A MODEL STRUCTURE FOR QUASI-CATEGORIES

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1. Introduction

Quasi-categories live at the intersection of homotopy theory with category theory. In particular, they serve as a model for $(\infty, 1)$-categories, that is, weak higher categories with n-cells for each natural number n that are invertible when $n > 1$. Alternatively, an $(\infty, 1)$-category is a category enriched in ∞-groupoids, e.g., a topological space with points as 0-cells, paths as 1-cells, homotopies of paths as 2-cells, and homotopies of homotopies as 3-cells, and so forth.

The basic data for a quasi-category is a simplicial set. A precise definition is given below. For now, a simplicial set X is given by a diagram in **Set**

$$X_0 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; X_1 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \; X_2 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \cdots$$

with certain relations on the arrows. Elements of X_n are called *n-simplices*, and the arrows $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ are called *face* and *degeneracy* maps, respectively. Intuition is provided by simplical complexes from topology. There is a functor τ_1 from the category of simplicial sets to **Cat** that takes a simplicial set X to its *fundamental category* $\tau_1 X$. The objects of $\tau_1 X$ are the elements of X_0. Morphisms are generated by elements of X_1 with the face maps defining the source and target and $s_0 : X_0 \to X_1$ picking out the identities. Composition is freely generated by elements of X_1 subject to relations given by elements of X_2. More specifically, if $x \in X_2$, then we impose the relation that $d_1 x = d_0 x \circ d_2 x$.

This functor is very destructive. In particular, it only depends on the data of the simplicial set up to the 2-simplices. Quasi-categories provide a weaker notion of composition that is non-algebraic (2-simplices exhibiting *some* composite rather than *the* composite), while avoiding this sort of truncation. As before, the 0-simplices are interpreted as objects and the 1-simplices as morphisms. For given 1-simplices f and g with $d_0 f = d_1 g$, each 2-simplex x with $d_0 x = g$ and $d_2 x = f$ will be interpreted as giving a composite $d_1 x$ of f and g up to homotopy. The 3-simplices give homotopies between these homotopies, and so forth. In an ordinary simplicial set, the simplices exhibiting these composites need not exist; a quasi-category will be a simplicial set satisfying certain extra "horn-filling" conditions that suffice to define a non-algebraic composition of the simplices.

The functor τ_1 has a right adjoint N called the nerve functor that takes a category $\mathcal{C}$ to a simplicial set $N\mathcal{C}$. The nerve functor is full and faithful and all the simplicial sets in its image are quasi-categories. All of the important data of a category is contained in its nerve. In particular, $\tau_1 N\mathcal{C} = \mathcal{C}$ for any category, so categories can

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be studied via quasi-categories. The nerve functor will be described in more detail below.

A modern introduction to quasi-categories must note that they also serve as a model for the "homotopy theory of homotopy theories." In some sense a "homotopy theory" can be regarded as a category with some class of weak equivalences that one would like to formally invert. Any such homotopy theory gives rise to a simplicial category, and conversely simplicial categories arise from homotopy theories up to Dwyer-Kan equivalence. Thus a model structure on the category of simplicial categories has the interpretation as a homotopy theory of homotopy theories.

Simplicial categories are easily related to simplicial spaces, also known as bisimplicial sets. There are a number of models for the homotopy theory of homotopy theories with simplicial spaces as objects. Surprisingly, these models are all Quillen equivalent to the model structure for quasi-categories on **sSet**, a description of which is the main objective of this paper[1]. Quasi-categories, as noted below, are simplicial sets with a certain lifting property, and are thus much simpler objects than simplicial categories or simplicial spaces, suggesting that this model may prove most useful for performing actual computations. We say a few words about these equivalences in Section 6. A good summary is given in [2].

Alternatively, there is a direct functorial construction of a quasi-category from a category $\mathcal{C}$ with a class of weak equivalences $\mathcal{W}$. This construction is described at the end of Section 5.

The theory of quasi-categories has been developed extensively by André Joyal and Jacob Lurie, among others. We mostly follow Joyal's terminology; Lurie calls quasi-categories ∞-*categories*, which to a category theorist can be a bit misleading.

1.1. **Notation and Classical Results.** Classically, simplicial sets were introduced as a combinatorial model for the homotopy theory of topological spaces. One measure the success of this approach is two theorems by Quillen from his initial paper on model categories [13], which we record below.

First, we establish some notation and terminology. Let $\mathbf{\Delta}$ denote the category of finite non-empty ordinals and order preserving maps. We write $[n]$ for the set $\{0, 1, \ldots, n\}$, corresponding to the ordinal $n+1$. A simplicial set is a contravariant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and morphisms of simplicial sets are natural transformations. These form a category **sSet** that is complete and cocomplete (with limits and colimits formed levelwise), locally small, and cartesian closed.

The morphisms of $\mathbf{\Delta}$ are generated by the injective *coface* maps $d^i : [n-1] \to [n]$ whose image misses $i \in [n]$ and the surjective *codegeneracy* maps $s^i : [n+1] \to [n]$ for which $i \in [n]$ has two elements in its preimage. If X is a simplicial set, we write $d_i : X_n \to X_{n-1}$ for $Xd^i : X[n] \to X[n-1]$ and $s_i : X_n \to X_{n+1}$ for $Xs^i : X[n] \to X[n+1]$ and call these the *face* and *degeneracy* maps, respectively. A simplicial set may alternatively be described as a collection of sets $X_0, X_1, X_2, \ldots$ together with face and degeneracy maps satisfying certain relations dual to those satisfied by the d^i and s^i (see [5] or [12] for details).

