MIDTERM I- SOLUTIONS
MATH 109

1. Part I (25 points)

1. $\int x(\ln x)^2 = \frac{x^2}{2}(\ln x)^2 - \int x \ln x dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \int \frac{x}{2} dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$
Answer: $F(x) = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4}$.
2. With the substitution $x = 2\sin u$ we have:
$dx = 2\cos u du$ $\sqrt{4-x^2} = 2\cos u$ $u = \sin^{-1}(\frac{x}{2})$.
$\int \sqrt{4-x^2}dx = 4\int \cos^2 u du = \int 2 + 2\cos(2u)du = 2u + 2\sin(u)\cos(u) + C$.
Answer(get rid of u): $2\sin^{-1}(\frac{x}{2}) + 2\frac{x}{2}\sqrt{1-(\frac{x}{2})^2} = 2\sin^{-1}(\frac{x}{2}) + \frac{1}{2}x\sqrt{4-x^2} + C$.

2. Part II (25 points)

Reduced equation: $y'' - 6y' + 9y = 0$.
Characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow r_1 = r_2 = 3$.
Fundamental solutions: $u_1(x) = e^{3x}, \quad u_2(x) = xe^{3x}$.
Particular solution: search for $y_p(x) = Ax^2e^{3x}$; find $A = \frac{1}{2}$.
General solution: $y(x) = \frac{1}{2}x^2e^{3x} + C_1e^{3x} + C_2xe^{3x}$.
Solve for initial values: $y(0) = C_1 = -1$, $y'(0) = 3C_1 + C_2 = 1 \Rightarrow C_1 = -1, C_2 = 4$.
Answer: $y(x) = \frac{1}{2}x^2e^{3x} - e^{3x} + 4xe^{3x}$.

3. Part III (25 points)

2. $Area = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{16}{\pi^2}\theta^2 d\theta = \frac{2\pi}{3}$.

4. Part IV (25 points)

1. Parametrize $x = x(\theta) = \theta^2\cos\theta, y = y(\theta) = \theta^2\sin\theta$.
Slope at $\theta = \frac{\pi}{2}$: $m(\frac{\pi}{2}) = \frac{y'(\frac{\pi}{2})}{x'(\frac{\pi}{2})} = \frac{2\theta\sin\theta + \theta^2\cos\theta}{2\theta\cos\theta - \theta^2\sin\theta}\Big|_{\theta=\frac{\pi}{2}} = \frac{\pi}{-\frac{\pi^2}{4}} = -\frac{4}{\pi}$.

2. $Length = \int_\alpha^\beta \sqrt{\rho(\theta)^2 + [\rho'(\theta)]^2}d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}d\theta = \int_0^\pi \theta(\theta^2+4)^{\frac{1}{2}}d\theta$.
Answer: $\frac{1}{2}\frac{(\theta^2+4)^{3/2}}{3/2}\Big|_{\theta=0}^{\theta=\pi} = \frac{(\pi^2+4)^{3/2}-8}{3}$.