MIDTERM II-SOLUTIONS
MATH 109

Problem 1 a) **Answer:** $\frac{\pi^2}{2}$
$\lim n^2\big(\cos(\frac{\pi}{n})-1\big) = \lim_{n\to+\infty} \dfrac{\cos(\frac{\pi}{n})-1}{\frac{1}{n^2}} = \lim \dfrac{\cos(\frac{\pi}{n})-1}{\frac{1}{n^2}} = \lim_{x\to 0} \dfrac{\cos(\pi x)-1}{x^2}$
L'Hospital once $\lim_{x\to 0} \frac{\cos(\pi x)-1}{x^2} = \lim_{x\to 0} \frac{\pi\sin(\pi x)}{2x}$
L'Hospital twice $\lim_{x\to 0} \frac{\pi\sin(\pi x)}{2x} = \lim_{x\to 0} \frac{\pi^2\cos(\pi x)}{2} = \frac{\pi^2}{2}$
b) **Answer:** e^{-1}.
Logarithm: $\ln(\lim(1-\frac{1}{n})^n) = \lim n\ln(1-\frac{1}{n}) = \lim \frac{\ln(1-1/n)}{1/n} = \lim \frac{\ln(1-1/n)}{1/n} = \lim_{x\to 0} \frac{\ln(1-x)}{x}$
Apply L'Hopital $\lim_{x\to 0} \frac{\ln(1-x)}{x} = \lim_{x\to 0} \frac{-\frac{1}{1-x}}{1} = -1$
Conclusion $\lim(1-\frac{1}{n})^n = e^{-1}$
Problem 2 Answer: $\frac{17}{6}$
$\sum \frac{2^{k+1}+3^k}{5^k} = \sum_{k=1}^{\infty} \frac{2^{k+1}}{5^k} + \sum_{k=1}^{\infty} \frac{3^k}{5^k}$
$= \frac{2^2}{5}\sum_{k=0}^{\infty}(\frac{2}{5})^k + \frac{3}{5}\sum_{k=0}^{\infty}(\frac{3}{5})^k$
(geom. series) $= \frac{4}{5}\frac{1}{1-\frac{2}{5}} + \frac{3}{5}\frac{1}{1-\frac{3}{5}} = \frac{4}{3} + \frac{3}{2} = \frac{17}{6}$
Problem 3 Answer: $+\infty$
This is the s_n for $\sum \frac{1}{1+\sqrt{n}}$
Compare $\sum \frac{1}{1+\sqrt{n}} \sim \sum \frac{1}{\sqrt{n}}$

Justify: $\lim \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{n}} = \lim \frac{\sqrt{n}}{1+\sqrt{n}} = \lim \frac{1}{1+n^{-1/2}} = 1$

(p-series, $p = 1/2$) $\sum \frac{1}{\sqrt{n}}$ is divergent, therefore $\sum \frac{1}{1+\sqrt{n}}$ is divergent.

Divergent and positive series$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} = +\infty$, in other words $\lim s_n = +\infty$
Problem 4 [20 pts] **Answer: convergent;** $\lim a_n = 0$
a) $\frac{a_{n+1}}{a_n} = \frac{2}{(\frac{n+1}{n})^n}$
$\lambda = \lim \frac{a_{n+1}}{a_n} = \frac{2}{e}$
Hence $\lambda < 1$ (since $e > 2$) $\Rightarrow$ convergent series
b) Application of the Basic divergence test.
Problem 5 Answer: true
The series is convergent, ($p = 4$)
[from the proof of the integral test] $s_n \le 1 + \int_1^n \frac{dx}{x^4} \Rightarrow \sum \frac{1}{n^4} = \lim s_n \le 1 + \int_1^{\infty} \frac{dx}{x^4}$
Evaluate improper integral get $\le 1 + \frac{1}{3} = \frac{4}{3}$