Let Δ^n be the represented simplicial set $\mathbf{\Delta}(-, [n])$. By the Yoneda lemma n-simplices of a simplicial set X correspond bijectively to maps $\Delta^n \to X$ of simplicial sets and our notation deliberately conflates the two. Δ^n has several important

[1]Note that the full subcategory $\mathbf{QCat} \subset \mathbf{sSet}$ spanned by the quasi-categories is neither complete nor cocomplete, so the model structure for quasi-categories will actually be a model structure on **sSet**

simplicial subsets: the i-th face $\partial_i\Delta^n$, which is the image of $d^i : \Delta^{n-1} \to \Delta^n$; the n-sphere $\partial\Delta^n$, which is the union of the $n+1$ faces of Δ^n; and the (n,k)-horn Λ^n_k, which is the union of all faces except for $\partial_k\Delta^n$. We often write I for Δ^1, as this simplicial set is analogous to the topological interval, and we write $*$ for the terminal simplicial set Δ^0. The geometric realizations of each of these simplicial subsets are the topological spaces suggested by their names.

Like all categories of presheaves of a small category, **sSet** is cartesian closed; we denote the internal-hom by X^A. By the defining adjunction, maps $\Delta^n \to X^A$ correspond to maps $A \times \Delta^n \to X$. By the Yoneda lemma, we can take the set of the latter to be the definition of n-simplices $[X^A]_n$, if we wish. The face and degeneracy maps are given by precomposition by $1 \times d^i$ and $1 \times s_i$, respectively.

A map $\Lambda^n_k \to X$ of simplicial sets is called a *horn of* X. We say a simplicial set X is a *Kan complex* if every horn of X has a filler, that is, if every $\Lambda^n_k \to X$ can be extended along the inclusion $h^n_k : \Lambda^n_k \to \Delta^n$. A (small) *quasi-category* is a simplical set such that every *inner* horn (i.e., Λ^n_k with $0 < k < n$) has a filler.

Theorem 1.1 (Quillen Model Structure)**.** **sSet** *has a model structure* $(\mathcal{C}, \mathcal{F}_k, \mathcal{W}_h)$, *where* $\mathcal{C}$ *is the class of monomorphisms,* $\mathcal{F}_k$ *is the class of Kan fibrations, and* $\mathcal{W}_h$ *is the class of weak homotopy equivalences. This model structure is cofibrantly generated by the sets*

$$\mathcal{I} = \{i_n : \partial\Delta^n \hookrightarrow \Delta^n \mid \forall n \geq 0\}$$

of generating cofibrations and

$$\mathcal{J} = \{h^n_k : \Lambda^n_k \hookrightarrow \Delta^n \mid \forall n \geq 1\}$$

of generating trivial cofibrations.

We will refer to $(\mathcal{C}, \mathcal{F}_k, \mathcal{W}_h)$ as the *Quillen* or *classical model structure* on **sSet**. It is cartesian (see Theorem 4.1) and proper. The following theorem is the reason why simplicial sets provide a useful model for the homotopy theory of topological spaces.

Theorem 1.2 (Quillen)**.** *Geometric realization and the total singular complex functor induce a Quillen equivalence between* **sSet** *with the above model structure and* **Top** *with the model structure where weak equivalences are weak homotopy equivalences and fibrations are Serre fibrations.*

The *geometric realization* functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ has a concise categorical description as the left Kan extension of the functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ that takes the ordinal $[n] = \{0, 1, \ldots, n\}$ to the standard topological n-simplex Δ_n, along the Yoneda embedding $y : \mathbf{\Delta} \hookrightarrow \mathbf{sSet}$. Because the Yoneda embedding is full and faithful, $|-|$ will be a literal extension of Δ, meaning that $|y[n]| = \Delta[n]$; in more conventional notation this says that $|\Delta^n| = \Delta_n$. The right adjoint of geometric realization is the *total singular complex functor* $S : \mathbf{Top} \to \mathbf{sSet}$, with the n-simplices of the total singular complex of a topological space X given by the set

$$SX_n := \mathbf{Top}(\Delta_n, X).$$

This type of adjunction involving simplicial sets is quite general. Given any $F : \mathbf{\Delta} \to \mathcal{E}$ where $\mathcal{E}$ is cocomplete and locally small, the functor $\mathrm{Lan}_y F : \mathbf{sSet} \to \mathcal{E}$ has a right adjoint R where the n-simplices of Re for any $e \in \mathrm{ob}\,\mathcal{E}$ are defined to be the set

$$Re_n := \mathcal{E}(F[n], e),$$

with the face and degeneracy maps given by precomposition by the corresponding morphisms in the image of F.

Other examples that fit this paradigm include the categorical nerve $N : \mathbf{Cat} \to \mathbf{sSet}$ and its left adjoint τ_1, subdivision and extension, and also the simplicial nerve functor $\mathbb{N}$ that takes a simplicial category (that is a category enriched in **sSet**) to a simplicial set and its left adjoint $\mathbb{C}$, both of which are discussed in Section 6. Under suitably nice hypotheses (see [9]) the left adjoints will preserve finite products.

Finally, we write $f \boxslash g$ to mean that a morphism f has the left lifting property with respect to a morphism g, that is, that every commutative square

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] \arrow[ur, dashed, "w"] & \cdot \end{tikzcd}$$

has a lift w such that both triangles commute. Lifting properties are central to the definition of quasi-categories as well as many other related concepts. For example, X is a quasi-category iff the map $q : X \to 1$ has the property $h^n_k \boxslash q$ for all $0 < k < n$, for all $n > 1$.

2. Basic Quasi-categories

Categories provide important examples of quasi-categories, as we see below. Because the nerve embedding $N : \mathbf{Cat} \to \mathbf{sSet}$ is full and faithful, we can regard the theory of quasi-categories as an extension of the theory of categories. Also for this reason, we will occasionally use the same notation for a category and its nerve.

Lemma 2.1. *If $\mathcal{K}$ is a category, every inner horn of $N\mathcal{K}$ has a unique filler; hence, the nerve of a category is a quasi-category.*

Proof. Horns $\Lambda^n_k \to N\mathcal{K}$ correspond bijectively to functors $\tau_1\Lambda^n_k \to \mathcal{K}$ by the adjunction $\tau_1 \dashv N$. The inner horn inclusion h^n_k induces an isomorphism of categories; hence, there is a unique extension of $\tau_1\Lambda^n_k \to \mathcal{K}$ along the functor $\tau_1\Lambda^n_k \to \tau_1\Delta^n$. The result follows by passing back across the adjunction. □

In fact, the converse also holds: if X is a quasi-category such that every inner horn has a unique filler, then X is isomorphic to the nerve of a category.

