A SURVEY OF CATEGORICAL CONCEPTS

EMILY RIEHL

Abstract. This survey is intended as a concise introduction to the basic concepts, terminology, and, most importantly, philosophy of category theory by carefully examining their implications in a number of elementary examples. We particularly emphasize the importance of the concept of representability, which is often imperfectly understood at first acquaintance. The categorical material included here is chosen to provide a foundation for discussion of more sophisticated topics, in particular, for enriched category theory.

Atiyah described mathematics as the "science of analogy"; in this vein, the purview of category theory is *mathematical analogy*. Specifically, category theory provides a unifying language that can be deployed to describe phenomena in any mathematical context. Surprisingly given its level of generality, these concepts are neither meaningless and nor in many cases so clearly visible prior to their advent. In part, this is accomplished by a subtle shift in perspective. Rather than characterize mathematical objects directly, the categorical approach emphasizes the morphisms, which give comparisons between objects of the same type. Structures associated to particular objects can frequently be characterized by their *universal properties*, i.e., by the existence of certain canonical morphisms to other objects of a similar form.

A great variety of constructions can be described at this level of generality: products, kernels, and quotients for instance are all *limits* or *colimits* of a particular shape, a characterization that has the advantage of describing the universal property associated to each construction. Tensor products, free objects, and localizations are also uniquely characterized by universal properties in appropriate categories. Important technical differences between particular sorts of mathematical objects can be described by the distinctive properties of their categories: that it has certain limits and colimits, but not others, that certain classes maps are *monomorphisms* or *epimorphisms*. Constructions that take mathematical objects of one sort and produce mathematical objects of another sort are often morphisms between categories, called *functors*. Functors can then be said to *preserve*, or not, various categorical structures. Of particular interest is when these functors describe an *equivalence* of categories, which means that objects of the one sort can be translated back and forth between those of the other without losing any information.

1. Categories

A *category* is an organizational and linguistic tool providing a context in which to describe the interaction between mathematical objects of any description and the morphisms that encode comparisons between them. The mathematical objects themselves are called the *objects* of the category and comparisons between them take

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the form of *morphisms* aka *arrows* aka *maps* with specified domain and codomain. Any pair of arrows such that the domain of the latter equals the codomain of the former can be composed and this composition law is associative. Finally, each object has a specified identity endo-arrow that acts trivially with respect to pre- and postcomposition.

$$\begin{tikzcd}
& y \arrow[dr, "hg"] \arrow[dd, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h(gf) = (hg)f"] \arrow[dr, "gf"'] & & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "1_x"] \arrow[dr, "f"'] & x \arrow[d, "f"] \arrow[dr, "f"] & \\
& y \arrow[r, "1_y"'] & y
\end{tikzcd}$$

Notation. We write $x, y \in \mathcal{C}$ to indicated that x and y are objects of $\mathcal{C}$; a morphism f with domain x and codomain y is depicted as $f\colon x \to y$. We write $\mathcal{C}(x, y)$ for the set of arrows in $\mathcal{C}$ from x to y.

Examples abound: **Set**, with objects sets and arrows functions; $\mathbf{Vect}_k$, with objects vector spaces over a fixed field k and arrows linear transformations; $\mathbf{Mod}_R$, with objects left modules with respect to a fixed ring R and arrows R-linear maps; **Gp**, with objects groups and arrows homomorphisms; **Ab**, the subcategory of abelian groups and homomorphisms; $\mathbf{Ch}_*(\mathbb{Z})$ whose objects are chain complexes of abelian groups and whose morphisms are chain maps; **Top**, with objects topological spaces and arrows continuous functions; $\mathbf{Top}_*$, with objects based spaces and arrows based maps; **Ban**, with objects Banach spaces and arrows either bounded linear maps or linear contractions.

These examples are all *concrete* meaning the objects are sets with structure and the arrows are, among other things, set functions. But this is not always the case. For instance, a group (or, most generally, a monoid) forms a category with a single object: the arrows are the elements of the group, composition is multiplication, and the identity is the unit. Or, for instance, the elements of a poset (or, most generally, a preorder) form the objects of a category which has a unique arrow from one object to another if and only if the domain is less than or equal to the codomain. In this way, a topology on a set X, regarded as a poset $\mathcal{O}(X)$ of open sets ordered by inclusion, forms a category.

A group acting on a set gives rise to its *orbit category*, whose objects are the set elements and whose arrows are labeled by elements of the group that act on the domain to return the codomain. Or there is a category whose objects are closed manifolds and whose arrows are diffeomorphism classes of cobordisms. Or there are categories whose objects are either chain complexes or spaces and whose arrows are (chain) homotopy classes of maps. Or whose objects are points in a fixed topological space and whose arrows are boundary preserving homotopy classes of paths. Or there is a category, commonly denoted $\mathbf{\Delta}$, whose objects are finite non-empty ordinals and whose arrows are order-preserving maps. And so on.

Aside 1.1. In certain examples, the primary role played by the objects of a category is to parameterize composability, as is familiar in the distinction between a *group* (a one object category with all arrows invertible) and a *groupoid* (a many object category with all arrows invertible). In the first case, the single object records the fact that all arrows are composable.[1] But when considering, for instance, homotopy classes of paths in a fixed topological space, only certain compositions are naturally defined; the domains and codomains assigned to arrows specify precisely which.

Consider $x, y \in \mathcal{C}$ and $f \colon x \to y$, $g \colon y \to x$ such that $gf = 1_x$ and $fg = 1_y$. In this case, we say that x and y are *isomorphic* and f and g are *isomorphisms*. An elementary but important point is made by the following exercise.

Exercise 1.2. If x and y are isomorphic, then the sets $\mathcal{C}(x,z)$ and $\mathcal{C}(y,z)$ are isomorphic for any z. More precisely, if $f \colon x \to y$ is an isomorphism, then precomposition with f induces an isomorphism $\mathcal{C}(y,z) \to \mathcal{C}(x,z)$ of hom-sets for any $z \in \mathcal{C}$.

In general, given arrows f, g, m such that $mg = mf$ there is no reason to suppose that $f = g$. If an arrow m is left-cancelable in this way, we say m is a *monomorphism* or that m is *monic*. Similarly, if $fe = ge$ implies $f = g$, then we say e is an *epimorphism* or e is *epic*. Isomorphisms are necessarily both epic and monic, though there are certain categories in which the converse does not hold. For example, a continuous homeomorphism need not admit an inverse and hence might not be an isomorphism in **Top**. It is a useful exercise to determine the monomorphisms and epimorphisms in particular categories.

Categories beget categories. There are a number of ways to obtain new categories from existing ones. Most basically, a *subcategory* of a given category consists of a subcollection of objects, together with their identities, and a subcollection of arrows, closed under domains, codomains, and composition.

Example 1.3. The *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects and arrows, but domains and codomains and direction of composition are reversed. This leads to an important principle of *duality* in category theory: every theorem has a dual theorem, whose statement and proof are obtained by replacing the relevant category by its opposite.

Example 1.4. The *arrow category* $\mathcal{C}^{\mathbf{2}}$ has arrows in $\mathcal{C}$ as objects; a morphism $f \to g$ is a *commutative square*

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

The above *diagram* indicates that u and v are arrows whose domains and codomains agree with those of f and g in the way suggested by the figure (the "$\cdot$"s serving as placeholders for generic objects of $\mathcal{C}$ that may or may not coincide) such that $gu = vf$. More generally, a directed graph whose edges and vertices are labelled by arrows and objects of a category $\mathcal{C}$ is said to *commute* if the composites of the arrows along any paths between two fixed vertices agree in $\mathcal{C}$.

Example 1.5. The *product* $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$ is the category whose objects are pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$. A morphism $(f, g) \colon (c, d) \to (c', d')$ consists of a morphism $f \colon c \to c'$ of $\mathcal{C}$ and a morphism $g \colon d \to d'$ of $\mathcal{D}$. Composition

[1] Interestingly, the higher-dimensional analogs of this construction — to be precise, a 2-category with one object, one arrow, and many 2-dimensional automorphisms of this arrow — forces commutativity by the Eckmann-Hilton argument; the 2-dimensional morphisms form an *abelian* group.

and identities are defined componentwise. It follows that for any pair of arrows f and g, the diagram

$$\begin{tikzcd} (c,d) \arrow[r, "(f,1_d)"] \arrow[d, "(1_c,g)"'] \arrow[dr, "(f,g)"] & (c',d) \arrow[d, "(1_{c'},g)"] \\ (c,d') \arrow[r, "(f,1_{d'})"'] & (c',d') \end{tikzcd}$$

commutes.

Example 1.6. Any category $\mathcal{C}$ has a *skeleton* $\mathbf{sk}\mathcal{C}$ defined uniquely up to isomorphism. The skeleton of $\mathcal{C}$ has exactly one object from each isomorphism class of $\mathcal{C}$. Hom-sets are well-defined by Exercise 1.2; composition and identities are inherited from $\mathcal{C}$.

