QUASI-CATEGORIES AS $(\infty, 1)$-CATEGORIES

EMILY RIEHL

Contents

My goal today is less to give a comprehensive introduction to quasi-categories as a model for $(\infty, 1)$-categories[1] but rather to give one that is as close to the ground as possible. For every statement that appears below, I'll try to either explain the proof or at least give some indication of how it is proven. This strongly influences the order of the topics. Some of what will appear below is self-plagiarized from [Rie13, Part IV], written for a class I taught here last spring. Some of the rest is copied from some joint papers with Dominic Verity, which I hope will appear soon.

Here we go!

Basic notions

Suppose a simplicial set is a quasi-category unless explicitly stated otherwise. An important feature of quasi-categories that isn't true for generic simplicial sets is that for every relation in the homotopy category and any choice of representing 1-simplices, there exists a 2-simplex that witnesses the relation. More precisely:

Proposition 0.1. *Given 1-simplicies $f, g, h \in X$, $h = gf$ in $\mathrm{ho}X$ if and only if there exists a 2-simplex in X with boundary*

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, bend left=20, "="] & & \cdot \end{tikzcd}$$

Date: Spring 2013.

[1] A (m, n)-category is a (weak) category with cells up to dimension m so that every cell above dimension n is (weakly) invertible.

We have an adjunction $\mathrm{ho}\colon \mathbf{qCat} \rightleftarrows \mathbf{Cat}\colon N$ whose right adjoint, the nerve functor, is fully faithful. Sometimes it's conventional to regard categories as quasi-categories without writing the "N." In every case we know of (certainly in every example we will mention) the quasi-categorical notion, when restricted to the full subcategory of categories, will coincide exactly with the categorical notion bearing the same name. So category theory is really a subset of quasi-category theory.

Proposition 0.2. **qCat** *is cartesian closed (and admits cotensors by arbitrary simplicial sets) with the internal hom (cotensor) given by the internal hom for simplicial sets.*

There is a bit of combinatorics that goes into the proof of this, which we will address momentarily. The obvious fact is that the larger **sSet** is cartesian closed. To my mind, the reason quasi-categories are such a convenient model of $(\infty, 1)$-categories owes largely to the fact that **sSet**, as a presheaf category, is so well behaved (in particular complete and cocomplete closed symmetric monoidal). We'll see later that a number of the objects used to build the category theory of quasi-categories are modeled by the analogous simplicial weighted limits.

Let us think what is being asserted by this statement. From the definition, we are asked to show that for any quasi-category X and simplicial set A there exist extensions

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X^A \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \times \Lambda^n_k \arrow[r] \arrow[d] & X \\ A \times \Delta^n \arrow[ur, dashed] \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & X^{\Delta^n} \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & X^{\Lambda^n_k} \end{tikzcd}$$

for all $n \geq 2$, $0 < k < n$.[2] The two lifting problems correspond by adjunction. Let us think what is being asserted by the latter. We are asked to choose cylinders $\Delta^m \times \Delta^n \to X$ for each m-simplex in A in a way that is compatible with the specified horn $\Delta^m \times \Lambda^n_k \to X$ and also with previously specified cylinders $\partial\Delta^m \times \Delta^n \to X$ corresponding to the boundary of the m-simplex. In other words, inductively, we must choose extensions

$$\begin{tikzcd} \partial\Delta^m \arrow[rr] \arrow[d, "i_m"'] & & X^{\Delta^n} \arrow[d, "X^{j^n_k}"] \\ \Delta^m \arrow[r] \arrow[urr, dashed] & A \arrow[r] & X^{\Lambda^n_k} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta^m \times \Delta^n \coprod_{\partial\Delta^m \times \Lambda^n_k} \Delta^m \times \Lambda^n_k \arrow[r] \arrow[d, "i_m \hat{\times} j^n_k"'] & X \\ \Delta^m \times \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

The indicated lifting problems are again transposes, on account of the **Leibniz construction** applied to the two variable adjunction between the cartesian product and internal hom.[3] Assuming the ambient categories have the necessary pullbacks and pushouts, any two-variable adjunction

$$C(a \times b, c) \cong C(a, \underline{\mathrm{hom}}(b, c))$$

(such as a closed monoidal structure) gives rise to a two-variable adjunction

$$C^2(f \hat{\times} g, h) \cong C^2(f, \underline{\hat{\mathrm{hom}}}(g, h))$$

on the arrow categories. The left adjoint is the pushout product bifunctor $-\hat{\times}-$ and the right adjoint, defined dually, might be called the pullback hom (or Leibniz hom) $\underline{\hat{\mathrm{hom}}}(-, -)$. For example, the map $X^{\Delta^n} \to X^{\Lambda^n_k}$ is the Leibniz hom of $\Lambda^n_k \to \Delta^n$ with $X \to *$.

Such extensions always exist on account of the following result.

[2] With apologies to Mike, I have to change notation. I'll write Δ^n for his $\Delta[n]$ and write Λ^n_k for his $V_k[n]$.