The 0-simplices of a simplicial set are often called its vertices and the 1-simplices its edges. If a and b are vertices of a simplicial set X, we write $X(a, b)$ for the simplicial set of 1-simplices f with $d_1 f = a$ and $d_0 f = b$. Alternatively, $X(a, b)$ is the fiber of the map $X^I \to X^{\{0,1\}}$ induced by the inclusion $\{0, 1\} \hookrightarrow I$ at the vertex (a, b) of $X^{\{0,1\}}$. The simplicial set $X(a, b)$ is a Kan complex if X is a quasi-category; Theorem 4.2 will imply that its a quasi-category and the rest follows from results characterizing Kan complexes not contained in this paper.

The fundamental category of a quasi-category is isomorphic to its homotopy category, described below.

Definition 2.2. Let X be a quasi-category and let $f, g \in X(a, b)$. We write $f \sim_a g$ if there is a 2-simplex with boundary $(f, g, 1_a)$ and $f \sim_b g$ if there is a 2-simplex with boundary $(1_b, g, f)$. The four relations $f \sim_a g$, $f \sim_b g$, $g \sim_a f$ and $g \sim_b f$ are equivalent when X is a quasi-category, and we denote the common relation they define by $f \sim g$ and say that f is *homotopic to* g. We write $[f]$ for the equivalence class of f up to homotopy. The homotopy category $\mathrm{ho}X$ has objects X_0, morphisms homotopy classes of arrows $f \in X_1$, with degenerate arrows acting as identities, and composition given by filling the horns h^2_1, which is well-defined.

Theorem 2.3. *When X is a quasi-category, the natural map $\tau_1 X \to \mathrm{ho}X$ is a canonical isomorphism.*

The proof is straightforward.

3. Weak Categorical Equivalences

The weak equivalences of Joyal's model structure for quasi-categories are called *weak categorical equivalences*, which we describe below. Joyal's definition is analogous to a combinatorial description of weak homotopy equivalences of simplicial sets, which we also give below. By contrast, Lurie defines weak categorical equivalences (which he calls categorial equivalences) to be those maps whose image under $\mathbb{C}$ is an equivalence of simplicial categories. These definitions are equivalent.

Taking a cue from enriched category theory, when a category $\mathcal{E}$ is cartesian closed and we are given a product preserving functor $\tau : \mathcal{E} \to \mathbf{Set}$, we can define a cartesian closed category $\mathcal{E}^\tau$ with the same objects as $\mathcal{E}$ and with hom-sets given by $\mathcal{E}^\tau(A, B) := \tau(B^A)$.

For **sSet**, we have two relevant product preserving functors: $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ that takes a simplicial set X to the set of path components of vertices and $\tau_0 : \mathbf{sSet} \to \mathbf{Set}$ that takes X to the set of isomorphism classes of objects of $\tau_1 X$. Note that τ_0 is the composite

$$\mathbf{sSet} \overset{\tau_1}{\to} \mathbf{Cat} \overset{J}{\to} \mathbf{Gpd} \hookrightarrow \mathbf{Cat} \overset{N}{\to} \mathbf{sSet} \overset{\pi_0}{\to} \mathbf{Set}$$

In the above J is the functor that takes a category to its groupoid of isomorphisms. We've remarked already that π_0 and τ_1 preserve finite products; the other functors all have left adjoints. Hence, τ_0 preserves finite products.

There is a natural transformation $\alpha : \tau_0 \Rightarrow \pi_0$ that takes an isomorphism class to its path component, which gives rise to a functor $\mathbf{sSet}^{\tau_0} \to \mathbf{sSet}^{\pi_0}$.

Definition 3.1. A map of simplicial sets is a *homotopy equivalence* if its image in $\mathbf{sSet}^{\pi_0}$ is an isomorphism. We say a map $u : A \to B$ is a *weak homotopy equivalence* if

$$\mathbf{sSet}^{\pi_0}(u, X) : \mathbf{sSet}^{\pi_0}(B, X) \to \mathbf{sSet}^{\pi_0}(A, X)$$

is a bijection for all Kan complexes X.

Classically, a weak homotopy equivalence between simplicial sets is a map whose geometric realization is a weak homotopy equivalence of topological spaces — or equivalently, by Whitehead's theorem, a homotopy equivalence of topological spaces — because the geometric realization of a simplicial set is a CW complex. Formally, $|A| \overset{|u|}{\to} |B|$ is a homotopy equialence of CW complexes iff π_0 applied to $\mathrm{Map}(|B|, |X|) \to \mathrm{Map}(|A|, |X|)$ is an isomorphism. When X is Kan,

$$\pi_0 X^B \cong \pi_0 |X^B| \cong \pi_0 \operatorname{Map}(|B|, |X|),$$

so which proves that this definition is equivalent to the usual one.

This combinatorial definition of a weak homotopy equivalence has a clear analogy with the following definition.

Definition 3.2. A map of simplicial sets is a *categorical equivalence* if its image in $\mathbf{sSet}^{\tau_0}$ is an isomorphism. We say a map $u : A \to B$ is a *weak categorical equivalence* if

$$\mathbf{sSet}^{\tau_0}(u, X) : \mathbf{sSet}^{\tau_0}(B, X) \to \mathbf{sSet}^{\tau_0}(A, X)$$

is a bijection for all quasi-categories X.

A categorical equivalence is necessarily a weak categorical equivalence. Conversely, a weak categorical equivalence between quasi-categories is a categorical equivalence as a consequence of the Yoneda lemma applied to $\mathbf{QCat}^{\tau_0} \subset \mathbf{sSet}^{\tau_0}$, the full subcategory spanned by the quasi-categories. The analogous facts are true for homotopy equivalences and Kan complexes. Note that all of these classes satisfy the 2 of 3 property.