Example 1.7. Fixing an object $c \in \mathcal{C}$, the *slice category* or *comma category* $c/\mathcal{C}$ has arrows $c \to c' \in \mathcal{C}$ as objects and morphisms commutative triangles

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ c' \arrow[rr] & & c'' \end{tikzcd}$$

under c. For instance, if $\mathcal{C} = \mathbf{Top}$ (or $\mathbf{Set}$) and c is the one-object space (or set), this slice category is the category $\mathbf{Top}_*$ of based spaces (pointed sets) and basepoint-preserving maps. Of equal importance is the *dual* notion: slicing over an object. We write $\mathcal{C}/c$ for $(c/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Slice categories are the starting point to define categories of vector bundles, sheaves, and a variety of more exotic algebro-geometric structures.

It is obvious that the category $\mathcal{C}/c$ should be isomorphic to the subcategory of the arrow category $\mathcal{C}^{\mathbf{2}}$ containing only those objects whose codomain is c and those morphisms whose codomain component is the identity at c. Indeed, this identification can be realized as an isomorphism *in* a particular category, which we shall now introduce.

2. Functoriality and naturality

A particularly important example is the category $\mathbf{Cat}$ whose objects are categories, which, to avoid set-theoretical paradoxes, we take to be *small*, meaning with only a set's worth of objects and arrows. This excludes all the concrete categories mentioned above. We have yet to describe the appropriate notion of arrow comparing two categories. Experience teaches us that the appropriate comparisons between mathematical objects are those that preserve their structures; in this case, the objects, arrows, domains, codomains, compositions, and identities of the category. A *functor* does precisely this: given categories $\mathcal{C}$ and $\mathcal{D}$ a functor $F\colon \mathcal{C} \to \mathcal{D}$ assigns an object of $\mathcal{D}$ to each object of $\mathcal{C}$ and an arrow of $\mathcal{D}$ to each arrow of $\mathcal{C}$ in such a way that domains, codomains, identities, and composites are preserved. One easily checks that functors can themselves be composed, and in this way $\mathbf{Cat}$ becomes a category with functors as its arrows. Two categories are *isomorphic* if they are isomorphic as objects of $\mathbf{Cat}$. We shall see below that this is not a very useful notion.

In common parlance, the adjective *functorial* means that a construction on objects can be extended to a construction on arrows that preserves composition and identies (preservation of domains and codomains being regarded as obvious). For instance, one launching point of algebraic topology is the fundamental group functor, which associates to each based topological space X the abelian group $\pi_1 X$ of basepoint preserving homotopy classes of loops in X at the base point. A continuous map $f\colon X \to Y$ of based spaces induces a group homomorphism $\pi_1 f\colon \pi_1 X \to \pi_1 Y$ and this construction is functorial: $\pi_1 g \cdot \pi_1 f = \pi_1(gf)$ and $\pi_1(1_X) = 1_{\pi_1 X}$. This last equations says that the identity function induces the identity (not to be confused with the trivial) homomorphism. In other words, the fundamental group defines a functor $\pi_1\colon \mathbf{Top}_* \to \mathbf{Gp}$ from based topological spaces to groups.

The functoriality axioms have real mathematical power, enabling for instance a slick proof of the Brouwer fixed point theorem.

Theorem 2.1 (Brouwer fixed point theorem)**.** *Any continuous endomorphism of the closed unit disk has a fixed point.*

Proof. Give the disk D^2 and its boundary S^1 the same basepoint, so that the inclusion $i\colon S^1 \to D^2$ is a map in $\mathbf{Top}_*$. Suppose an (unbased) continuous function $f\colon D^2 \to D^2$ has no fixed point and define a retraction $r\colon D^2 \to S^1$ by mapping $x \in D^2$ to the intersection of the ray from $f(x)$ to x with S^1. This retraction fixes the boundary, so r is a map of based spaces such that $r \circ i = 1_{S^1}$.

A well-known calculation shows that $\pi_1 D^2 = 0$, $\pi_1 S^1 = \mathbb{Z}$; the functor π_1 defines group homomorphisms

$$\pi_1 S^1 \xrightarrow{\pi_1 i} \pi_1 D^2 \xrightarrow{\pi_1 r} \pi_1 S^1 \ .$$

By the first fucntoriality axiom, $\pi_1 r \cdot \pi_1 i = \pi_1(ri) = \pi_1(1_{S^1})$; by the second, $\pi_1(1_{S^1}) = 1_{\mathbb{Z}}$. But the identity homomorphism cannot factor through the trivial group 0. $\square$

Many familiar constructions organize themselves as functors. For instance, there are functors

$$\mathbf{Top} \xrightarrow{S} \mathbf{sSet} \xrightarrow{F_*} \mathbf{sAb} \xrightarrow{\sum(-1)^i} \mathbf{Ch}_*(\mathbb{Z}) \xrightarrow{H_n} \mathbf{Ab}$$

that map a space to its total singular complex; a simplicial set to its free simplicial abelian group; a simplicial abelian group to a chain complex under the Dold-Kan correspondence; and a chain complex to its nth homology group. The composite functor is the nth singular homology of a space, as usually defined.

Here are some important elementary examples.

Example 2.2. For any category $\mathcal{C}$, there are functors $\mathbf{sk}\mathcal{C} \to \mathcal{C}$ and $\mathcal{C} \to \mathbf{sk}\mathcal{C}$ between the category and its skeleton (Example 2.2); the first is the obvious inclusion. The second maps an object to its isomorphism class. A choice of a representative for each class and an isomorphism between every other object and the representative is required define the functor on arrows.

Example 2.3. The concrete categories mentioned above are equipped with *forgetful functors* to $\mathbf{Set}$, commonly denoted U for “underlying.” More generally, there are forgetful functors $U\colon \mathbf{Vect}_k \to \mathbf{Ab}$ that forget the scalar multiplication but not addition.

Example 2.4. Consider the category whose objects are finite dimensional real vector spaces equipped with their standard basis, and whose morphisms $f\colon \mathbb{R}^n \to \mathbb{R}^m$ are m-dimensional vector-valued continuously differentiable functions of n variables. Consider another category with the same objects but whose maps $\mathbb{R}^n \to \mathbb{R}^m$ are $m \times n$ matrices of continuous real-valued functions of n-variables. There is an identity-on-objects functor from the first category to the second that associates to f the $m \times n$ matrix of its partial derivatives. Here, functoriality of this construction encodes the chain rule for vector-valued functions.

Example 2.5. There is a functor $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ that maps a set to its powerset and a function $f\colon A \to B$ to the function $f_*\colon \mathcal{P}A \to \mathcal{P}B$ that takes a subset of A to its image in B.

There is another natural powerset functor that maps $f\colon A \to B$ to the function $f^*\colon \mathcal{P}B \to \mathcal{P}A$ that maps a subset of B to its preimage in A. Such functors are called *contravariant*: a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is an ordinary, aka *covariant*, functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. A functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is equivalently a functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$, but we prefer the former representation so that when we draw arrows in the target they point in the most natural direction.

For example, a presheaf over a space X is exactly a contravariant functor on $\mathcal{O}(X)$ valued in $\mathbf{Set}$, $\mathbf{Ab}$, or any appropriate category. The term *presheaf* is used more generally in category theory to mean any contravariant $\mathbf{Set}$-valued functor on a small category. For instance, a *simplicial set* is just a presheaf on the category $\mathbf{\Delta}$ defined above.

Example 2.6. For a fixed group G, a functor from its one-object category G to $\mathbf{Vect}_k$ is exactly a representation of G; a functor to $\mathbf{Set}$ is exactly a set with a (left) group action. A contravariant functor from G to $\mathbf{Set}$ is a set with a right G-action.

Natural transformations. More subtly, there is an important notion of arrow between two functors with common domain and codomain. This is perhaps the first real insight of category theory, and indeed it was the motivation for the subjects' foundational paper. We have seen that mathematical constructions frequently assemble into functors; parallel functors $F, G\colon \mathcal{C} \to \mathcal{D}$ could conceivably be compared. Sometimes these comparisons are ad-hoc, but in other instances, they are colloquially described as *natural*. These latter comparisons then typically assemble into a *natural transformation* between the functors F and functor G.

Recalling that "comparisons" in category theory take the form of arrows, the data of a natural transformation $\alpha\colon F \Rightarrow G$ consists of a collection of arrows $\alpha_c\colon Fc \to Gc$ in $\mathcal{D}$ for each object c of $\mathcal{C}$. "Naturality" means that these comparisons commute with those arising from the arrows of $\mathcal{C}$. That is, for each arrow $h\colon c \to c'$ of $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd} Fc \arrow[r, "Fh"] \arrow[d, "\alpha_c"'] & Fc' \arrow[d, "\alpha_{c'}"] \\ Gc \arrow[r, "Gh"'] & Gc' \end{tikzcd} \qquad \text{i.e.,} \quad \alpha'_c \cdot Fh = Gh \cdot \alpha_c. \tag{2.7}$$

In this way, functors $\mathcal{C} \to \mathcal{D}$, with $\mathcal{C}$ and $\mathcal{D}$ fixed, and natural transformations assemble into a category $\mathcal{D}^{\mathcal{C}}$, the essential point being that natural transformations compose "vertically" in a strictly associative manner. For example, when $\mathcal{C} = \mathbf{1}$, the category with a single object and only its identity arrow, $\mathcal{D}^{\mathbf{1}}$ is isomorphic to $\mathcal{D}$ itself. If $\mathcal{C} = \mathbf{2}$, the category with two objects and one non-identity morphism from the first to the second, then $\mathcal{D}^{\mathbf{2}}$ is the arrow category defined above. The category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ is the category $\mathbf{sSet}$ of simplicial sets.