[3] The name, propagandized by Dominic Verity, is inspired by Leibniz' formula for the boundary of a product of polygons.

Proposition 0.3 (Joyal). *The pushout-product of an inner anodyne map with a cofibration is inner anodyne.*

Proof. It suffices to show this is true of the $(\partial\Delta^m \to \Delta^m)\hat{\times}(\Lambda^n_k \to \Delta^n)$'s because the bifunctor $-\hat{\times}-$ preserves colimits in each variable and the inner anodyne maps, as the left class of a weak factorization system, is weakly saturated. This can be proven directly by decomposing these monomorphisms into pushouts of inner horns (see [DS11, A.1]) or via a slick, but non-constructive, argument that proves the result as stated but doesn't tell us whether the maps $(\partial\Delta^m \to \Delta^m)\hat{\times}(\Lambda^n_k \to \Delta^n)$ are *cellular* inner anodyne (relative cell complexes built from the inner horn inclusions). □

Remark. By easy formalities involving two-variable adjunctions and lifting properties there are actually three equivalent statements here, i.e., Proposition 0.3 is equivalent to either of the following two statements:

- the pullback-hom of a cofibration with an inner fibration is an inner fibration
- the pullback-hom of an inner anodyne map with an inner fibration is a trivial fibration.

In particular, the pullback-hom of $\emptyset \to A$ and $X \to *$ is $X^A \to *$, proving that X^A is a quasi-category if X is. We have another immediate corollary.

Corollary 0.4. *If X is an ∞-category, then $X^{\Delta^n} \to X^{\Lambda^n_k}$ is a trivial fibration.*

In particular, the fiber over any point is a contractible Kan complex. This says that the spaces of fillers to a given horn is a contractible Kan complex. This is the common form taken by a homotopical uniqueness statement in ∞-category theory and is what is meant by saying something is "well defined up to a contractible space of choices."

Equivalences between quasi-categories

By an observation of Joyal, the cofibrations and fibrant objects completely determine a model structure, supposing one exists. As it turns out, again by work of Joyal, the monomorphisms and quasi-categories give rise to a model structure on simplicial sets whose weak equivalences, called simply **equivalences** when between quasi-categories, are a good notion.

Theorem 0.5 (Joyal). *The cofibrations and fibrant objects completely determine a model structure.*

The following argument parallels his proof of this theorem in our particular case of interest. Supposing there is such a model structure for quasi-categories, the weak equivalences must be characterized representably as maps $f : A \to B$ that induce bijections on hom-sets in the homotopy category when homming into any quasi-category X. Because all objects are cofibrant, we can characterize the hom-sets in the homotopy category of the hypothesized model structure by use of a good cylinder object.

To that end write J for the nerve of the free-standing isomorphism.[4] Observe that $J \to *$ and hence any projection $A \times J \to A$, as its pullback, is a trivial fibration. Consequently,

$$A \sqcup A \rightarrowtail A \times J \xrightarrow{\sim} \!\!\!\!\rightarrow A$$

[4]This is a simplicial model for $S^\infty = B(\mathbb{Z}/2, \mathbb{Z}/2, *)$, the total space of the classifying space $K(\mathbb{Z}/2, 1) = B\mathbb{Z}/2 = \mathbb{R}P^\infty$.

defines a very good cylinder object. Using this, by a theorem of Quillen the hom-set from A to X in the homotopy category is isomorphic to the set $[A,X]_J$ defined to be the quotient of $\hom(A,X)$ by the relation generated[5] by $f \sim g$ if there exists a diagram

$$\begin{tikzcd} A \arrow[d, "j_0"'] \arrow[dr, "f"] & \\ A \times J \arrow[r] & X \\ A \arrow[u, "j_1"] \arrow[ur, "g"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} * \arrow[d, "j_0"'] \arrow[dr, "f"] & \\ J \arrow[r] & X^A \\ * \arrow[u, "j_1"] \arrow[ur, "g"'] & \end{tikzcd} \tag{0.6}$$

So we declare a map $f\colon A \to B$ of simplicial sets to be an **weak equivalence** if and only if it induces a bijection $[B,X]_J \to [A,X]_J$ for all quasi-categories X. We follow Lurie and call these maps **categorical equivalences**[6] or simply **equivalences** if the source and target are quasi-categories because no ambiguity is possible in that case. A good exercise for the reader is to show that inner anodyne maps and trivial fibrations are weak equivalences using this definition.

Theorem 0.7 (Joyal)**.** *There is a left proper cofibrantly generated model structure on simplicial sets whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories.*

Fibrations between fibrant objects, which we shall call **isofibrations** are characterized by the right lifting property against the inner horn inclusions and the map $* \to J$, which is the nerve of the functor whose right lifting property chracterizes the isofibrations in **Cat** (hence the name). Note that the trivial fibrations are the same in Joyal's and in Quillen's model structures. Some closing remarks:

- $\mathrm{ho} \dashv N$ is a Quillen adjunction with the folk model structure on **Cat**.
- As a corollary, both adjoint functors preserve equivalences. A functor between categories is an equivalence if and only if its nerve is an equivalence.
- Categorical equivalences are weak homotopy equivalences.