An easy exercise shows that if X is a Kan complex, then its homotopy category is a groupoid (and the converse holds when X is a quasi-category, though this is somewhat harder to prove). This result enables the following lemma.

Lemma 3.3. *If X is a Kan complex, $\tau_0 X = \pi_0 X$.*

If X is a Kan complex, so is X^A for any simplicial set A as a consequence of the first part of Theorem 4.2, which is a classical result. So the previous lemma implies that $\mathbf{sSet}^{\tau_0}(A, X) = \mathbf{sSet}^{\pi_0}(A, X)$ for any simplicial set A when X is Kan. With these tools, we can prove the following important theorem.

Theorem 3.4. *A (weak) categorical equivalence is a (weak) homotopy equivalence.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
 & \mathbf{sSet} \arrow[dl, "\rho"'] \arrow[dr, "\rho"] & \\
\mathbf{sSet}^{\tau_0} \arrow[rr, "\alpha"'] & & \mathbf{sSet}^{\pi_0}
\end{tikzcd}$$

By functoriality of α, an arrow that becomes invertible in $\mathbf{sSet}^{\tau_0}$ remains so in $\mathbf{sSet}^{\pi_0}$, so a categorical equivalence is a homotopy equivalence.

Now suppose $u : A \to B$ is a weak categorical equivalence. Then $\mathbf{sSet}^{\tau_0}(u, X) : \tau_0(X^B) \to \tau_0(X^A)$ is a bijection for all Kan complexes X. By Lemma 3.3 it follows that $\mathbf{sSet}^{\pi_0}(u, X)$ is also a bijection. Hence u is a weak homotopy equivalence. □

Quillen's model structure on simplicial sets has the monomorphisms as cofibrations and weak homotopy equivalences as weak equivalences. Joyal's model structure for quasi-categories also has the monomorphisms as cofibrations and has weak categorical equivalences as weak equivalences. So Theorem 3.4 will imply that the Quillen model structure is a Bousfield localization of the Joyal model structure. This fact is recorded as Proposition 5.9.

The next result will be used to establish the Joyal model structure.

Proposition 3.5. *A trivial Kan fibration is a categorical equivalence, and thus a weak categorical equivalence.*

Proof. If $f : X \to Y$ is a trivial Kan fibration, it has a section s constructed as a lift of the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

Let J be the groupoid with two objects and exactly one morphism in each hom-set. By an abuse of notation we also refer to NJ as J. Let $j : \{0,1\} \hookrightarrow J$ be the monomorphism of simplicial sets that includes the vertices of J into J. We obtain a commutative square

$$\begin{tikzcd} X \times \{0,1\} \arrow[r, "{(1,sf)}"] \arrow[d, "1\times j"'] & X \arrow[d, "f"] \\ X \times J \arrow[r, "f\cdot \mathrm{proj}_1"'] \arrow[ur, dashed, "h"'] & Y \end{tikzcd}$$

which has a lift because the left hand side is monic. The lift h is adjoint to an arrow $k : J \to X^X$ which takes the non-trivial isomorphisms in J to arrows between the vertices sf and 1_X of X^X, which become isomorphisms in $\tau_1(X^X)$. Hence, s is an inverse for f in $\mathbf{sSet}^{\tau_0}$, which makes f a categorical equivalence. □

Proposition 3.6. *A functor between categories is a categorical equivalence if and only if its nerve is a categorical equivalence of simplicial sets. Furthermore, τ_1 takes weak categorical equivalences to categorical equivalences.*

Proof. A functor in **Cat** is an equivalence of categories if and only if it is invertible in $\mathbf{Cat}^{\tau_0}$. Because τ_1 preserves finite products, N preserves exponentials and the natural bijection $\tau_0(N(\mathcal{L}^{\mathcal{K}})) \to \tau_0(N\mathcal{L}^{N\mathcal{K}})$ implies that the functor $\mathbf{Cat}^{\tau_0} \to \mathbf{sSet}^{\tau_0}$ induced by the nerve is fully faithful. So a functor is an equivalence iff its nerve is a categorical equivalence. Similarly, $(N\mathcal{K})^X \cong N(\mathcal{K}^{\tau_1 X})$ if $\mathcal{K}$ is a category and X is a simplicial set. If u is a weak categorical equivalence, then $\mathbf{sSet}^{\tau_0}(u, N\mathcal{K})$ and thus also $\mathbf{Cat}^{\tau_0}(\tau_1 u, \mathcal{K})$ is an isomorphism for any category $\mathcal{K}$, and Yoneda implies that $\tau_1 u$ is invertible in $\mathbf{Cat}^{\tau_0}$. □

4. Mid, Right, and Left Fibrations

The results in the section are only tangentially relevant to the model structure for quasi-categories but encode a lot of the combinatorial work necessary to prove things such as the fact that X^A is a quasi-category if X is a quasi-category and A is a simplicial set. The main point of this section is to describe five weak factorization systems on **sSet** that have nice properties encoded in Theorems 4.1 and 4.2. Two of these are classical: the weak factorization systems $(\mathcal{C}, \mathcal{F}_k \cap \mathcal{W}_h)$ and $(\mathcal{C} \cap \mathcal{W}_h, \mathcal{F}_k)$ arising from Quillen's model structure. These are both cofibrantly generated by sets $\mathcal{I}$ and $\mathcal{J}$ (see Theorem 1.1). Morphisms in $\mathcal{C} \cap \mathcal{W}_h$ are called *anodyne*.

Let $\mathcal{J}_m \subset \mathcal{J}$ be the set of inner horn inclusions described in Section 1.1. Let $\mathcal{F}_m$ be the class of *mid fibrations* (Lurie writes *inner fibrations*), that is, those maps that have the right lifting property with respect to $\mathcal{J}_m$. As the domains of the maps in $\mathcal{J}_m$ are small in **sSet**, it follows from Quillen's small object argument that there is a cofibrantly generated weak factorization system $(\mathcal{A}_m, \mathcal{F}_m)$ on **sSet**, where $\mathcal{A}_m$ is the saturated class generated by $\mathcal{J}_m$. The elements of $\mathcal{A}_m$ are exactly those morphisms that have the left lifting property with respect to $\mathcal{F}_m$; we call these morphisms *mid anodyne* (Lurie writes *inner anodyne*).

