SECTION 11.1: SOLUTIONS TO SELECTED PROBLEMS

1. Section 11.1: Infinite Series

Ex. 27
Since \( \frac{1}{k} - \frac{1}{k+3} = \frac{3}{k(k+3)} \), the \( n \)th partial sum of the series \( \sum_{k=1}^{\infty} \frac{1}{k(k+3)} \) is

\[
s_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+3} = \frac{1}{3} \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+3} \right)\]

\[
= \frac{1}{3} \left( \frac{1}{1} - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \cdots + \frac{1}{n} - \frac{1}{n+3} \right)
\]

Most of the terms cancel out, so that

\[
s_n = \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right)
\]

\[
= \frac{1}{3} \left( \frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right)
\]

It is now clear that \( \lim_{n \to \infty} s_n = \frac{11}{18} \), therefore we conclude that \( \sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \frac{11}{18} \).

Ex. 31

\[
\sum_{k=0}^{\infty} \frac{1-2^k}{3^k} = \sum_{k=0}^{\infty} \frac{1}{3^k} - \sum_{k=0}^{\infty} \left( \frac{2}{3} \right)^k = \frac{1}{1-1/3} - \frac{1}{1-2/3} = \frac{3}{2} - 3 = -\frac{3}{2}
\]

Ex. 34

\[
\sum_{k=2}^{\infty} \frac{3^{k-1}}{4^{3k+1}} = \sum_{k=0}^{\infty} \frac{3^{k+1}}{4^{3k+7}} = \frac{3}{4^7} \cdot \sum_{k=0}^{\infty} \left( \frac{3}{4^7} \right)^k = \frac{3}{4^7} \cdot \frac{1}{1 - \frac{3}{4^7}} = \ldots?
\]

Ex. 43-48
1) The starting point is the identity \( \sum_{k=0}^{\infty} u^k = \frac{1}{1-u} \) whenever \( |u| < 1 \).
2) For \( u = -x \) \( \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x^2} \).
3) For \( u = -x^2 \) yields \( \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2} \).
4) Multiplying both terms of 3) by \( x \) we obtain \( \sum_{k=0}^{\infty} (-1)^k x^{2k+1} = \frac{x}{1+x^2} \).
5) Replacing \( x \) by \( 2x \) in 3) yields \( \sum_{k=0}^{\infty} (-1)^k 4^k x^{2k} \). Multiplying both sides by \( x \) yields \( \frac{x}{1+4x^2} = \sum_{k=0}^{\infty} (-1)^k 4^k x^{2k+1} \).

Ex. 49
The series diverges since the general term \( \frac{x^m}{2^n} = \left( \frac{1}{2} \right)^n \) does not tend to 0.

Ex. 50
The general term of the series is \( \frac{1}{k}(\frac{-1}{2})^k \) which does not tend to 0 (it is an unbounded sequence). The series is therefore divergent (Basic Divergence Test).