Aside 2.8. In fact, natural transformations admit a richer compositional structure, precisely described by the statement that $\mathbf{Cat}$ itself forms a 2-*category*, meaning a $\mathbf{Cat}$-enriched category, with 0-, 1-, and 2-dimensional data. This is the reason for the notation $\alpha\colon F \Rightarrow G$ for natural transformations.

Example 2.9. Writing G for the one-object category associated to a group G, the functor categories $\mathbf{Set}^G$ and $\mathbf{Vect}_k^G$ are the categories of G-sets and G-representations with G-equivariant maps. Functor categories have a number of nice categorical properties, which are useful in these particular examples.

Example 2.10. Regarding groups G and H as 1-object categories, a functor $G \to H$ is simply a group homomorphism $G \to H$. Given two such $\phi, \psi\colon G \to H$, a natural transformation $\phi \Rightarrow \psi$ is determined by a single arrow (element) $h \in H$, necessarily an isomorphism such that for all $g \in G$, $h \cdot \phi(g) \cdot h^{-1} = \psi(g)$. In other words, ϕ and ψ are naturally isomorphic if and only if they are conjugate, in which case the natural isomorphism is given by the element of H that conjugates ϕ into ψ.

Example 2.11. Two basic homotopy invariants of spaces are given by the fundamental group and reduced homology, functors $\pi_1, \tilde{H}_1\colon \mathbf{Top}_* \to \mathbf{Gp}$. There is a canonical natural transformation $\pi_1 \to \tilde{H}_1$ that might be called "abelianization"; for each space X, the map $\pi_1 X \to \tilde{H}_1 X$ is the canonical quotient map of a group by its commutator subgroup, which vanishes in first homology.

Example 2.12. The canonical inclusion of a vector space into its double dual is a natural transformation ι from the identity functor on $\mathbf{Vect}_k$ to the double dual functor

$$\begin{tikzcd} v \arrow[d, mapsto] & & V \arrow[r, "T"] \arrow[d, "\iota_V"'] & W \arrow[d, "\iota_W"] \\ \mathrm{ev}_v\colon f \mapsto f(v) & & V^{**} \arrow[r, "T^{**}"] & W^{**} \end{tikzcd}$$

No similar comparison exists for single duals for two reasons. A minor one has to do with variance—the identity functor is covariant while the functor $(-)^*$ is contravariant.[2] The second failure is the main point, an essential failure of naturality. The isomorphisms $V \cong V^*$ that exist when V is finite dimensional require the choice of a basis, which will be preserved by essentially no linear maps, indeed by no non-identity linear endomorphism.

Another familiar isomorphism that is not natural arises in the classification of finitely generated abelian groups, objects of a category $\mathbf{Ab_{fg}}$. Let TA denote the torsion subgroup of an abelian group A.

Proposition 2.13. *Every finitely generated abelian group decomposes as a direct sum $A \cong TA \oplus (A/TA)$ but these isomorphisms are not natural in $A \in \mathbf{Ab_{fg}}$.*

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[2] A more flexible notion of *extranatural transformation* can accommodate functors with conflicting variance [**?**, IX.4].

Proof. The result follows from the claim that every natural endomorphism α of the identity functor on $\mathbf{Ab_{fg}}$ is multiplication by some $n \in \mathbb{Z}$. Clearly the component of α at $\mathbb{Z}$ has this description for some n. But note that homomorphisms $\mathbb{Z} \xrightarrow{a} A$ correspond bijectively to elements $a \in A$ by choosing a to be the image of $1 \in \mathbb{Z}$. Thus, commutativity of

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\alpha_{\mathbb{Z}} = n \cdot -"] \arrow[d, "a"'] & \mathbb{Z} \arrow[d, "a"] \\ A \arrow[r, "\alpha_A"'] & A \end{tikzcd} \tag{2.14}$$

forces us to define $\alpha_A(a) = n \cdot a$.

Now a natural isomorphism $A \cong TA \oplus (A/TA)$ would induce a natural transformation $A/TA \to A$ because $\oplus$ is the coproduct in $\mathbf{Ab_{fg}}$; see §4 below. Precomposing with the quotient map $A \to A/TA$, which is natural (exercise), we would obtain a natural endomorphism of the identity functor, which by the previous claim must be multiplication by some $n \in \mathbb{Z}$. Now consider $A = \mathbb{Z}/2n\mathbb{Z}$. This group is torsion, so any map, such as $\alpha_{\mathbb{Z}/2n\mathbb{Z}}$, which factors through the quotient by its torsion subgroup is zero. But $n \neq 0 \in \mathbb{Z}/2n\mathbb{Z}$, a contradiction. □

Natural transformations bear close analogy with the notion of homotopy from topology with one important difference: natural transformations are not generally invertible.[3] As above, let $\mathbf{1}$ denote the category with a single object and identity arrow and let $\mathbf{2}$ denote the category with two objects $0, 1 \in \mathbf{2}$ and a single non-identity arrow $0 \to 1$. There are two evident functors $i_0, i_1 \colon \mathbf{1} \to \mathbf{2}$. A natural transformation $\alpha \colon F \Rightarrow G$ between functors $F, G \colon \mathcal{C} \to \mathcal{D}$ is precisely a functor $H \colon \mathcal{C} \times \mathbf{2} \to \mathcal{D}$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathcal{C} \times \mathbf{2} \arrow[d, "H"] & \mathcal{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathcal{D} & \end{tikzcd} \tag{2.15}$$

commutes. The analogy with the notion of homotopy is evident.

For example, if $\mathcal{C} = \mathbf{2}$, each functor $F, G \colon \mathbf{2} \to \mathcal{D}$ picks out an arrow of $\mathcal{D}$, which we'll denote by F and G. The directed graph underlying the category $\mathbf{2} \times \mathbf{2}$ looks like

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

together with four identity arrows not depicted here; the diagonal serves as the common composite of the edges of the square. The functor H necessarily maps the top and bottom arrows of (2.15) to F and G, respectively. The vertical arrows define α_0 and α_1 and the diagonal arrow witnesses that the square analogous to (2.7) commutes.

If, in the above discussion, the category $\mathbf{2}$ were replaced by the category $\mathbb{I}$ with two objects and a single arrow in each hom-set, necessarily an isomorphism, then

[3]Note however, that if α is a natural transformation in which each of the constituent arrows is an isomorphism, then the pointwise inverses assemble into a natural transformation (exercise).

"homotopies" with this interval would be precisely natural *isomorphisms*. One use of this analogy is it that leads to the correct notion of equivalence for categories mirroring homotopy equivalence of spaces.

An *equivalence of categories* consists of functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G$ together with natural isomorphisms $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. A useful theorem characterizes those functors forming part of an equivalence of categories.

Theorem 2.16. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ defines an equivalence of categories if and only if F is*

- full: *for each $x, y \in \mathcal{C}$, the map of hom-sets $\mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy)$ is surjective;*
- faithful: *for each $x, y \in \mathcal{C}$, the map of hom-sets $\mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy)$ is injective;*
- *and* essentially surjective: *for every $d \in \mathcal{D}$ there is some $c \in \mathcal{C}$ such that d is isomorphic to Fc.*

For example, any possible functor $\mathbf{1} \to \mathbb{I}$ is an equivalence of categories, despite not being surjective on objects or arrows. Similarly, the unique functor $\mathbb{I} \to \mathbf{1}$ is an equivalence of categories despite failing to be injective on both objects and arrows. One might express the equivalence of $\mathbb{I}$ to $\mathbf{1}$ by saying this category is *contractible*.

It is strongly recommended that the reader work out the details necessary to complete the following sketch proof.

Proof/Exercise. The harder direction is the converse. Here are a few hints: using essential surjectivity (and the axiom of choice) choose, for each $d \in \mathcal{D}$ an object of $\mathcal{C}$, cleverly denoted Gd, and an isomorphism $\epsilon_d\colon FGd \cong d$. The first two properties of F can be used to define the action of G on morphisms in such a way that ϵ is natural. The essential point is that the map G so-defined is necessarily *functorial*; one should work out why this is the case and also how to define η. $\square$

Example/Exercise 2.17. Give a precise definition of the functors of Example 2.2 and show that they define an equivalence between $\mathcal{C}$ and its skeleton.

Example 2.18. Temporarily restrict the category $\mathbf{Top}_*$ to the full subcategory of path connected based spaces. There is a natural transformations between the functors

$$\pi_1\colon \mathbf{Top}_* \xrightarrow{\pi_1} \mathbf{Gp} \hookrightarrow \mathbf{Cat} \qquad \text{and} \qquad \Pi_1\colon \mathbf{Top}_* \xrightarrow{U} \mathbf{Top} \xrightarrow{\Pi_1} \mathbf{Gpd} \hookrightarrow \mathbf{Cat}.$$

The first regards the fundamental group as a one object category; the second forgets the basepoint, computes the fundamental groupoid, and again regards this as a category. The inclusion of the fundamental group into the fundamental groupoid defines a natural transformation $\pi_1 \Rightarrow \Pi_1$ such that each component $\pi_1 X \to \pi_1 X$, itself a functor, is furthermore an equivalence of categories. Consequently, for each based space X, there exists an inverse equivalence $\pi_1 X \to \pi_1 X$, but its definition requires the choice of paths connecting each point to the basepoint, hence is not a natural transformation.