Analogously, we have cofibrantly generated weak factorization systems $(\mathcal{A}_l, \mathcal{F}_l)$ and $(\mathcal{A}_r, \mathcal{F}_r)$ generated by sets

$$\mathcal{J}_l = \{h^n_k : \Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k < n, n \geq 1\} \qquad \text{and}$$
$$\mathcal{J}_r = \{h^n_k : \Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k \leq n, n \geq 1\}$$

respectively. The left classes are the *left* and *right anodyne* maps and the right classes are the *left* and *right fibrations*, respectively. Recalling that the weak factorization system $(\mathcal{C} \cap \mathcal{W}_h, \mathcal{F}_k)$ is cofibrantly generated by $\mathcal{J} \subset \mathcal{J}_l, \mathcal{J}_r \subset \mathcal{J}_m$, we have the obvious inclusions

$$\mathcal{F}_m \subset \mathcal{F}_l, \mathcal{F}_r \subset \mathcal{F}_k.$$

If $\mathcal{E}$ is a complete and cocomplete category, for any morphism u we have an adjunction

$$u \,\square\, - : \mathcal{E}^{\mathbf{2}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E}^{\mathbf{2}} : \langle u, - \rangle$$

on the arrow category, where $u \,\square\, v \cong v \,\square\, u$ is the pushout product and $\langle u, f \rangle$ is the dual *pullback product*, as illustrated below

$$\begin{tikzcd}
A \times K \arrow[r, "1 \times v"] \arrow[d, "u \times 1"'] & B \times K \arrow[d] \arrow[ddr, bend left, "u \times 1"] & \\
A \times L \arrow[r] \arrow[drr, bend right, "1 \times v"'] & \cdot \arrow[dr, dashed, "u \,\square\, v"] \arrow[ul, phantom, "\ulcorner"] & \\
& & B \times L
\end{tikzcd}
\qquad
\begin{tikzcd}
X^B \arrow[dr, dashed, "\langle u, f \rangle"] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "u^*"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y^B \arrow[d, "u^*"] \\
& X^A \arrow[r, "f_*"'] & Y^A
\end{tikzcd}$$

The product $-\square-$ gives $\mathbf{sSet}^{\mathbf{2}}$ the structure of a closed symmetric monoidal category, with $\langle -, - \rangle$ acting as the internal hom.

If u and v are monic, it is easy to verify that the pushout product $u \,\square\, v$ is monic. The following results are less trivial.

Theorem 4.1. *If u is monic and v is anodyne (resp. mid anodyne, left anodyne, right anodyne), then so is $u \,\square\, v$.*

Proof. This can be proved directly by a lot of messy combinatorics, which is worth it because it makes a lot of other proofs relatively easy (the next theorem is one example). See [10, §2.3.2] for a direct proof or [8, §4] for a more high level approach. $\square$

Theorem 4.2. *If f is a Kan fibration (resp. mid fibration, left fibration, right fibration), then so is $\langle u, f \rangle$ for any monic u. Moreover, $\langle u, f \rangle$ is a trivial Kan fibration if in addition u is anodyne (resp. mid anodyne, left anodyne, right anodyne).*

Proof. Suppose f is a mid fibration and u is monic. Then $(u \,\square\, v) \boxslash f$ for all mid anodyne v by Theorem 4.1. By adjunction, this is equivalent to $v \boxslash \langle u, f \rangle$, which tells us that $\langle u, f \rangle$ is mid anodyne. The other proofs of the first statement are analogous.

Now suppose f is a mid fibration and u is mid anodyne. Then $(v \,\square\, u) \boxslash f$ for all monic v by Theorem 4.1. By adjunction, this is equivalent to $v \boxslash \langle u, f \rangle$, which says that $\langle u, f \rangle$ is a trivial Kan fibration. The other proofs are analogous. $\square$

An easy and important corollary is that if X is a quasi-category, then so is X^A for any simplicial set A. This allows us to prove the following alternative characterization of weak categorical equivalences.

Corollary 4.3. *A map $u : A \to B$ is a weak categorical equivalence iff*

$$X^u : X^B \to X^A$$

is an equivalence of quasi-categories for every quasi-category X.

Proof. Follows from the Yoneda lemma and the fact that $\mathbf{sSet}^{\tau_0}$ is cartesian closed. □

Corollary 4.4. *Every mid anodyne map is a weak categorical equivalence.*

Proof. If X is a quasi-category and u is mid anodyne then X^u is a trivial fibration by Theorem 4.2 and thus a categorical equivalence by Proposition 3.5. The result follows from Corollary 4.3. □

5. Quasi-fibrations and Model Structure

In Section 3, we noted that Joyal's model structure for categories will have monomorphisms for cofibrations and weak categorical equivalences for weak equivalences. In this section, we will attempt (with only partial success) to characterize the fibrations, called *quasi-fibrations*, more precisely, and we will complete the proof that these classes constitute a model structure on **sSet**.

Let J be the nerve of the groupoid with two objects and exactly one morphism in each hom-set. Call a map between quasi-categories a *quasi-fibration* if it is a mid fibration that has the right lifting property with respect to the inclusion $j_0 : \{0\} \hookrightarrow J$. This lifting property characterizes those maps between quasi-categories that have lifts for quasi-isomorphisms, which are arrows that become invertible in the homotopy category. This lifting property is exactly analogous to that of iso-fibrations in **Cat**. In fact, a functor is an iso-fibration if and only if its nerve is a quasi-fibration.

We have the following analog of 4.2.

Lemma 5.1. *Let f be a quasi-fibration between quasi-categories. Then so is $\langle u, f\rangle$ for any monomorphism u.*

Quasi-fibrations between quasi-categories are characterized by the following theorem.