Quasi-categories as $(\infty,1)$-categories

A quick inductive definition of an $(\infty,1)$-category is that it's something (weakly) enriched over $(\infty,0)$-categories, i.e., ∞-groupoids, i.e., homotopy types.

Aside (the homotopy category of spaces as a base for enrichment). Because I like knowing why these types of things are true, permit me a digression on why it makes sense to enrich over the homotopy category of spaces. Everyone knows that simplicial sets is a closed symmetric monoidal category and has a compatible model structure which makes it a simplicial model category. This is Quillen equivalent to a simplicial model structure on your favorite convenient category of spaces, e.g., k-spaces or compactly generated spaces. The Quillen equivalence descends to an equivalence between the homotopy categories, which we'll call the **homotopy category of spaces** and denote by $\mathcal{H}$.

Using this simplicial model structure, there is a uniform way to construct point-set level and total derived functors of left and right Quillen functors, bifunctors, etc: Just precompose with cofibrant replacement or fibrant replacement, as appropriate. The fact

[5] Indeed, the "generated" here is unnecessary because X, and hence X^A, is a quasi-category; any f and g in the same equivalence class admit such diagrams, as we shall prove momentarily.

[6] Joyal calls these "weak categorical equivalences."

that the model structure is closed monoidal implies that the cartesian product and internal hom are amenable to such deformations, so have derived functors constructed in this way.

We'd like to say that the total derived functors of the closed symmetric monoidal structure on **sSet** define a closed symmetric monoidal structure on $\mathcal{H}$. To prove this we need to show that we can also derived the natural isomorphisms expressing coherence of the derived monoidal product, existence of the derived adjunction, and so forth. Now composing derived functors is somewhat non-trivial but in this case the axioms that establish that **sSet** is a monoidal model category (plus Ken Brown's lemma) say things like the hom-space from a cofibrant object to a fibrant object is again fibrant which imply that everything works out. (See [Rie13, Chapter 10] for more details.)

Furthermore, the localization functor **sSet** $\to \mathcal{H}$ is lax monoidal which means any simplicial enrichment descends to an $\mathcal{H}$-enrichment.

Our goal is define $\underline{\mathrm{ho}}X$ as an $\mathcal{H}$-category so the underlying category — whose arrows are homotopy classes of maps $* \to \underline{\mathrm{ho}}X(a, b)$, i.e., whose hom-sets can be computed by applying π_0 to the hom-spaces — is $\mathrm{ho}X$.

The first, to my mind most obvious construction, makes use of the quasi-category X^{Δ^1} of paths in X; vertices are 1-simplices in X, and n-simplices are cylinders $\Delta^n \times \Delta^1 \to X$. One candidate mapping space between two fixed vertices $x, y \in X$ is the pullback

$$\begin{tikzcd} \mathrm{Hom}_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta^1} \arrow[d] \\ * \arrow[r, "{(x,y)}"'] & X \times X \cong X^{\partial\Delta^1} \end{tikzcd}$$

By the combinatorics encoded above by Proposition 0.3, $\mathrm{Hom}_X(x, y)$ is a quasi-category. An n-simplex is $\mathrm{Hom}_X(x, y)$ is a map $\Delta^n \times \Delta^1 \to X$ such that the image of $\Delta^n \times \{0\}$ is degenerate at x and and the image of $\Delta^n \times \{1\}$ is degenerate at y. In particular, 1-simplices look like

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr] \arrow[dr, phantom, "\sim", shift left=2ex] \arrow[dr, phantom, "\sim"', shift right=2ex] & y \arrow[d, equal] \\ x \arrow[r, "g"'] & y \end{tikzcd} \tag{0.8}$$

from which we see that $\pi_0\mathrm{Hom}_X(x, y)$ is the hom-set from x to y in hX.

A less symmetric but more efficient construction is also possible. Let $\mathrm{Hom}^R_X(x, y)$ be the simplicial set whose 0-simplices are 1-simplices in X from x to y, whose 1-simplices are 2-simplices of the form

$$\begin{tikzcd} & x \arrow[dr] & \\ x \arrow[ur, equal] \arrow[rr] & & y \end{tikzcd}$$

and whose n-simplices are $(n+1)$-simplices whose last vertex is y and whose $(n+1)$th face is degenerate at x. Dually, $\mathrm{Hom}^L_X(x, y)$ is the simplicial set whose n-simplices are $(n + 1)$-simplices in X whose first vertex is x and whose d^0-face is degenerate at y. Once again, note that $\pi_0\mathrm{Hom}^L_X(x, y) = \pi_0\mathrm{Hom}^R_X(x, y) = hX(x, y)$.

Remark. The spaces $\mathrm{Hom}^L_X(x, y)$ and $\mathrm{Hom}^R_X(x, y)$ are dual in the sense that $\mathrm{Hom}^L_X(x, y) = (\mathrm{Hom}^R_{X^{\mathrm{op}}}(y, x))^{\mathrm{op}}$. The annoying fact, from the perspective of homotopy (co)limits, that a simplicial set is not isomorphic to its opposite, in which the conventions on ordering of vertices in a simplex are reversed, is technically convenient here.

