regardless of the initial conditions) up to two coefficients C_1 and C_2, and after that input the initial conditions to determine C_1 and C_2 explicitly.

3.1. **Example.** Solve the differential equation

$$y'' - 4y' + 4 = 8x + 4 \tag{4}$$

subject to the initial conditions $y(0) = 5, y'(0) = 3$.

Answer.

Step 1. The reduced equation is $y'' - 4y' + 4y = 0$ and the associated characteristic equation is

$$r^2 - 4r + 4 = 0, \quad r_1 = r_2 = 2$$

Therefore the fundamental solutions of the reduced equation are

$$u_1(x) = e^{2x}, \quad u_2(x) = xe^{2x}$$

Step 2. Determine a particular solution to the complete equation (4). Try $y_p(x) = Ax + B$, and solve

$$y_p'' - 4y_p' + 4y_p = -4A + 4Ax + 4B = 4Ax + (4B - 4A) = \phi(x) = 8x + 4$$

therefore $A = 2, B = 3$, and hence $y_p(x) = 2x + 3$.

Step 3. Determine the general solution to the complete equation (4). This is

$$\begin{aligned} y(x) &= y_p(x) + C_1 u_1(x) + C_2 u_2(x) \\ &= 2x + 3 + C_1 e^{2x} + C_2 x e^{2x} \end{aligned}$$

Step 4. Solve for the initial conditions and determine C_1, C_2. We have

$$y(0) = 3 + C_1 = 5$$

$$y'(0) = 2 + 2C_1 + C_2 = 3$$

It follows that $C_1 = 2, C_2 = -3$ so the solution is

$$y(x) = 2x + 3 + 2e^{2x} - 3xe^{-2x}$$