Theorem 5.2. *If $f : X \to Y$ is a map between quasi-categories, the following are equivalent:*

(i) *f is a quasi-fibration.*

(ii) *f has the right lifting property with respect to every monic weak categorical equivalence.*

(iii) *$\langle u, f\rangle$ is a trivial Kan fibration for every monic weak categorical equivalence u.*

Proof. (ii) $\Leftrightarrow$ (iii) follows easily from Theorem 4.2 and the adjunction on $\mathbf{sSet}^{\mathbf{2}}$. (ii) $\Rightarrow$ (i) is also straightforward: by Corollary 4.4, the inner horn inclusions are monic weak categorical equivalences. The monomorophism j_0 is the image under the nerve functor of a categorical equivalence, so it is a monic weak categorical equivalence by Proposition 3.6. (i) $\Rightarrow$ (iii) follows easily from two facts: Lemma 5.1 and a result that will be strengthened in Theorem 5.5, which says that a quasi-fibration between quasi-categories is a trivial fibration iff it is a categorical equivalence. See [8, §7]. □

Lemma 5.3. *$\mathcal{C}\cap\mathcal{W}_c = {}^{\boxslash}\mathcal{F}_0$ where $\mathcal{F}_0$ is the class of quasi-fibrations between quasi-categories and $\mathcal{W}_c$ is the class of weak categorical equivalences.*

Proof. In light of Theorem 5.2, we need only show that ${}^{\boxslash}\mathcal{F}_0 \subset \mathcal{C}\cap\mathcal{W}_c$. Given a $u : A \to B$ with this lifting property, we obtain a lifting problem

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r] & * \end{tikzcd}$$

by factoring $A \to *$ using the weak factorization system $(\mathcal{A}_m, \mathcal{F}_m)$. In particular, X is a quasi-category and for the reasons explained in the proof of Theorem 5.7, $f \in \mathcal{F}_0$. So $u \boxslash f$ and hence v factors through u; since v is monic, u is as well. It follows from the adjunction on $\mathbf{sSet^2}$ and Lemma 5.1 that $\langle u, f\rangle$ is a trivial Kan fibration for any $f \in \mathcal{F}_0$. In particular, $X^u : X^B \to X^A$ is a trivial Kan fibration for all quasi-categories X, and thus a weak categorical equivalence by Proposition 3.5. □

More generally, call a map of simplicial sets a *quasi-fibration* if it has the right lifting property with respect to $\mathcal{C}\cap\mathcal{W}_c$. By Theorem 5.2, this extends the previous definition for $\mathbf{QCat} \subset \mathbf{sSet}$. Let $\mathcal{F}_q$ denote the class of quasi-fibrations.

Theorem 5.4. *If f is a quasi-fibration then so is $\langle u, f\rangle$ for any monic u. Moreover, $\langle u, f\rangle$ is a trivial fibration if in addition u is a weak categorical equivalence.*

Proof. Given $u \in \mathcal{C}$ and $v \in \mathcal{C}\cap\mathcal{W}_c$, we can use Lemmas 5.1 and 5.3 to show that $u \square v \in \mathcal{C}\cap\mathcal{W}_c$. By adjunction, this tells us that $\langle u, f\rangle \in \mathcal{F}_q$. The same adjunction proves the second part. □

Theorem 5.5. *A quasi-fibration is a weak categorical equivalence iff it is a trivial Kan fibration.*

Proof. Half of this was done already in Proposition 3.5. Given $f \in \mathcal{F}_q \cap \mathcal{W}_c$ factor f as qu with $u \in \mathcal{C}$ and q a trivial Kan fibration. By 3.5, $q \in \mathcal{W}_c$ so $u \in \mathcal{W}_c$ by the 2 of 3 property. Hence the lifting problem

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "u"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "q"'] \arrow[ur, dashed, "s"] & \cdot \end{tikzcd}$$

has a solution s, which is used to display f as a retract of q. The classes of a weak factorization system are closed under retracts, so f is a trivial Kan fibration. □

In other words, $\mathcal{F}_k\cap\mathcal{W}_h = \mathcal{F}_q\cap\mathcal{W}_c$, which says that we can use the phrase *trivial fibration* unambiguously to describe members of either of these classes. It follows that $(\mathcal{C}, \mathcal{F}_q\cap\mathcal{W}_c)$ is a cofibrantly generated weak factorization system.

The class $\mathcal{C}\cap\mathcal{W}_c$ of monic weak categorical equivalences is not known to be generated by an easily described set, as is the case for the classical model structure.

However, this class is accessible (see [8]), which means that it is cofibrantly generated by *some* set. This allows us to apply the small object argument to obtain factorizations, completing the proof of the following.

Theorem 5.6. *If $\mathcal{F}_q$ is the class of quasi-fibrations then* $(\mathcal{C} \cap \mathcal{W}_c, \mathcal{F}_q)$ *is a cofibrantly generated weak factorization system.*

As remarked previously, $\mathcal{W}_c$ satisfies the 2 of 3 property. Thus, we have proven that $(\mathcal{C}, \mathcal{F}_q, \mathcal{W}_c)$ is a model structure on **sSet**, which we call *Joyal's model structure for quasi-categories*. In general, the pushout along a cofibration of a weak equivalence between cofibrant objects is a weak equivalence. Since every object of **sSet** is cofibrant, this model structure in left proper. However, it is not right proper. The inclusion $d^1 : I \hookrightarrow \Delta^2$ is a quasi-fibration between quasi-categories, which can be checked by verifying the required lifting properties directly. As noted in Section 2, $h^2_1 : \Lambda^2_1 \to \Delta^2$ induces an equivalence of categories, and thus must be a weak categorical equivalence. However, the pullback of h^2_1 along d^1 is the inclusion $i_1 : \partial I \to I$, which is not even a weak homotopy equivalence.

To see that the model structure $(\mathcal{C}, \mathcal{F}_q, \mathcal{W}_c)$ is indeed a model structure for quasi-categories, we need the following theorem.