By Theorem 2.16, a full and faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence onto its *essential image*, the full subcategory of objects isomorphic to Fc for some $c \in \mathcal{C}$. Such functors have a useful property: if F is full and faithful and Fc and Fc' are isomorphic in $\mathcal{D}$, then c and c' are isomorphic in $\mathcal{C}$. We will introduce what is easily the most important full and faithful functors in category theory toward the end of the next section: the covariant and contravariant Yoneda embeddings.

3. Representability and the Yoneda lemma

Implicit in the proof of Proposition 2.13 are two deep ideas: that of *representable functors* and an unreasonably useful result, called the Yoneda lemma, that classifies natural transformations with representable domain.

Representables. Tacitly, we have been assuming that all our categories are *locally small*, which is to say that the collection of arrows between fixed objects forms a *set* and not a proper class. Locally small categories are precisely categories *enriched in the category* **Set**; we introduce this terminology to suggest that other options might be available. We want this to be our basic notion of category and won't always mention this implicit assumption in the future.

Given a locally small category $\mathcal{C}$ and an object $x \in \mathcal{C}$, there are two *representable functors* taking values in **Set**, one covariant and one contravariant. The notation

$$\begin{aligned} \mathcal{C}(x,-)\colon \mathcal{C} &\to \mathbf{Set} \\ y &\mapsto \mathcal{C}(x,y) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{C}(-,x)\colon \mathcal{C}^{\mathrm{op}} &\to \mathbf{Set} \\ z &\mapsto \mathcal{C}(z,x) \end{aligned}$$

suggests the definition: the covariant representable at x maps an object y to the set of arrows from x to y; the contravariant representable is defined dually. A morphism $f\colon y \to z$ acts on the left of the set $\mathcal{C}(x,y)$ and the right of the set $\mathcal{C}(z,x)$ by post- and precomposition respectively. Note that the functors $\mathcal{C}(-,c)$ and $\mathcal{C}^{\mathrm{op}}(c,-)$ are canonically naturally isomorphic.

More generally, a functor $F\colon \mathcal{C} \to \mathbf{Set}$ is *representable* if it is naturally isomorphic (i.e., isomorphic in the category $\mathbf{Set}^{\mathcal{C}}$) to $\mathcal{C}(c,-)$ for some $\mathcal{C}$. In this case, we say that F is *represented* by the object c, though, as we shall emphasize below, the data of a representation also includes the choice of isomorphism. In general, this definition only makes sense for **Set**-valued functors, though if the category $\mathcal{C}$ is *enriched over* some other category $\mathcal{V}$, meaning loosely that the hom-sets of $\mathcal{C}$ have the structure of objects in the category $\mathcal{V}$, then there is an analogous notion of representable functors $F\colon \mathcal{C} \to \mathcal{V}$.

Example 3.1. The forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Set}$ is represented by the integers: this is to say, the set of elements of the abelian group A is isomorphic to the set of group homomorphisms $\mathbb{Z} \to A$, and this isomorphism is natural in A.

Because the set of group homomorphisms $A \to B$ is itself an abelian group, with addition defined pointwise in B, it makes sense to assert that the identity functor $\mathrm{id}\colon \mathbf{Ab} \to \mathbf{Ab}$ is also represented by $\mathbb{Z}$. The proof is an *enrichment* of the previous assertion: the abelian group of homomorphisms $\mathbb{Z} \to A$ is naturally isomorphic to the group A.

Example 3.2. The forgetful functors $U\colon \mathbf{Mod}_R \to \mathbf{Set}$ and $U\colon \mathbf{Vect}_k \to \mathbf{Set}$ are represented by R and k respectively. As above, the set-level isomorphisms respect various enrichments of these categories: these same objects represent the forgetful funtors $\mathbf{Mod}_R \to \mathbf{Ab}$ and $\mathbf{Vect}_k \to \mathbf{Ab}$ or the identity functors on $\mathbf{Mod}_R$ and $\mathbf{Vect}_k$.

Example 3.3. The functor $\mathbf{Cat} \to \mathbf{Set}$ that maps a small category to its underlying set of objects is represented by the category $\mathbf{1}$: i.e., functors $\mathbf{1} \to \mathcal{C}$ correspond bijectively to objects of $\mathcal{C}$. Similarly, the functor $\mathbf{Cat} \to \mathbf{Set}$ that maps a small category to its underlying set of arrows is represented by the category $\mathbf{2}$.

Example 3.4. The identity functor on $\mathbf{Set}$ is represented by the singleton set. The forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ is represented by the one-point space. The forgetful functor $U\colon \mathbf{Top}_* \to \mathbf{Set}$ is represented by the space S^0 with two points and the discrete topology.

Exercise 3.5. What contravariant functor does the Sierpinkski space represent?

Example 3.6. The contravariant power-set functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is represented by the set with two elements $\{0,1\}$. In other words, set functions $A \to \{0,1\}$ correspond to subsets of A, say the subset of elements that map to 1, and this correspondence is preserved by precomposition by an arbitrary function $f\colon B \to A$.

We'll see in the next section that the covariant power-set functor $\mathbf{Set} \to \mathbf{Set}$ is not representable.

Example 3.7. Consider the category of CW-complexes and homotopy classes of maps. The contravariant functor assigning a space (the underlying set of) its nth cohomology group with coefficients in a fixed abelian group A is represented by a CW-complex, commonly denoted $K(A,n)$. We'll prove shortly that representing objects are unique up to isomorphism in the relevant category. Here this means that $K(A,n)$ is unique up to homotopy equivalence.

More generally, every generalized cohomology theory is represented by some spectrum in the stable homotopy category.

The Yoneda lemma. Let us examine the natural isomorphism of Example 3.1 more closely, taking care to write UA when we mean the underlying set of an abelian group A. The natural transformation $\mathbf{Ab}(\mathbb{Z},-) \Rightarrow U$ is determined by the element $1 \in U\mathbb{Z}$ in the following manner. A generic homomorphism $f\colon \mathbb{Z} \to A$ is mapped to the element $Uf(1) \in UA$. Note that the element $1 \in U\mathbb{Z}$ is associated to the identity map at $\mathbb{Z}$.

No part of this discussion required that the map $\mathbf{Ab}(\mathbb{Z},-) \Rightarrow U$ was an isomorphism. In the general case, given $F\colon \mathcal{C} \to \mathbf{Set}$ and $c \in \mathcal{C}$, one could imagine trying to build a natural transformation $\mathcal{C}(c,-) \Rightarrow F$ by following a similar prescription: choose $x \in Fc$ and map $f\colon c \to d$ to the element $Ff(x) \in Fd$. This is the essential idea behind the following result.

Lemma 3.8 (Yoneda lemma)**.** *Let $c \in \mathcal{C}$ and $F\colon \mathcal{C} \to \mathbf{Set}$. Natural transformations $\mathcal{C}(c,-) \Rightarrow F$ correspond bijectively to elements of the set Fc, and this correspondence is natural in both F and c.*

Proof. Given $\alpha\colon \mathcal{C}(c,-) \Rightarrow F$ we associate the element $\alpha_c(1_c) \in Fc$. Conversely, given $x \in Fc$, we claim there is a unique natural transformation that sends the identity at c to x: naturality of α at $f\colon c \to d$, i.e., commutativity of the square

$$\begin{tikzcd} \mathcal{C}(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\ \mathcal{C}(c,d) \arrow[r, "\alpha_d"'] & Fd \end{tikzcd} \tag{3.9}$$

forces us to define $\alpha_d(f) = Ff(x)$. These assignments are inverses.

Naturality in F asserts that, given $\beta\colon F \Rightarrow F'$, the element of $F'c$ associated to $\beta\alpha\colon \mathcal{C}(c,-) \Rightarrow F'$ is the image of the element of Fc associated to α under β_c. Both elements are $\beta_c\alpha_c(1_c)$ by definition.

Naturality in c asserts that, given $h\colon c \to c'$, the element associated to the composite

$$\mathcal{C}(c',-) \xRightarrow{h^*} \mathcal{C}(c,-) \xRightarrow{\alpha} F$$

is the image of the element associated to α under $Fh\colon Fc \to Fc'$. By definition, the former element is

$$\alpha_c(h^*(1_{c'})) = \alpha'_c(h) = Fh(\alpha_c(1_c))$$

as desired. □

In light of Lemma 3.8, we say a *representation* for a representable functor $F\colon \mathcal{C} \to$ **Set** consists of an object $c \in \mathcal{C}$ together with an element $x \in Fc$ determining a natural isomorphism $\mathcal{C}(c,-) \cong F$.

Example 3.10. Consider a group G as a one-object category. The unique represented functor in the category of G-sets $\mathbf{Set}^G$ is the Cayley representation of G: the set G acted on by left-multiplication. The Yoneda lemma says that this is the free G-set with one generator, i.e., G-equivariant maps $G \to X$ correspond to elements of X.

By the Yoneda lemma again, a G-set X is representable if and only if it is a G-*torsor*, i.e., if and only if G acts freely and transitively on X. Necessarily the sets X and G are isomorphic. A choice of representation amounts to a choice of a point in X that correspond to the identity of G, so we think of a G-torsor as a copy of G where we have "forgotten the identity."