In fact all three of these candidate hom-spaces are good models: they're all Kan complexes (the explanation for which we'll postpone for now) and they're all equivalent. To explain the equivalence, let us think geometrically about the difference.[7] Each simplicial set has the same zero simplices. An n-simplex in $\mathrm{Hom}^L_X(x,y)$ or $\mathrm{Hom}^R_X(x,y)$ is an $(n+1)$-simplex in X one of whose faces is degenerate. Thus the relevant shapes are given by the quotients

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[d, "d^0"'] & \Delta^0 \arrow[d] \\ \Delta^{n+1} \arrow[r] & C^n_L \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \Delta^n \arrow[r] \arrow[d, "d^{n+1}"'] & \Delta^0 \arrow[d] \\ \Delta^{n+1} \arrow[r] & C^n_R \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This simplicial set has two vertices and has a non-degenerate k-simplex for each non-degenerate k-simplex of Δ^n whose image surjects onto Δ^1.

Similarly, the shape of an n-simplex in $\mathrm{Hom}_X(x,y)$ is given by

$$\begin{tikzcd} \Delta^n \times \partial\Delta^1 \arrow[r] \arrow[d] & \partial\Delta^1 \cong * \sqcup * \arrow[d] \\ \Delta^n \times \Delta^1 \arrow[r] & C^n_{\mathrm{cyl}} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We have canonical maps

$$\begin{tikzcd} C^n_L \arrow[r] \arrow[dr] & C^n_{\mathrm{cyl}} \arrow[d] & C^n_R \arrow[l] \arrow[dl] \\ & \Delta^1 & \end{tikzcd}$$

where the horizontal maps are induced by the inclusions of $\Delta^{n+1} \rightrightarrows \Delta^n \times \Delta^1$ as the first and last shuffles respectively.[8]

These constructions define three cosimplicial objects $C^\bullet_L, C^\bullet_{\mathrm{cyl}}, C^\bullet_R$ taking values in the category of simplicial sets and maps preserving two chosen basepoints. Write $\mathbf{sSet}_{*,*}$ for this slice category $\partial\Delta^1/\mathbf{sSet}$. The simplicial set X with chosen vertices x, y becomes an object of $\mathbf{sSet}_{*,*}$. The three hom-spaces introduced above are defined from these cosimplicial objects and the hom-sets of $\mathbf{sSet}_{*,*}$ by the equalities

$$\begin{aligned} \mathrm{Hom}^L_X(x,y) &= \mathbf{sSet}_{*,*}(C^\bullet_L, X) \\ \mathrm{Hom}_X(x,y) &= \mathbf{sSet}_{*,*}(C^\bullet_{\mathrm{cyl}}, X) \\ \mathrm{Hom}^R_X(x,y) &= \mathbf{sSet}_{*,*}(C^\bullet_R, X). \end{aligned}$$

The natural maps $\mathrm{Hom}^L_X(x,y) \leftarrow \mathrm{Hom}_X(x,y) \rightarrow \mathrm{Hom}^R_X(x,y)$ come from the maps between the cosimplicial objects. We would like to show that these are equivalences. Morally, this follows because $C^\bullet_L$, $C^\bullet_{\mathrm{cyl}}$, and $C^\bullet_R$ are cofibrant resolutions of Δ^1 in the Joyal model structure. Let us give just a few more details.

Remark. The category $\mathbf{sSet}_{*,*}$, defined as a slice category, inherits a model structure from the quasi-categorical model structure on $\mathbf{sSet}$: A map of twice-based simplicial sets is a cofibration, fibration, or weak equivalence just when the underlying map of simplicial sets

[7]This proof is due to Daniel Dugger and David Spivak with some modifications by Verity.

[8]Recall simplices in $\Delta^n \times \Delta^m$ correspond bijectively to totally ordered collections of vertices (i, j) with $i \in [n]$ and $j \in [m]$. Simplices of maximal dimension are called **shuffles**. The first shuffle is the unique one containing the vertices $(0,0), \ldots, (n,0), \ldots (n,m)$. The last is the unique one containing the vertices $(0,1), \ldots, (0,m), \ldots, (n,m)$.

is one. Fibrant objects are quasi-categories with chosen basepoints. An object is cofibrant if and only if its two chosen basepoints are distinct.

Lemma 0.9. *$C^\bullet_R$, $C^\bullet_L$, $C^\bullet_{\mathrm{cyl}}$ are Reedy cofibrant.*

Proof. There's a simple criterion ("unaugmentable" in [BK72]) that detects when a cosimplicial object is Reedy cofibrant, and if you know it, it's easy to check that it's true here. □

The geometrical heart of the argument is in the proof of the following result.