Theorem 5.7. *The fibrant objects of* $(\mathcal{C}, \mathcal{F}_q, \mathcal{W}_c)$ *are the quasi-categories.*

Proof. If X is a quasi-category, then $X \to 1$ is a mid fibration between quasi-categories. Given a vertex $x \in X$, $s_0 x \in X_1$ is a quasi-isomorphism with source x. This shows that $X \to 1$ has the right lifting property with respect to $j_0 : \{0\} \to J$ and by Theorem 5.2, X is fibrant.

Conversely, if X is fibrant it has the right lifting property with respect to monic weak categorical equivalences, which by Corollary 4.4 includes the mid anodyne maps. So $X \to 1$ is a mid fibration and X is a quasi-category. □

Every simplicial set is cofibrant in the Joyal model structure and the quasi-categories are the fibrant objects. For any simplicial set A, $A \times J$ is a cylinder object. For any quasi-category X, X^J is a path object.

A Quillen adjunction $F : \mathcal{K} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{L} : G$ is a *homotopy localization* if the right derived functor $\mathbb{R}G : \mathrm{Ho}\mathcal{L} \to \mathrm{Ho}\mathcal{K}$ is full and faithful. This happens if and only if the counit of the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ is an isomorphism.

Proposition 5.8. *The adjunction $\tau_1 \dashv N$ is a homotopy localization between model structures* $(\mathcal{C}, \mathcal{F}_q, \mathcal{W}_c)$ *on* **sSet** *and the model structure (functors injective on objects, iso-fibrations, categorical equivalences) on* **Cat**.

Proof. τ_1 takes monomorphisms to functors which are injective on objects. The nerve takes an iso-fibration to a map between quasi-categories that has the right lifting property with respect to j_0 because this is exactly the condition for lifting of isomorphisms in **Cat**. Thus $\tau_1 \dashv N$ is a Quillen pair. The homotopy localization follows from the fact that $\tau_1 N = 1$. □

A model structure $(\mathcal{C}, \mathcal{F}_2, \mathcal{W}_2)$ is a *Bousfield localization* of a model structure $(\mathcal{C}, \mathcal{F}_1, \mathcal{W}_1)$ on the same category if $\mathcal{W}_1 \subset \mathcal{W}_2$. The following result is immediate from Theorem 3.4.

Proposition 5.9. *The model structure* $(\mathcal{C}, \mathcal{F}_k, \mathcal{W}_h)$ *is a Bousfield localization of* $(\mathcal{C}, \mathcal{F}_q, \mathcal{W}_c)$.

For any Bousfield localization with $\mathcal{W}_1 \subset \mathcal{W}_2$, the identity functors form a homotopy localization $(\mathcal{K}, \mathcal{W}_1) \underset{\leftarrow}{\overset{\perp}{\rightarrow}} (\mathcal{K}, \mathcal{W}_2)$. An analog of the following proposition holds for any Bousfield localization, with "Kan complex" and "quasi-category" replaced by the appropriate fibrant objects for each model structure.

Proposition 5.10. *A map between Kan complexes is a Kan fibration if and only if it is a quasi-fibration and a (weak) homotopy equivalence if and only if it is a (weak) categorical equivalence.*

From the small object argument, we have a fibrant replacement functor R that, for any simplicial set X, yields a monic weak categorical equivalence $X \to RX$, with RX a quasi-category. Using this functor, we can describe the construction of a quasi-category modeling a particular small ordinary category $\mathcal{K}$ with weak equivalences $\mathcal{W}$, promised in the introduction.[2] Given such a category, form the pushout

$$\begin{tikzcd} \coprod_{w \in \mathcal{W}} I \arrow[r, "w"] \arrow[d, hook] & N\mathcal{K} \arrow[d] \\ \coprod_{w \in \mathcal{W}} J \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Because τ_1 is cocontinuous, the pushout X is a simplicial set whose fundamental category $\tau_1 X$ is equivalent to the homotopy category $\mathcal{K}[\mathcal{W}^{-1}]$. Taking a fibrant replacement of X yields a monic weak categorical equivalence $X \to RX$, which τ_1 takes to an equivalence of categories. So RX is a quasi-category whose fundamental category is equivalent to the homotopy category $\mathcal{K}[\mathcal{W}^{-1}]$, as desired.

6. Quillen Equivalences

A *simplicial category* is a category enriched in **sSet**, though for sake of consistency it would be better if this terminology described functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$, which are instead called simplicial objects in **Cat**. Indeed the two notions are related. A simplicial category $\mathcal{C}$ gives rise to a simplicial object $D : \mathbf{\Delta}^{op} \to \mathbf{Cat}$ in which each category D_n has the same objects as $\mathcal{C}$ and all face and degeneracy functors are constant on objects. Conversely, any simplicial object in **Cat** satisfying these properties gives rise to a simplicial category. The category **sCat** of small simplicial categories can be given a model structure with weak equivalences the *Dwyer-Kan equivalences*. These are **sSet**-enriched functors F such that the component maps $\mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$ are weak homotopy equivalences of simplicial sets and such that the induced map $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ on the so-called *component categories* of the simplicial categories $\mathcal{C}$ and $\mathcal{D}$ are equivalences of categories. Similarly, the fibrations are enriched functors such that the component maps are Kan fibrations that satisfy a lifting condition for homotopy equivalences in $\mathcal{D}(Fa, b)_0$.

As mentioned in the introduction, simplicial categories provide one model for homotopy theories, but there are others. Two other models are Segal categories and complete Segal spaces, both of which are simplicial spaces satisfying certain additional properties. A *simplicial space*, also known as a *bisimplicial set*, is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$. Simplicial categories are closely related to simplicial spaces.

[2] Note that this construction has the usual size issues if $\mathcal{K}$ is large. In particular, $\mathcal{K}[\mathcal{W}^{-1}]$ will not in general be locally small and the resulting X and RX will not be simplicial *sets*. However, we can sensibly regard RX as a large quasi-category.

As mentioned above, a simplicial category gives rise to a simplicial object in **Cat**. This construction is functorial and the resulting functor is full and faithful. Post-composing with the nerve functor takes a simplicial object in **Cat** to a simplicial space.