Example 3.11. Recall, simplicial sets are **Set**-valued contravariant functors on the category $\mathbf{\Delta}$ of non-empty finite ordinals and order-preserving maps. The represented functors, variously denoted Δ^n or $\Delta[n]$, play an important role in describe constructions in this category, precisely because maps $\Delta^n \to X$ correspond bijectively to n-simplices in the simplicial set X, by Yoneda's lemma.

A more-sophisticated consequence of Yoneda's lemma is the following: a colimit-preserving functor (see §4) whose domain is the category of simplicial sets, is uniquely determined by its values on the full subcategory of represented functors. For instance, the *geometric realization* functor from simplicial sets to topological spaces is characterized by the condition that the geometric realization of Δ^n is the standard topological n-simplex.

Example 3.12. Fix a topological group G and let **Man** be the category of paracompact manifolds and homotopy classes of smooth maps. There is a functor $P\colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ that sends paracompact manifolds B to the set of isomorphism classes of principle G-bundles over B. This functor is contravariant: given $f\colon B' \to B$, the associated set-function forms principle G-bundles over B' by pulling back each G-bundle over B.

The functor P is represented by an object $BG \in \mathbf{Man}$ called the *classifying space* of the group G and an element of $P(BG)$, i.e., a principle G-bundle commonly denoted $EG \to BG$. The Yoneda lemma says that any G-bundle $E \to B$ is obtained up to isomorphism as the image of $EG \to BG$ under the action of the functor P on some map $B \to BG$, i.e., $E \to B$ is obtained by pulling back $EG \to BG$ along

$B \to BG$. In this way, homotopy classes of maps $B \to BG$ classifying principle G-bundles over B up to isomorphism.

The Yoneda embedding. A morphism $f\colon c \to d$ in $\mathcal{C}$ gives rise to natural transformations between both the covariant and contravariant functors represented by these objects. The natural transformation $f_*\colon \mathcal{C}(-,c) \Rightarrow \mathcal{C}(-,d)$ is defined by postcomposing with the arrow f; dually, $f^*\colon \mathcal{C}(d,-) \Rightarrow \mathcal{C}(c,-)$ is defined by precomposing with f. We shall see shortly that all natural transformations between represented functors arise this way.

Exercise 3.13. Verify that $f_*\colon \mathcal{C}(-,c) \Rightarrow \mathcal{C}(-,d)$ and $f^*\colon \mathcal{C}(d,-) \Rightarrow \mathcal{C}(c,-)$ are natural transformations and use these to define functors

$$\mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \quad \text{and} \quad \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}. \tag{3.14}$$

These functors are referred to as the *covariant* and *contravariant Yoneda embeddings*, respectively. The adjective "embedding" is usually reserved for functors that are injective on objects and also full and faithful. These latter characteristics are a consequence of the Yoneda lemma.

Corollary 3.15. *The Yoneda embeddings (3.14) are full and faithful.*

Exercise 3.16. Prove this.

Many applications of the Yoneda lemma are in fact consequences of this corollary; indeed sometimes this result is called the Yoneda lemma. One surprisingly useful consequence is that to prove that two objects in $\mathcal{C}$ are isomorphic, it suffices to prove that their represented functors are naturally isomorphic. For instance, this is ho we concluded above that representing objects are unique up to isomorphism.

Remark 3.17. The key step in the proof of Proposition 2.13 involved Corollary 3.15: because the identity functor on abelian groups is represented by $\mathbb{Z}$, any natural endomorphism of that functor arise from an endomorphism of $\mathbb{Z}$, and these are classified by the integers. Compare (2.14) with (3.9).

For another application, consider a group G as a one-object category. The Yoneda embedding defines a functor

$$G^{\mathrm{op}} \to \mathbf{Set}^{G}.$$

Its image is the unique represented G-set; in Example 3.10 we saw this is the set G with left multiplication. Corollary 3.15 implies that the only endomorphisms of this set that commute with the action of G are given by right multiplication by G.

We'll see further applications of the Yoneda lemma and its corollary in the next sections.

4. Limits, colimits, and universal properties

One of the main themes of category theory is that mathematical structures that are often defined set-theoretically can be completely characterized by a description of the morphisms to or from the structure qua object of a particular category. Such a characterization, called a *universal property* of the object so-described, can immediately be generalized from **Set** to any other category. More precisely, the generalization from a **Set**-based universal property to a generic one, makes use of the idea of representability, introduced in the previous section. But first let's examine a concrete example.

Definition via universal property. For instance, the cartesian product of two sets is a set equipped with two projection arrows with the following property: a function to the cartesian product is uniquely determined by the functions obtained by composing with these projections, and furthermore any similar pair of maps factors uniquely through the cartesian product in this way. Similarly:

Definition 4.1. The *product* of objects $A, B \in \mathcal{C}$ consists of an object P together with arrows $p_1 \colon P \to A$, $p_2 \colon P \to B$ such that the triple (P, p_1, p_2) is *universal* in then following sense: given any $Z \in \mathcal{C}$ and arrows $f \colon Z \to A$, $g \colon Z \to B$, f and g factor uniquely through p_1 and p_2 along a common arrow.

(4.2)
$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "g"] & \\ A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \end{tikzcd}$$

The utility of this abstract definition is that it helps identify the right notion in a generic category. For instance, interpreting this definition in **Top** endows the product of topological spaces with the product topology. To define the product of spaces X and Y, we must endow the set $X \times Y$ with a suitable topology.[4] In order for the projections to be continuous, the preimage $U \times Y$ of each open $U \subset X$ must be open in $X \times Y$; similarly the sets $X \times V$ must be open for all open $V \subset Y$. Taking intersections, the product of any open sets in X and Y must be open in $X \times Y$.

Now we'll show the topology assigned the product of X and Y can't be any finer. If there were additional open sets in the topology assigned the product $X \times Y$, then this space would not satisfy the required universal property (4.2) when we take Z to be $X \times Y$ with the product topology. Commutativity with the projections requires that the underlying set function $X \times Y \to X \times Y$ is the identity; the identity is only continuous when the topology on the domain is finer than that on the codomain.

More generally, if X_α are spaces indexed by elements in some possibly infinite set, the previous discussion says that we should endow their product $\prod_\alpha X_\alpha$ with the topology generated under finite intersections by the open sets $U_\alpha \times (\prod_{\beta \neq \alpha} X_\beta)$ for each open $U_\alpha \subset X_\alpha$.

Here's a somewhat more sophisticated example:

Definition 4.3. The *tensor product* of $V, W \in \mathbf{Vect}_k$ is a vector space $V \otimes W$ equipped with a bilinear map

$$V \times W \xrightarrow{-\otimes-} V \otimes W$$

that is *universal* in the following sense. Any bilinear map $V \times W \to Z$, factors through $\otimes$ along a unique linear map $V \otimes W \to Z$.

Indeed, the defining universal property of the tensor product gives a recipe for its construction. Supposing the vector space $V \otimes W$ exists, consider its quotient by the vector space spanned by the image of the bilinear map[5] $-\otimes-$. By definition the quotient map $V \otimes W \to V \otimes W / \langle v \otimes w \rangle$ precomposes with $-\otimes-$ to yield the zero

[4] One might quibble that the product of spaces might a priori have a different underlying set than the product of the underlying sets, or different projection maps, but in fact there is no other possibility; see Theorem 5.9 below.

[5] The image of a bilinear map is not itself a sub-vector space, so closing under span is necessary.

bilinear map. But the zero map $V \otimes W \to V \otimes W/\langle v \otimes w \rangle$ also has this property, so by the universal property of $V \otimes W$, these linear maps must agree. Because the quotient map is surjective, this implies that $V \otimes W$ is isomorphic to the span of the vectors $v \otimes w$ for all $v \in V$ and $w \in W$ modulo the bilinearity relations satisfied by $- \otimes -$. This is of course the usual constructive definition.

Exercise 4.4. Show that scalar multiplication $k \times V \to V$ is a bilinear map satisfying the universal property of the tensor product. Hence $k \otimes V \cong V$.

Just for fun, let's give one more definition.

Definition 4.5. Let R be a commutative ring and $S \subset R$ a subset not containing the additive identity. The *localization* of R at S is a ring A equipped with a homomorphism $R \to A$ such that elements of S map to units in A and universal with this property: given another homomorphism $R \to B$ that carries elements of S to units, then there is a unique ring homomorphism $A \to B$ such that

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & A \arrow[d, dashed, "\exists"] \\ & B \end{tikzcd}$$

commutes.

The following lemma should not be a surprise.

Lemma 4.6. *Any object defined by a universal property, when it exists, is unique up to unique isomorphism.*

A slick general proof appears below. But for now, let's examine this statement in a specific case because there is some possibility for confusion. Let $A, B \in \mathbf{Set}$. The sets

$$A \times B = \{(a, b) \mid a \in A, b \in B\} \qquad \text{and} \qquad B \times A = \{(b, a) \mid b \in B, a \in A\}$$

both satisfy the universal property of the product of A and B. Lemma 4.6 asserts they are uniquely isomorphic.