Lemma 0.10. *The canonical maps $C^\bullet_L \to C^\bullet_{\mathrm{cyl}} \leftarrow C^\bullet_R$ are pointwise categorical equivalences.*

Proof. Those with patience for combinatorics can check that $C^n_L \to \Delta^1$, $C^n_{\mathrm{cyl}} \to \Delta^1$, and $C^n_R \to \Delta^1$ by showing that sections are (cellular) inner-anodyne maps [DS11]. □

We might think about these cosimplicial spaces as "weights" whose weighted limits define our three candidate mapping spaces. To use this information to obtain our desired conclusion, the starting point is that one can define simplicial mapping spaces for $\mathbf{sSet}_{*,*}$ so that when X is a quasi-category $\underline{\mathrm{hom}}(-, X)\colon \mathbf{sSet}^{\mathrm{op}}_{*,*} \to \mathbf{sSet}$ is a right Quillen functor. By Ken Brown's lemma, it follows that this functor preserves equivalences between objects with distinct basepoints. The proof is completed by some Reedy category theory.

Consider a cosimplicial object $C^\bullet\colon \mathbb{\Delta} \to \mathbf{sSet}_{*,*}$. Latching and matching objects can be defined to be certain (dual) weighted colimits and limits from which it is clear that

$$M_n \underline{\mathrm{hom}}(C^\bullet, X) \cong \underline{\mathrm{hom}}(L^n C^\bullet, X).$$

If $C^\bullet$ is Reedy cofibrant, the maps $L^n C^\bullet \to C^n$ are cofibrations and hence

$$\underline{\mathrm{hom}}(C^\bullet, X) \to \underline{\mathrm{hom}}(L^n C^\bullet, X) \cong M_n \underline{\mathrm{hom}}(C^\bullet, X)$$

are fibrations because $\underline{\mathrm{hom}}(-, X)$ is right Quillen. This says that $\underline{\mathrm{hom}}(C^\bullet, X)$ is Reedy fibrant. Applying this result to the cosimplicial objects $C^\bullet_L, C^\bullet_{\mathrm{cyl}}, C^\bullet_R$ we see that we have pointwise weak equivalences between Reedy fibrant objects

$$\underline{\mathrm{hom}}(C^\bullet_L, X) \leftarrow \underline{\mathrm{hom}}(C^\bullet_{\mathrm{cyl}}, X) \to \underline{\mathrm{hom}}(C^\bullet_R, X)$$

in the category of bisimplicial sets.

Remembering only the vertices of each simplicial set in the simplicial objects — a process which might be called "taking vertices pointwise" — we are left with the diagram of simplicial sets $\mathrm{Hom}^L_X(x, y) \leftarrow \mathrm{Hom}_X(x, y) \to \mathrm{Hom}^R_X(x, y)$ that is actually of interest. The proof that these maps are weak equivalences is completed by the following lemma.

Lemma 0.11. *Suppose $f\colon X \to Y$ is a weak equivalence between Reedy fibrant bisimplicial sets. Then the associated map of simplicial sets $X_{\bullet,0} \to Y_{\bullet,0}$ obtained by taking vertices pointwise is a weak equivalence.*

Proof. By Ken Brown's lemma, it suffices to prove that if $f\colon X \to Y$ is a Reedy trivial fibration of bisimplicial sets then the associated map $X_{\bullet,0} \to Y_{\bullet,0}$ is a weak equivalence. Indeed, this map is a trivial fibration of simplicial sets. Because f is a Reedy trivial fibration, each relative matching map $X_n \to Y_n \times_{M_n Y} M_n X$ is a trivial fibration of simplicial sets, and in particular, the map on vertices $X_{n,0} \to (Y_n \times_{M_n Y} M_n X)_0 = Y_{n,0} \times_{(M_n Y)_0} (M_n X)_0$ is a surjection in **Set**. But this says exactly that any lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X_{\bullet,0} \arrow[d] \\ \Delta^n \arrow[r] & Y_{\bullet,0} \end{tikzcd}$$

has a solution. □

Thus, we have proven:

Theorem 0.12. *The natural maps* $\mathrm{Hom}^L_X(x,y) \leftarrow \mathrm{Hom}_X(x,y) \to \mathrm{Hom}^R(x,y)$ *are equivalences of quasi-categories.*

Using retractions to the maps $C^n_L \to C^n_{\mathrm{cyl}} \leftarrow C^n_R$, which can be defined as quotients of the appropriate projections $\Delta^n \leftarrow \Delta^n \times \Delta^1 \to \Delta^n$, there are also equivalences $\mathrm{Hom}^L_X(x,y) \to \mathrm{Hom}_X(x,y) \leftarrow \mathrm{Hom}^R_X(x,y)$. We'll see shortly that any equivalence $X \to Y$ of quasi-categories has an inverse equivalence $Y \to X$.

Q (for the audience)**.** Reedy category theory is good for this sort of thing and for proving simplified formulas for homotopy limits and colimits. What else?