There exist model structures for Segal categories and for complete Segal spaces as subcategories of larger categories in the sense that there is a model structure for quasi-categories on **sSet**. The model structure on **sCat** is Quillen equivalent to the model structure for Segal categories, which is Quillen equivalent to the model structure for complete Segal spaces, though these Quillen equivalences go in opposite directions and so cannot be composed.

More surprisingly, Joyal's model structure for quasi-categories is directly Quillen equivalent to all three model categories. We only describe the Quillen equivalence with **sCat**, which is a simplicially enriched analog of the adjunction $\tau_1 \dashv N$ between **sSet** and **Cat** that also fits in to the paradigm described toward the end of Section 1. We will begin by defining a functor $\mathbb{C}\Delta^{-} : \mathbf{\Delta} \to \mathbf{sCat}$. Given this functor, the simplicial nerve $\mathbb{N} : \mathbf{sCat} \to \mathbf{sSet}$ is given by

$$\mathbb{N}\mathcal{C}_n := \mathbf{sCat}(\mathbb{C}\Delta^{[n]}, \mathcal{C}).$$

Its left adjoint $\mathbb{C} : \mathbf{sSet} \to \mathbf{sCat}$ is, as usual, the left Kan extension of $\mathbb{C}\Delta^{-}$ along the Yoneda embedding. As the notation suggestions, by construction $\mathbb{C}\Delta^n = \mathbb{C}\Delta^{[n]}$.

We have yet to define the simplicial category $\mathbb{C}\Delta^n$. The objects are the elements of the set $[n]$. We define the simplicial sets

$$\begin{aligned} \mathbb{C}\Delta^n(i,j) &:= \emptyset \text{ for all } j < i, \\ \mathbb{C}\Delta^n(i,i) = \mathbb{C}\Delta^n(i,i+1) &:= *, \text{ and} \\ \mathbb{C}\Delta^n(i,i+k+1) &:= I^k \end{aligned}$$

to be the enriched homs. Each of these simplicial sets is the nerve of a partially ordered set. More precisely, the simplicial set $\mathbb{C}\Delta^n(i,j)$ is the nerve of the poset of subsets of

$$P_{i,j} := \{k \in [n] \mid i \leq k \leq j\}$$

that include the endpoints. Composition maps $\mathbb{C}\Delta^n(j,l) \times \mathbb{C}\Delta^n(i,j) \to \mathbb{C}\Delta^n(i,l)$ need only be defined when $i \leq j \leq l$ (otherwise at least one of these hom-objects is empty), in which case they are induced by taking unions of the corresponding subsets. Similarly, a morphism $f : [n] \to [m]$ in $\mathbf{\Delta}$ gives the object function for the corresponding enriched functor $\mathbb{C}\Delta^n \to \mathbb{C}\Delta^m$. The maps of simplicial sets $\mathbb{C}\Delta^n(i,j) \to \mathbb{C}\Delta^m(fi,fj)$ are induced by the map $S \mapsto fS$ of subsets.

For example, $\mathbb{C}\Delta^2$ consists of three objects — 0, 1, and 2 — and non-empty hom-objects: $\mathbb{C}\Delta^2(0,0) = \mathbb{C}\Delta^2(0,1) = \mathbb{C}\Delta^2(1,1) = \mathbb{C}\Delta^2(1,2) = \mathbb{C}\Delta^2(2,2) = *$ and $\mathbb{C}\Delta^2(0,2) = I$. Consequently, for any simplicial category $\mathcal{C}$, $\mathbb{N}\mathcal{C}_2 = \mathbf{sCat}(\mathbb{C}\Delta^2, \mathcal{C})$ consists of the following data: $x, y, z \in \mathcal{C}$; $f \in \mathcal{C}(x,y)_0$, $g \in \mathcal{C}(y,z)_0$, $h \in \mathcal{C}(x,z)_0$; and a 1-simplex in $\mathcal{C}(x,z)_1$ of the form $h \Rightarrow gf$.

Some work is required to show that the adjoint pair of functors

$$\mathbb{C} : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sCat} : \mathbb{N}$$

defined above is a Quillen equivalence. As mentioned above, Lurie presents Joyal's model structure for quasi-categories on **sSet** with this objective in mind. Describing the details of his proof would nearly double the length of this paper, so we refer the reader to [10, §2.2] instead.

A simplicial category $\mathcal{C}$ is *locally Kan* when each hom-object $\mathcal{C}(a,b)$ is a Kan complex. In particular, this is true for the subcategory of fibrant-cofibrant objects of a simplicial model category as a consequence of one of the axioms or any topological category, which can be regarded as a simplicial category by applying the total singular complex functor S to each hom-object. When $\mathcal{C}$ is locally Kan, the simplicial set $\mathbb{N}\mathcal{C}$ is a quasi-category: by adjunction $\mathbb{N}\mathcal{C}$ is a quasi-category iff every simplicially enriched functor $F : \mathbb{C}\Lambda^n_k \to \mathcal{C}$ can be extended along $\mathbb{C}\Lambda^n_k \to \mathbb{C}\Delta^n$. On objects, these simplicial categories agree and the inclusion functor is the identity. Each hom-object also coincides with the lone exception of $\mathbb{C}\Lambda^n_k(0,n) \to \mathbb{C}\Delta^n(0,n)$. The latter simplicial set is the cube I^{n-1} while the former can be identified with the same cube with the interior and one of its faces removed. This inclusion is manifestly anodyne, so the desired extension exists since $\mathcal{C}(F0, Fn)$ is Kan, which proves the claim.

As a consequence of this result, the simplicial nerve functor $\mathbb{N}$ is an important part of a procedure that forms a quasi-category from a model category because the simplicial localization of a model category is a simplicial category that is locally Kan [4]. Simplicial localization preserves higher homotopical information that is lost by the ordinary localization $\mathcal{K} \to \mathrm{Ho}\mathcal{K}$ of a model category. This information is retained when we pass to quasi-categories, which is part of the reason why their study is so fruitful.

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