At this point an alert and skeptical reader might object: if A and B are finite, say of cardinality m and n, then there are $(mn)!$ isomorphisms between these sets. But recall the data of a *product* of A and B is not just an object but also a pair of "projection arrows"

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B \qquad\qquad A \xleftarrow{p_2} B \times A \xrightarrow{p_1} B$$

and an isomorphism between the objects qua products must commute with these projections.

Now that the statement is understood, the proof is easy. By the universal property of $B \times A$ in the left diagram and $A \times B$ in the right, there are unique arrows ϕ and ψ such that

$$\begin{tikzcd} & A \times B \arrow[dl, "p_1"'] \arrow[d, dashed, "\phi"] \arrow[dr, "p_2"] & \\ A & B \times A \arrow[l, "p_2"] \arrow[r, "p_1"'] & B \end{tikzcd} \qquad \begin{tikzcd} & B \times A \arrow[dl, "p_2"'] \arrow[d, dashed, "\psi"] \arrow[dr, "p_1"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

commute. But then $\psi\phi$ and the identity at $A \times B$ can both take the place of the dotted arrow in the following commutative diagram

$$\begin{tikzcd} & A \times B \arrow[dl, "p_1"'] \arrow[d, dashed] \arrow[dr, "p_2"] & \\ A & A \times B \arrow[l, "p_2"'] \arrow[r, "p_1"] & B \end{tikzcd}$$

so by the uniqueness statement in the universal property of $A \times B$, $\psi\phi = \mathrm{id}$. Similarly, $\phi\psi = \mathrm{id}$. Hence, ϕ and ψ are the desired isomorphisms.

Note we've made no explicit use of the category of **Set** (after the discussion of the purported "counterexample") nor have we made essential use of the sort of universal property involved. These observations should give some indication of the generality of the argument.

Limits and colimits. Objects defined by universal property can be classified as either *limits* or *colimits* in an appropriate category. These notions are dual. A slogan is that limits are built from other objects by imposing additional coherence conditions. Colimits are formed by gluing objects together. Before giving precise definitions we must introduce some ancillary notions.

A *diagram* in a category $\mathcal{C}$ is simply a functor $F \colon \mathcal{J} \to \mathcal{C}$ where the domain category $\mathcal{J}$ is assumed to be small. A *cone* over a diagram F with *summit* $c \in \mathcal{C}$ is a natural transformation from the constant functor at c to F. In other words, a cone consists of arrows $c \to Fj$ for each $j \in \mathcal{J}$, called the *legs* of the cone, such that each triangle formed by the image of some morphism in $\mathcal{J}$ and the appropriate legs of the cone commutes.

A *limit* is a cone over F that is universal in the sense that any other cone over F factors uniquely through the limit. Concretely, a limit consists of an object in $\mathcal{C}$, often written $\lim F$, together with a cone $\lim F \to Fj$ such that for any other cone, say with summit c, there is a unique morphism $c \to \lim F$ such that each leg of the cone factors along this map through the limit cone.

Dually, a cone under F with summit $c \in \mathcal{C}$ is a natural transformation from F to the constant functor at c; its legs are arrows $Fj \to c$. A *colimit* is a cone under F that is universal in the sense that any other cone factors uniquely through the colimit. This is to say, for any other cone, say with summit c, there is a unique morphism $\operatorname{colim} F \to c$ such that the legs of the cone factor uniquely through this map along the legs $Fj \to \operatorname{colim} F$ of the colimit cone.

Example 4.7. A *product* of several objects, generalizing the notion introduced above, is the limit of a diagram on a *discrete*[6] category $\mathcal{J}$. The dual notion is a *coproduct*. The coproduct of sets A and B is simply their disjoint union. The coproduct of groups G and H is the so-called free product $G * H$, the group whose elements are finite lists of elements in G and H and whose composition is by concatenation. Note that the underlying set of the group $G * H$ is much larger than the coproduct of the underlying sets of G and H.

Example 4.8. Products in $\mathbf{Vect}_k$ are direct products: Recall from Example 3.2 that the vector space k represents the forgetful functor $U \colon \mathbf{Vect}_k \to \mathbf{Set}$. Hence, elements of the direct product $\prod_\alpha V_\alpha$ correspond to linear maps $k \to \prod_\alpha V_\alpha$, which by the universal property of the product correspond to maps $k \to V_\alpha$ for each α.

[6]Containing no non-identity morphisms.

In other words, an element of $\prod_\alpha V_\alpha$ corresponds to a vector in each V_α, subject to no further restrictions. Because each of the canonical projections $\prod_\alpha V_\alpha \to V_\alpha$ must be linear, we see that addition and scalar multiplication in the direct product are component-wise.

Coproducts in $\mathbf{Vect}_k$ are direct sums, which are smaller than direct products if the index set is infinite. The direct product admits canonical injections $V_\alpha \to \prod_\alpha V_\alpha$ that map $v \in V_\alpha$ to the vector with v in the αth component and zero vectors elsewhere.[7] These define a canonical map

$$\bigoplus_\alpha V_\alpha \to \prod_\alpha V_\alpha$$

from the direct sum to the direct product. It is instructive to use the defining universal properties to understand why this map is not generally an isomorphism.

Exercise 4.9. Describe the data and universal property of the limits of diagrams of shape

$$\mathcal{J} = \bullet \rightrightarrows \bullet \qquad \text{or} \qquad \mathcal{J} = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

Limits of the first sort are called *equalizers*, and limits of the second sort are called *pullbacks*. Colimits over the dual categories $\mathcal{J}^{\text{op}}$ are called *coequalizers* and *pushouts*.

Suppose $\mathcal{J}$ is the empty category. There is a unique diagram $\mathcal{J} \to \mathcal{C}$ for any $\mathcal{C}$. A limit consists of an object $1 \in \mathcal{C}$, called a *terminal object*, such that for any $c \in \mathcal{C}$ there is a unique arrow $c \to 1$. Dually, a colimit is an object $0 \in \mathcal{C}$, called an *initial object*, such that for any $c \in \mathcal{C}$ there is a unique arrow $0 \to c$.

For example, the empty set is initial and the singleton set is terminal in **Set** or in **Top**. In $\mathbf{Top}_*$, the singleton space is both initial and terminal; in **Gp** or **Ab** the trivial group is similarly both initial and terminal. In the category of rings with identity, the ring $\mathbb{Z}$ is initial because ring homomorphisms necessarily preserve both the additive and multiplicative identity. The zero ring is terminal. Neither category depicted in Exercise 4.9 above has an initial object; the middle object in the latter category is terminal.

Let $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$ be, respectively, the posets of positive and negative integers, i.e., the categories with these objects and with a unique arrow $i \to j$ if and only if $i \leq j$. These categories are generated by the directed graphs

$$\begin{array}{ll} \mathbb{Z}_{\geq 0} & \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \\ \mathbb{Z}_{\leq 0} & \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \end{array}$$

A limit of a diagram $\mathbb{Z}_{\leq 0} \to \mathcal{C}$ is sometimes called an *inverse limit* and a colimit of a diagram $Z_{\geq 0} \to \mathcal{C}$ is somties called a *direct limit*, but we try to avoid this terminology. For example, the limit of the diagram

$$\cdots \twoheadrightarrow \mathbb{Z}/p^4 \twoheadrightarrow \mathbb{Z}/p^3 \twoheadrightarrow \mathbb{Z}/p^2 \twoheadrightarrow \mathbb{Z}/p$$

in **Ab** is the group $\mathbb{Z}_p$ of p-adic integers. The colimit of

$$\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^3 \hookrightarrow \mathbb{Z}/p^4 \hookrightarrow \cdots$$

is the group $\mathbb{Z}/p^\infty$.

There is a good reason why mathematicians are seldom interested in colimits of diagrams of shape $\mathbb{Z}_{\leq 0}$ or limits of diagrams of shape $\mathbb{Z}_{\geq 0}$. Because the category

[7] Using the universal property of the product, we could define these to be the maps that are determined by the identity map to V_α and the zero linear map $V_\alpha \to V_\beta$ for all $\beta \neq \alpha$.

$\mathbb{Z}_{\leq 0}$ itself has a terminal object 0, the colimit of any diagram of shape $\mathbb{Z}_{\leq 0}$ is necessarily isomorphic to the value of that diagram on this object. Dually, because 0 in an initial object in the category $\mathbb{Z}_{\geq 0}$, the limit of any diagram of shape $\mathbb{Z}_{\geq 0}$ is isomorphic to the value of that diagram on this object. A generalization of these observations yields the theory of *cofinal functors*: functors $\mathcal{I} \to \mathcal{J}$ such that the colimit of any diagram over $\mathcal{J}$ is isomorphic to the colimit of the restriction to a diagram over $\mathcal{I}$.

Example 4.10. Let $\mathbb{R}$ be the category whose objects are real numbers and which has a unique morphism $x \to y$ if and only if $x \leq y$. A functor $\mathbb{Z}_{\leq 0} \to \mathbb{R}$ is a non-increasing sequence of real numbers. This functor has a limit if and only if the sequence is bounded below and has a limit, in the classical "ϵ-δ" sense.

An important theorem says that limits over small categories of any shape can be constructed by first forming two products and then taking an equalizer. In this way, we can prove that a number of familiar categories are *complete*, that is, have all small limits. These include **Set**, **Top**, **Gp**, **Ab**, $\mathbf{Vect}_k$, $\mathbf{Mod}_R$, **Cat**, and the categories $\mathbf{Set}^{\mathcal{C}}$ for any small category $\mathcal{C}$. Each of these examples is also *cocomplete*, having all small colimits. But not all categories have all limits and colimits. For example, the category **Met** of metric spaces has neither binary coproducts nor infinite products.

