THE COMPLETE EQUATION OF ORDER TWO

1. General principles

Consider the complete equation
\[ y'' + ay' + by = \phi(x) \]
where \( a, b \) are real numbers and \( \phi(x) \) is a given function. The general solution to this equation is of the form
\[ y(x) = y_p(x) + y_0(x) \]
where \( y_p \) is a particular solution of the complete equation (1) and \( y_0 \) is a general solution to the reduced equation
\[ y'' + ay' + by = 0 \]
in other words
\[ y(x) = y_p(x) + C_1 u_1(x) + C_2 u_2(x) \]
where \( u_1, u_2 \) are the fundamental solutions of the reduced equation, and \( C_1, C_2 \) are some constants. Therefore all that remains is to find a particular solution \( y_p \) to the complete equation (1).

2. Search of \( y_p \): method of undetermined coefficients

This is the method of guessing \( y_p \) by starting with a general form which is "related" somehow to \( \phi(x) \). If this method doesn’t work, one should then try the formulas from the method of the variation of parameters.

Few examples:

a) \( y'' + 2y' + 5y = 10e^{-2x} \). Try \( y_p(x) = Ae^{-2x} \) and determine \( A \).

b) \( y'' - 3y' + 2y = e^x \). Here looking for \( y_p \) of the form \( Ae^x \) will not work, since \( u_1(x) = e^x \) is a fundamental solution of the reduced equation. So in this case one should try \( y_p = Axe^x \).

c) \( y'' - 4y' + 4y = x^3 + x + 1 \). Try \( y_p(x) = Ax^3 + Bx^2 + Cx + D \) and determine the coefficients \( A, B, C, D \).

d) Principle of superposition. To determine a particular solution to the equation, say,
\[ y'' - 2y' + y = x^2 + e^{-x} \sin(x) \]
it is better to take the following steps:

i) Find a particular solution \( y_1 \) to the equation \( y'' - 2y' + y = x^2 \), and of course one should try the form \( y_1(x) = Ax^2 + Bx + C \) and determine \( A, B, C \).

ii) Find a particular solution \( y_2 \) to the equation \( y'' - 2y' + y = e^{-x} \sin(x) \), and one should try the form \( y_2(x) = Ae^{-x} \sin(x) + Be^{-x} \cos(x) \), and determine \( A, B \).

iii) A particular solution of the original equation (3) is then given by \( y_p(x) = y_1(x) + y_2(x) \).
3. Initial value problems

When solving a differential equation subject to initial conditions, we first need to determine the general form of the solution to the differential equation (regardless of the initial conditions) up to two coefficients $C_1$ and $C_2$, and after that input the initial conditions to determine $C_1$ and $C_2$ explicitly.

3.1. Example. Solve the differential equation

\[ y'' - 4y' + 4 = 8x + 4 \]  \hspace{1cm} (4)

subject to the initial conditions $y(0) = 5, y'(0) = 3$.

Answer.

**Step 1.** The reduced equation is $y'' - 4y' + 4y = 0$ and the associated characteristic equation is

\[ r^2 - 4r + 4 = 0, \quad r_1 = r_2 = 2 \]

Therefore the fundamental solutions of the reduced equation are

\[ u_1(x) = e^{2x}, \quad u_2(x) = xe^{2x} \]

**Step 2.** Determine a particular solution to the complete equation (4). Try $y_p(x) = Ax + B$, and solve

\[ y_p'' - 4y_p' + 4y_p = 4A + 4Ax + 4B = 4Ax + (4B - 4A) = \phi(x) = 8x + 4 \]

therefore $A = 2, B = 3$, and hence $y_p(x) = 2x + 3$.

**Step 3.** Determine the general solution to the complete equation (4). This is

\[ y(x) = y_p(x) + C_1u_1(x) + C_2u_2(x) \]

\[ = 2x + 3 + C_1e^{2x} + C_2xe^{2x} \]

**Step 4.** Solve for the initial conditions and determine $C_1, C_2$. We have

\[ y(0) = 3 + C_1 = 5 \]

\[ y'(0) = 2 + 2C_1 + C_2 = 3 \]

It follows that $C_1 = 2, C_2 = -3$ so the solution is

\[ y(x) = 2x + 3 + 2e^{2x} - 3xe^{-2x} \]