SOLUTIONS TO 8.4 PRACTICE PROBLEM SET

1. I

A) $\int \sec^3 u\ du = \int \sec u (\tan u)'\ du = \sec u \tan u - \int \sec u \tan^2 u\ du$
$= \sec u \tan u - \int \sec u (\sec^2 u - 1)\ du$, therefore
$2 \int \sec^3 u\ du = \sec u \tan u + \int \sec u\ du \Rightarrow \int \sec^3 u\ du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \int \sec u\ du$,
so

$$\int \sec^3 u\ du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + c$$

B) $\int \sec^5 u\ du = \int \sec^3 u (\tan u)'\ du = \sec^3 u \tan u - 3 \int \sec^3 u \tan^2 u\ du$
$= \sec^3 u \tan u - 3 \int \sec^3 u (\sec^2 u - 1)\ du$, therefore
$4 \int \sec^5 u\ du = \sec^3 u \tan u + 3 \int \sec^3 u\ du$ and hence

$$\int \sec^5 u\ du = \frac{1}{4} \sec^3 u \tan u + \frac{3}{2} \int \sec^3 u\ du$$

2. II

We will use the substitutions indicated in 8.4:
1) $\int \sqrt{1+x^2} dx = \int \sec u \text{sèc}^2 u\ du = \int \sec^3 u\ du$
$= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C = \frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2} \ln|x + \sqrt{1+x^2}| + C.$
2) $\int \frac{dx}{x^2 \sqrt{1+x^2}} = \int \frac{\sec^2 u\ du}{\sec u \tan^2 u} = \int \frac{\sec u\ du}{\tan^2 u} = \int \frac{\cos u\ du}{\sin^2 u} = -\frac{1}{\sin u} + C = -\frac{\sqrt{1+x^2}}{x} + C.$
3) $\int \frac{dx}{x^2 \sqrt{x^2-1}} = \int \frac{\sec u \tan u\ du}{\sec^2 u \tan u} = \int \cos u\ du = \sin u + C = \frac{\sqrt{x^2-1}}{x} + C.$
4) $\int \frac{dx}{x^2 \sqrt{1-x^2}} = \int \frac{\cos u\ du}{\sin^2 u \cos u} = \int \frac{du}{\sin^2 u} = -\cot u + C = -\frac{\sqrt{1-x^2}}{x} + C.$
5) $\int \frac{dx}{x \sqrt{x^2-1}} = \int \frac{\sec u \tan u\ du}{\sec u \tan u} = u + C = \sec^{-1} x + C.$
6) $\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2 u\ du}{\sec u} = \int \sec u\ du = \ln|\sec u + \tan u| + C = \ln|x + \sqrt{1+x^2}| + C.$
7) $\int x^2 \sqrt{1+x^2} dx = \int \tan^2 u \sec^3 u\ du = \int (\sec^2 u - 1) \sec^3 u\ du$
$= \int \sec^5 u - \int \sec^3 u\ du = \frac{1}{4} \sec^3 u \tan u - \frac{1}{4} \int \sec^3 u\ du$ (see I.B)
$= \frac{1}{4} \sec^3 u \tan u - \frac{1}{8} \sec u \tan u - \frac{1}{8} \ln|\sec u + \tan u| + C$
$= \frac{1}{4} x (1+x^2)^{3/2} - \frac{1}{8} x \sqrt{1+x^2} - \frac{1}{8} \ln|x + \sqrt{1+x^2}| + C.$
8) $\int x^2 \sqrt{1-x^2} dx = \int \sin^2 u \cos^2 u\ du = \frac{1}{4} \int \sin^2(2u)\ du = \frac{1}{4} \int \frac{1-\cos(4u)}{2}\ du$
$= \frac{1}{8} u - \frac{1}{32} \sin(4u) + C$. But
$\sin(4u) = 2 \sin(2u) \cos(2u) = 4 \sin u \cos u [1 - 2 \sin^2 u] = 4 \sin u \cos u - 8 \sin^3 u \cos u,$
hence our integral equals
$= \frac{1}{8} u - \frac{1}{8} \sin u \cos u + \frac{1}{4} \sin^3 u \cos u + C = \frac{1}{8} \sin^{-1} x - \frac{1}{8} x \sqrt{1-x^2} + \frac{1}{4} x^3 \sqrt{1-x^2} + C.$
9) $\int x \sqrt{1-x^2} dx = \int \sin u \cos^2 u\ du = -\frac{1}{3} \cos^3 u + C = -\frac{1}{3} (1-x^2)^{3/2} + C.$
10) $\int \frac{x dx}{\sqrt{1+x^2}} = \int \frac{\tan u \sec^2 u\ du}{\sec u} = \int \tan u \sec u\ du = \int \frac{\sin u\ du}{\cos^2 u} = \frac{1}{\cos u} + C$
$= \sqrt{1+x^2} + C.$
11) $\int \frac{x^2 dx}{\sqrt{1+x^2}} = \int \frac{\tan^2 u \sec^2 u\ du}{\sec u} = \int \tan^2 u \sec u\ du = \int (\sec^2 u - 1) \sec u\ du =$
$\int \sec^3 u\ du - \int \sec u\ du = \frac{1}{2} \sec u \tan u - \frac{1}{2} \ln|\sec u + \tan u| + C$

$= \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln|x+\sqrt{1+x^2}| + C.$

13) $\int \frac{x^2dx}{\sqrt{1-x^2}} = \int \frac{\sin^2 u \cos u \; du}{\cos u} = \int \sin^2 u \; du = \int \frac{1-\cos(2u)}{2} \; du = \frac{1}{2}u - \frac{1}{2}\sin u \cos u + C = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C.$

14) $\int \frac{x^2dx}{(1+x^2)^{3/2}} = \int \frac{\tan^2 u \sec^2 u \; du}{\sec^3 u} = \int \frac{\tan^2 u}{\sec u} \; du = \int \frac{\sec^2 u - 1}{\sec u} \; du = \int \sec u \; du - \int \frac{du}{\sec u} = \int \sec u \; du - \int \cos u \; du = \ln|\sec u + \tan u| - \sin u + C = \ln|x + \sqrt{1+x^2}| - \frac{x}{\sqrt{1+x^2}} + C.$