Representability of limits and colimits. Fixing a diagram $D\colon \mathcal{J} \to \mathcal{C}$, there is a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ that sends an object of c to the set of cones over D with summit c. Similarly, there is a functor $\mathcal{C} \to \mathbf{Set}$ that sends c to the set of cones under D with summit c. Unpacking the definitions given above, a limit for D is exactly a representation for the former functor and a colimit is a representation for the latter. This observation already has some mileage. By the Yoneda lemma, representing objects qua representations are unique up to unique isomorphism. Hence, limits qua limits are unique up to unique isomorphism, extending the concrete argument given above.

Given a **Set**-valued functor F, there is a category called the *category of elements* of F obtained by a procedure known as the Grothendieck construction. A covariant functor $F\colon \mathcal{C} \to \mathbf{Set}$ is representable if and only if its category of elements has an initial object and a contravariant functor is representable if and only if its category of elements has a terminal object. Applying these facts to the functors of cones over/under D described above, we see that any limit is simply a terminal object in the appropriate category and dually that any colimit is simply an initial object in the appropriate category.

A more satisfying description of the "cones over D" functor is possible. For each $j \in \mathcal{J}$, there is a functor $\mathcal{C}(-, Dj)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ that sends an object of c to the set of arrows $c \to Dj$ in $\mathcal{C}$. Maps in $\mathcal{J}$ induce natural transformations between these functors. By Example 5.12 below, this diagram of functors $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has a limit. Unpacking the universal property of the limit reveals that the resulting functor

$$\lim_{j \in \mathcal{J}} \mathcal{C}(-, Dj)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

is precisely the functor that takes c to the set of cones over D introduced above. This is what's meant by the assertion that "limits are defined representably": the limit is an object equipped with an isomorphism

$$\mathcal{C}(c, \lim_{\mathcal{J}} D) \cong \lim_{\mathcal{J}} \mathcal{C}(c, Dj)$$

natural in c. Note that the right-hand limit takes place in **Set**. Dually, the colimit is an object equipped with a natural isomorphism

$$\mathcal{C}(\operatorname{colim}_{\mathcal{J}} D, c) \cong \lim_{\mathcal{J}} \mathcal{C}(Dj, c).$$

5. Adjunctions

Eilenberg and Mac Lane, authors of the foundational paper "General theory of natural equivalences," have asserted that the purpose of defining a category was to define a functor, and the purpose of defining a functor was to define a natural transformation. The modern consensus is that the definitive illustration of the power of the categorical perspective is in the definition of an *adjunction*, due to Daniel Kan.

To illustrate, suppose we have a set A and a topological space X. The key observation is that a function from A to the underlying set of the space X is equivalently a continuous function from the space A endowed with the discrete topology to X, and conversely, every continuous function from the discrete space A to X corresponds to a unique function between the underlying sets.

Write $U\colon \mathbf{Top} \to \mathbf{Set}$ for the functor that forgets the topology assigned a space and $D\colon \mathbf{Set} \to \mathbf{Top}$ for the functor that assigns a set the discrete topology. We have just observed that there is a bijection between the hom-sets

$$\mathbf{Top}(DA, X) \cong \mathbf{Set}(A, UX).$$

Furthermore, this bijection is natural in both variables: given a continuous map $h\colon X \to X'$, the underlying set function of a composite

$$DA \xrightarrow{g} X \xrightarrow{h} X'$$

is the composite of the set-function $g\colon A \to UX$ with Uh. Similarly, if $f\colon A' \to A$ is any set-function, the composite

$$A' \xrightarrow{f} A \xrightarrow{g} UX$$

corresponds to the continuous function defined by precomposing $g\colon DA \to X$ with Df. When this is the case, we say that the functor D is *left adjoint* to U. More generally,

Definition 5.1. An *adjunction* consists of a pair of functor

$$F\colon \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D}\colon G \tag{5.2}$$

together with, for each $c \in \mathcal{C}$ and $d \in \mathcal{D}$, a hom-set isomorphism

$$\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd) \tag{5.3}$$

natural in both variables.

Because the functor F appears on the left of (5.3), we say that F is *left adjoint*[8] to G. Corresponding arrows $Fc \to d$ and $c \to Gd$ are called *adjuncts*. An alternate formulation of the naturality condition, particularly convenient in homotopy theory because of its obvious relevance to "lifting problems," is that a square

$$\begin{tikzcd} Fc \arrow[r] \arrow[d, "Ff"'] & d \arrow[d, "g"] \\ Fc' \arrow[r] & d' \end{tikzcd} \quad \text{commutes in } \mathcal{D} \text{ if and only if} \quad \begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & Gd \arrow[d, "Gg"] \\ c' \arrow[r] & Gd' \end{tikzcd} \tag{5.4}$$

commutes in $\mathcal{C}$, where the unlabelled horizontal arrows in each diagram are adjuncts.

Example 5.5. The functor $U\colon \mathbf{Top} \to \mathbf{Set}$ is also a left adjoint. Set-functions $UX \to A$ correspond to continuous functions $X \to IA$, where $I\colon \mathbf{Set} \to \mathbf{Top}$ is the functor that endows a set A with the indiscrete topology. One might use the diagram

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "U" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{Set} \arrow[l, bend right=50, "D"'] \arrow[l, bend left=50, "I"] \end{tikzcd}$$

to indicate that $D \dashv U \dashv I$.

Another key example naturally motivates an equivalent definition. To explore this, let's consider what it would mean to have a left adjoint to the forgetful functor $U\colon \mathbf{Gp} \to \mathbf{Set}$. By Definition 5.1, if U has a left adjoint, then for each set A, the functor

$$\mathbf{Set}(A, U-)\colon \mathbf{Gp} \to \mathbf{Set}$$

that sends a group G to the set of maps $A \to UG$ would be representable. By the Yoneda lemma, a representation consists of a group, which we'll call FA, together with an element of the set $\mathbf{Set}(A, UFA)$, i.e., a functor $A \to UFA$ satisfying a particular universal property: For any group G, the map

$$\mathbf{Gp}(FA, G) \to \mathbf{Set}(A, UG)$$

that sends a homomorphism $h\colon FA \to G$ to the function $A \to UFA \xrightarrow{h} UG$ is an isomorphism. That is, every map $A \to UG$ uniquely factors through our specified function $A \to UFA$ along a group homomorphism $FA \to G$. Note that under this correspondence, the map $A \to UFA$ is associated to the identity homomorphism at FA.

In this example, FA is the free group on the set A whose elements are words whose letters are elements of A and whose group operation is concatenation. The function $A \to UFA$ maps each element of A to the corresponding one-letter word. The universal property says that any set function $A \to UG$ extends to a unique group homomorphism $FA \to G$.

A second equivalent definition of an adjunction is proposed by the following exercise.

[8] The direction is indicated by the symbol "$\perp$", also written $F \dashv G$. Many authors eschew this notation and instead attempt to organize the diagrams (5.2) so that the left adjoint always appears on the left, but in the presence of several functors, e.g. in the following example, this is not always possible.

Exercise 5.6. Suppose given $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ together with natural hom-set isomorphisms (5.3). Define natural transformations $\eta\colon 1 \Rightarrow GF$, whose components η_c are adjunct to the identity at Gc, and $\epsilon\colon FG \Rightarrow 1$, whose components ϵ_d are adjunct to the identity at Fd. Show that the composites

$$Gd \xrightarrow{\eta_{Gd}} GFGd \xrightarrow{G\epsilon_d} Gd \quad \text{and} \quad Fc \xrightarrow{F\eta_c} FGFc \xrightarrow{\epsilon_{Fc}} Fc \tag{5.7}$$

are identities. Such η and ϵ are called the *unit* and *counit*, respectively.

Conversely, a pair of functors equipped with natural transformations η and ϵ, called the *unit* and *counit* respectively, that satisfy the *triangle identities* (5.7) uniquely determines an adjunction, with adjunct arrows corresponding via a procedure analogous to the one described above.

Adjunctions and (co)limits. Excluding the Yoneda lemma, the following theorem is perhaps the most useful result from category theory. First, we need a few definitions.

Definition 5.8. Fix a small category $\mathcal{J}$. A functor $F\colon \mathcal{C} \to \mathcal{D}$

- *preserves* limits of shape $\mathcal{J}$, if F takes any limit cone over some diagram $D\colon \mathcal{J} \to \mathcal{C}$ to a limit cone over the diagram $FD\colon \mathcal{J} \to \mathcal{D}$.
- *reflects* limits of shape $\mathcal{J}$, if a cone over D in $\mathcal{C}$ whose image under F is a limit cone over FD was necessarily a limit cone in $\mathcal{C}$.
- *creates* limits of shape $\mathcal{J}$ if, for every diagram of shape $\mathcal{J}$ in $\mathcal{C}$ whose image under F has a limit in $\mathcal{D}$, there is some limit cone in $\mathcal{C}$ whose image is isomorphic to the specified limit cone in $\mathcal{D}$.