Because equivalences between quasi-categories are homotopy equivalences, the objects $\mathrm{Hom}^L_X(x,y)$, $\mathrm{Hom}_X(x,y)$, and $\mathrm{Hom}^R_X(x,y)$ define weakly equivalent simplicial sets whose set of path components is the hom-set $\mathrm{ho}X(x,y)$. We would like to conclude that the homotopy category $\mathrm{ho}X$ is thereby enriched over the homotopy category of spaces — however, there is no natural composition law definable in **sSet** using any of these mapping spaces. These considerations motivate the introduction of a fourth candidate mapping space, which is weak homotopically equivalent (but not categorically equivalent) to these models, and associates to each simplicial set a simplicially enriched category.

Homotopy coherent diagrams

The point is there is an adjunction $\mathfrak{C}\colon \mathbf{sSet} \rightleftarrows \mathbf{sCat}\colon \mathbb{N}$ between simplicial sets and simplicial categories. It is a Quillen equivalence with respect to the Joyal and Bergner model structures. In particular, if $\underline{C}$ is a locally Kan simplicial category then $\mathbb{N}\underline{C}$ is a quasi-category. This is important source of quasi-categories in practice; for instance, the quasi-category associated to a simplicial model category is defined by applying $\mathbb{N}$ to the subcategory of fibrant-cofibrant objects. On the other side, if X is a quasi-category then the hom-spaces of $\mathfrak{C}X$, while not fibrant,[9] do have the same weak homotopy type as the mapping spaces introduced above. So we can use $\mathfrak{C}X$ to define $\underline{\mathrm{ho}}X$. In particular $\mathrm{ho}X = (\pi_0)_*\mathfrak{C}X$. A consequence of this Quillen equivalence, or really rather an ingredient in the proof, is that $X \to Y$ is a categorical equivalence (of simplicial sets even) if and only if $\mathfrak{C}X \to \mathfrak{C}Y$ is a DK-equivalence.

As an expository note, Lurie's entire approach to the proof of the model structure on quasi-categories is designed to facilitate the proof that this adjunction is a Quillen equivalence, which should serve as some indication of its importance [Lur09, Chapter 2].

A lot of you know a lot about this (and some subset of you have heard me talk about this before) so I'm not going to say too much except to remind you how this adjunction is

[9] Even though the hom-spaces of $\mathfrak{C}X$ aren't fibrant, they are, in some weird sense, close. More precisely, for any simplicial set X, the hom-spaces of $\mathfrak{C}X$ are 3-coskeletal, which implies that any horn of dimension 5 or higher will have a *unique* filler. But it is easy in toy examples to find low dimensional horns that cannot be filled.

defined. The reason I want to do this is that it connects back to the story about homotopy coherence mentioned by Mike last time that motivated the development of $(\infty, 1)$-category theory and quasi-categories in particular. In particular, the replacement of the indexing category $\mathcal{I}$ of a diagram by a simplicial category $\tilde{\mathcal{I}}_\bullet$ that was used to set up the obstruction theory for lifting diagrams in the homotopy category is an instance of cofibrant replacement in this model structure. Even more precisely, the map $\tilde{\mathcal{I}}_\bullet \to \mathcal{I}$ is isomorphic to the component of the counit of the adjunction $\mathfrak{C} \dashv \mathbb{N}$ at the discrete simplicial category $\mathcal{I}$.

There are two isomorphic descriptions of this cofibrant replacement. One, as I just claimed is $\mathfrak{C}N\mathcal{I}$.[10] But since we haven't defined these things yet, I'll give the other, which is the construction of Dwyer-Kan. There is a comonad F on **Cat** which replaces a category by the category freely generated by its underlying reflexive directed graph (forgetting composites but remembering identities). Note that $\mathcal{I}$ and $F\mathcal{I}$ have the same objects. Non-identity morphisms in $F\mathcal{I}$ are strings of composable non-identity morphisms. The counit $F\mathcal{I} \to \mathcal{I}$ composes the arrows in each string. The cosimplicial object in **Cat** that defines the simplicial category serving as the cofibrant replacement of $\mathcal{I}$ is the comonad resolution (augmented by this $F\mathcal{I} \to \mathcal{I}$). The n-th category is $F^{n+1}\mathcal{I}$. Its objects are the same as the objects of $\mathcal{I}$ and its morphisms are strings of composable arrows enclosed in exactly n pairs of parentheses (each indicating a layer of formal composition). The degeneracy maps "double up on parentheses" while the face maps remove parentheses, which should be thought of as a form of composition (because it is).