Theorem 5.9. *Right adjoints preserve limits and left adjoints preserve colimits.*

Proof 1. Let's use the Yoneda lemma and representability of limits. Given $G\colon \mathcal{D} \to \mathcal{C}$, a left adjoint $F \dashv G$, and a diagram $D\colon \mathcal{J} \to \mathcal{D}$ admitting a limit in $\mathcal{D}$, then we have a sequence of hom-set isomorphisms, natural in $c \in \mathcal{C}$

$$\mathcal{C}(c, U \lim_{\mathcal{J}} Dj) \cong \mathcal{D}(Fc, \lim_{\mathcal{J}} Dj) \cong \lim_{\mathcal{J}} \mathcal{D}(Fc, Dj) \cong \lim_{\mathcal{J}} \mathcal{C}(c, UDj) \cong \mathcal{C}(c, \lim_{\mathcal{J}} UDj)$$

The Yoneda lemma implies that $U \lim_{\mathcal{J}} D_j \cong \lim_{\mathcal{J}} UDj$, as desired. By naturality of the adjunctions and the representations for the limits, this diagram commutes with the natural maps to $\mathcal{C}(c, UDj) \cong \mathcal{D}(Fc, Dj)$ associated to some leg $\lim_{\mathcal{J}} Dj \to Dj$ of the limit cone, whence the desired result. $\square$

Proof 2. Let's dissect this argument in a specific case and show that right adjoints preserve products. Let $x, y \in \mathcal{D}$. We wish to show firstly that $U(x \times y)$ satisfies the universal property of the product of Ux and Uy and furthermore, that the image in $\mathcal{C}$ of the projections in $\mathcal{D}$ defines the limit cone.

Given another object $c \in \mathcal{C}$ with maps $c \to Ux$ and $c \to Uy$, their adjuncts define maps $Fc \to x$ and $Fc \to y$ in $\mathcal{D}$. By the universal property of the product $x \times y$, these maps factor uniquely through the projections via an arrow $Fc \to x \times y$. Its adjunct defines an arrow $x \to U(x \times y)$ that, by naturality of the adjunction, factors through the images of the projections. Given another such factorization, its adjunct would similarly factor through the cone on $x \times y$ in $\mathcal{D}$, by naturality again. Thus $Ux \leftarrow U(x \times y) \to Uy$ is a product in $\mathcal{C}$. $\square$

Remark 5.10. Note this is why we knew in §4 that the product of spaces had to be defined by suitably topologizing the product of their underlying sets, and furthermore why the projection maps had to align with the set-theoretic projections.

Remark 5.11. Because the underlying set of the direct sum of two abelian groups is not the same as the disjoint union of their underlying sets, we know the forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Set}$ does not admit a right adjoint.

Example 5.12. By Example 3.3, each object $c \in \mathcal{C}$ in a small category corresponds uniquely to a functor $\mathbf{1} \to \mathcal{C}$. Precomposition defines a functor

$$\mathrm{ev}_c\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}$$

that evaluates a functor $F\colon \mathcal{C} \to \mathbf{Set}$ at the object c. The functor ev_c admits both left and right adjoints given by *left* and *right Kan extension*, an exceedingly important concept that we will not describe here. By Theorem 5.9, it follows that limits and colimits in $\mathbf{Set}^{\mathcal{C}}$ are formed *pointwise*: i.e., evaluating a (co)limit of a diagram of functors at an object $c \in \mathcal{C}$ produces a (co)limit of the diagram formed by evaluating each functor at c. Conversely, it's reasonably easy to verify that the the limits at each object in $\mathcal{C}$ assemble into a functor that satisfies the universal property required of the limit of the diagram of functors.

Example 5.13. Let X be a topological space. It follows from Theorem 5.9 that any limits that exist in its category of sheaves are formed as in the category of presheaves on the poset $\mathcal{O}(X)$ of open subsets.[9] In fact, the existence of the adjunction

$$\begin{tikzcd} \mathbf{Sh}(X) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & {[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, shift right, "\text{sheafify}"'] \end{tikzcd}$$

both guarantees that $\mathbf{Sh}(X)$ has all colimits and tells us how to compute them. Any diagram of sheaves has a colimit as presheaves, because the latter category is cocomplete, with colimits formed pointwise, as in any functor category. The left adjoint preserves this colimit diagram, but the diagram is then naturally isomorphic to the original in the category of sheaves.[10]

Parameterized adjunctions. Kan's motivating example is somewhat more sophisticated. To illustrate, let $\mathcal{C}$ be any category with small products and coproducts. There is a *bifunctor*, i.e., functor of two variables,

$$- \cdot -\colon \mathbf{Set} \times \mathcal{C} \to \mathcal{C}$$

that takes a pair (A, c) to the A-indexed coproduct of copies of c, sometimes denoted $A \cdot c$. Furthermore, for each fixed set A, the functor $A \cdot -$ admits a right adjoint: by the universal property of the coproduct, maps $A \cdot c \to d$ in $\mathcal{C}$ are simply A-indexed maps $c \to d$, which are encoded by a single map from c to a single object d^A, the A-indexed product of copies of d. In other words, we have a hom-set isomorphism

$$\mathcal{C}(A \cdot c, d) \cong \mathcal{C}(c, d^A) \tag{5.14}$$

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[9]A more sophisticated result, that any adjunction with fully faithful right adjoint is *monadic*, implies that the category of sheaves has all limits that exist as presheaves.

[10]This last assertion is true because the sheafification of a sheaf is again a sheaf: the formal reason is that the counit of an adjunction is an isomorphism if (and only if) the right adjoint is full and faithful.

natural in both c and d. Inspecting the definitions, we see that these right adjoints also assemble into a bifunctor

$$(-)^{-} : \mathbf{Set}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

that is *contravariant* in the first variable: given a set function $f \colon B \to A$, we define $d^A \to d^B$, using the universal property of the latter product, to be the map defined on the component of $b \in B$ by projecting from d^A to the component of the product indexed by $f(b) \in A$.

Furthermore, the isomorphism (5.14) is natural in A. The composite of an arrow $c \to d^A$ with the "re-indexing" map $d^A \to d^B$ just described has the same description as the composite $B \cdot c \to A \cdot c \to d$: the component corresponding to $b \in B$ is the component of $c \to d^A$, or equivalently of $A \cdot c \to d$, corresponding to $f(b)$.

This example illustrates a general phenomenon, which is more clearly expressed for a generic bifunctor $F \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$.

Theorem 5.15. *Let $F \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and suppose for each $b \in \mathcal{B}$, the functor $F(-, b) \colon \mathcal{A} \to \mathcal{C}$ admits a right adjoint $G_b \colon \mathcal{C} \to \mathcal{A}$. These functors assemble into a bifunctor $G \colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ in a unique way so that $G_b(c) = G(b, c)$ and so that the isomorphisms*

$$\mathcal{C}(F(a, b), c) \cong \mathcal{A}(a, G(b, c))$$

are natural in all three variables.

Proof. We use G_b to define the action of $G(b, -)$ on morphisms $c \to c'$. To define the action of $G(-, c)$ on $g \colon b' \to b$, it suffices by the Yoneda lemma to define a natural transformation between the functors represented by $G(b, c)$ and $G(b', c)$. This natural transformation is the composite

$$\mathcal{A}(a, G(b, c)) \cong \mathcal{C}(F(a, b), c) \xrightarrow{F(a,g)^*} \mathcal{C}(F(a, b'), c) \cong \mathcal{A}(a, G(b', c)).$$

The desired naturality statement follows immediately from this definition. □

We have seen several other examples of bifunctors that admit pointwise right adjoints above. A category is *cartesian closed* if for each $c \in \mathcal{C}$, the product functor $- \times c \colon \mathcal{C} \to \mathcal{C}$ admits a right adjoint. By Theorem 5.15 these right adjoints assemble into a bifunctor $\hom(-, -) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ that we refer to as the *internal-hom*. For example, the category **Set** and the category $\mathbf{Set}^{\mathcal{C}}$ for any small $\mathcal{C}$ are both cartesian closed. The category **Top** of all topological spaces is not cartesian closed, but certain subcategories are. For instance, the subcategory of locally compact Hausdorff spaces is cartesian closed, with internal-homs given the compact-open topology. More famously, the category of compactly generated spaces is cartesian closed, though here the categorical product differs from the product in the category of all spaces. (This in particular, tells us that the inclusion functor from compactly generated spaces to all spaces is not a right adjoint.)

In the category $\mathbf{Mod}_R$ or $\mathbf{Vect}_k$, the tensor product described above admits pointwise right adjoints, which we also call internal-homs. Both examples behave similarly: the bifunctor

$$\hom(-, -) \colon \mathbf{Vect}_k^{\mathrm{op}} \times \mathbf{Vect}_k \to \mathbf{Vect}_k$$

assigns a pair V, W the vector space of linear functors $V \to W$.The adjoint correspondence

$$\mathbf{Vect}_k(U \otimes V, W) \cong \mathbf{Vect}_k(U, \hom(V, W))$$

says that linear maps $U \otimes V \to W$ correspond to linear maps $U \to \hom(V, W)$ and that this correspondence is natural in all three variables. Indeed, a linear map from U to the vector space of linear functions from V to W is precisely a bilinear function on U and V taking values in W, and hence a linear map $U \otimes V \to W$ by the defining universal property of the tensor product.