Now the adjunction $\mathfrak{C}\colon \mathbf{sSet} \rightleftarrows \mathbf{sCat}\colon \mathbb{N}$, like any adjunction so that the domain of the left adjoint is simplicial sets, is given by some "geometric realization–total singular complex"-type construction (or, if you will, "left Kan extension–nerve") with respect to some cosimplicial object $\mathbb{\Delta} \to \mathbf{sCat}$. This simplicial object is defined by taking the finite ordinal categories $[n]$ to their cofibrant replacements defined in this way. For example, let's compute the cofibrant replacement of $[2] = \mathbb{3}$, which is the category whose non-identity morphisms we might label as:

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "j"'] \arrow[rrr, bend right=60, "\ell"'] & \cdot \arrow[r, "g"] \arrow[rr, bend right, "k"'] & \cdot \arrow[r, "h"] & \cdot \end{tikzcd}$$

Let us describe the hom-space from the initial object to the terminal one. The vertices of this simplicial set are the paths of edges ℓ, kf, hj, hgf. The 1-simplices are once parenthesized strings of composable morphisms which are non-degenerate when there is more than one arrow inside some pair of parentheses. There are five such with boundary 0-simplices illustrated below

(0.13)
$$\begin{tikzcd} \ell \arrow[r, "(kf)"] \arrow[d, "(hj)"'] \arrow[dr, "(hgf)" description] \arrow[dr, phantom, shift left=4, "((hg)(f))"] \arrow[dr, phantom, shift right=4, "((h)(gf))"'] & kf \arrow[d, "(hg)(f)"] \\ hj \arrow[r, "(h)(gf)"'] & hgf \end{tikzcd}$$

[10]The nerve and the homotopy coherent nerve coincide for discrete simplicial categories.

There are only two non-degenerate 2-simplices whose boundaries are depicted above. Hence the hom-space is $\Delta^1 \times \Delta^1$.[11]

Those who are familiar with the classical literature on homotopy coherent diagrams will recognize a lot of these ideas. In the language of Cordier-Porter, Vogt, and others, a diagram of shape $\mathfrak{C}\mathbb{N}\mathcal{I}$ is exactly a **homotopy coherent diagram** of shape $\mathcal{I}$. In the context of quasi-category theory, Jacob defines a homotopy coherent diagram in a quasi-category X to be any map $N\mathcal{I} \to X$. (This makes sense geometrically if you think about the higher simplices of the nerve.) Note if X is one of these quasi-categories which arises as $\mathbb{N}\underline{C}$ for some locally Kan simplicial category $\underline{C}$ (and indeed all quasi-categories are equivalent to some such thing), then by adjunction $N\mathcal{I} \to \mathbb{N}\underline{C}$ is exactly $\mathfrak{C}\mathbb{N}\mathcal{I} \to \underline{C}$, i.e., a homotopy coherent diagram in the quasi-category is a homotopy coherent diagram in the associated simplicial category (which is another model for the $(\infty, 1)$-category).

Isomorphisms in quasi-categories

What I'm proposing here is not standard terminology but was suggested to me recently by Dominic Verity in the context of a paper we're writing. I thought I'd use it today to gauge reactions from the audience.

We say a 1-simplex in a quasi-category is an **isomorphism** if and only if it represents an isomorphism in ho$X$. By remarks made above, for any isomorphism $f \colon x \to y$ we can choose an inverse isomorphism $g \colon y \to x$ together with 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "="] & \\ y \arrow[ur, "g"] \arrow[rr, equal] & {} & y \end{tikzcd}$$

A key combinatorial lemma, due to Joyal, says that quasi-categories admit "special outer horn fillers," that is, any horn $\Lambda^n_0 \to X$ can be filled provided that its initial edge is an isomorphism and dually any $\Lambda^n_n \to X$ whose final edge is an isomorphism has a filler [Joy02]. Conversely (and this part is obvious) these extension properties characterize the isomorphisms. There is also this immediate corollary:

Corollary 0.14 (Joyal). *X is a Kan complex if and only if X is a quasi-category and* ho$X$ *is a groupoid.*

Another corollary is that the three models for mapping spaces mentioned above are Kan complexes. The spaces $\mathrm{Hom}^L_X(x, y)$ and $\mathrm{Hom}^R_X(x, y)$ are defined as pullbacks of right fibrations, which implies that all of their edges are isomorphisms. We've shown these are equivalent to $\mathrm{Hom}_X(x, y)$ which implies that their homotopy categories are equivalent which implies that $\mathrm{hoHom}_X(x, y)$ is a groupoid which implies that $\mathrm{Hom}_X(x, y)$ is also a Kan complex.

Also:

Lemma 0.15 (Joyal). *$f \colon \Delta^1 \to X$ is an isomorphism if and only if there exists an extension to $J = N(\bullet \cong \bullet)$.*

Proof. We make use of the following observation: an n-simplex in the nerve of a category is degenerate if and only if one of the edges along its spine is an identity. In particular,

[11] Those of you who have heard me talk about this sort of thing before will know that this isn't my favorite way to think about these hom-spaces: It's the "necklace" characterization of Dugger-Spivak.

there are only two non-degenerate simplices in each dimension in J and furthermore, if σ is a non-degenerate n-simplex, only its 0th and nth faces are non-degenerate.

The map f lands in ιX; it therefore suffices to show that $\Delta^1 \to J$ is anodyne. In fact, we will give a cellular decomposition of this inclusion, building J by attaching a sequence of outer horns. Abusing terminology, we will call the non-degenerate 1-simplex f. The first attaching map $\Lambda^2_2 \to \Delta^1$ has 0th face f and 1st face an identity. Call the 1-simplex obtained by pushing out

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \cdot \end{tikzcd}$$

g. This also defines the non-degenerate 2-simplex whose spine is fg. Next we use the Λ^3_3 horn whose boundary is depicted

$$\begin{tikzcd} & \cdot \arrow[dd, "g"] \arrow[dr, equal] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "f"] \arrow[dr, equal] & & \cdot \\ & \cdot \arrow[ur, "f"'] & \end{tikzcd}$$

to obtain the non-degenerate 2-simplex with spine gf and the non-degenerate 3-simplex with spine fgf. Next attach a Λ^4_4 horn and so on. □

We say that two objects in a quasi-category are **isomorphic** if and only if there is an isomorphism between them. (Exercise: check that this is an equivalence relation.) For instance, suppose $f\colon X \to Y$ is an equivalence between quasi-categories. In particular, it induces isomorphisms

$$[Y,X]_J \xrightarrow{f^*} [X,X]_J \qquad\qquad [Y,Y]_J \xrightarrow{f^*} [X,Y]_J.$$

Considering the first of these, we conclude that the identity on X is isomorphic in the quasi-category X^X to a vertex in the image of f. By Lemma 0.15, this isomorphism is represented by a map as displayed on the left

(0.16)
$$\begin{tikzcd} X \arrow[d] \arrow[dr, "1"] & \\ X \times J \arrow[r] & X \\ X \arrow[u] \arrow[r, "f"'] & Y \arrow[u, "g"'] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d] \arrow[dr, "1"] & \\ Y \times J \arrow[r] & Y \\ Y \arrow[u] \arrow[r, "g"'] & X \arrow[u, "f"'] \end{tikzcd}$$

Post-composing the equivalence with f we see that f and fgf are isomorphic in Y^X. From the second bijection, it follows that fg is isomorphic to the identity on Y via a map as displayed on the right above.

Some more facts whose proofs are now easy exercises:

- **qCat** has a full coreflective subcategory **Kan**. The coreflector takes a quasi-category to its maximal sub Kan complex spanning the isomorphisms.

- Any equivalence restricts to an equivalence between maximal sub Kan complexes.
- Conversely, any weak homotopy equivalence between maximal sub Kan complexes extends to a simplicial homotopy equivalence. The representing 1-simplex is an isomorphism in the hom Kan complex and hence this simplicial homotopy equivalence is a categorical equivalence (which is a priori stronger).

There are some other facts about isomorphisms that I want to mention though these are harder to prove. The proofs I know make use of the marked model structure, which again many of you know about, and in any case I don't want to get into.

Theorem 0.17 (pointwise natural isomorphisms are isomorphisms)**.** *Suppose given a natural transformation, i.e., a diagram $\Delta^1 \to X^A$. If this is a pointwise isomorphism (for each $a \in A$) then it's an isomorphism in X^A.*

This is a really awesome result, which follows essentially from the cartesian closure of the marked model structure.

Theorem 0.18 (inverting diagrams)**.** *Suppose K is any simplicial set and you have a diagram $K \to X$ in a quasi-category whose edges are taken to isomorphisms. Then this diagram admits an extension to the groupoidification $\tilde{K}$.*[12]

Any simplicial set is a colimit indexed over its category of simplicies of the Yoneda embedding. The groupoidification is formed by replacing the Yoneda embedding here by the nerves of the groupoidifications of the ordinal categories.

Quasi-categories and Rezk spaces

Actually what I want to talk about is an analog of the Segal condition and of the completeness condition. We're going to approach this via weighted limits and now seems as good a time as any since I've just secretly brought up weighted colimits. I learned about all of this from Dominic Verity, though it's likely that related ideas have appeared elsewhere.[13]

Let $\mathcal{M}$ be a combinatorial model category (so I can perform left Bousfield localization). You might be familiar with Dugger's procedure to replace this by a Quillen equivalent simplicial model category (which is how we'd get at the quasi-category that has the same homotopy theory). But I want to do something else.

Given a diagram, for us $\mathbb{\Delta}^{\mathrm{op}} \to \mathcal{M}$, a **weighted limit** is something that represents not just cones over the diagram but cones of some arbitrary shape. This is really important for enriched category theory but actually at the moment I don't need the enriched notion of a weighted limit, just the set-based one will do. So what I mean by cones of an arbitrary shape is that at each object of the diagram I can choose how many legs of the cone point toward it and then I can specify what sort of commutativity relations are satisfied by these legs and the maps in the diagram. This is all done by means of a functor $W \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ called the **weight**. The cardinality of the image of $[n]$ in the weight tells us how many legs should be above the object in the image of $[n]$ in the diagram. The maps then say which things compose with which maps in the diagram to which things.

Note of course that in this case the weight is just a simplicial set. Assuming $\mathcal{M}$ is complete, as is the case here, weighted limits always exist and can be computed as the functor cotensor product of the diagram with the weight.

---

[12]Gijs Heuts points out (again) that there is a simpler model categorical proof of this.

[13]Another disclaimer: My memory of his proof is imperfect, so any errors in the following are mine.

Example 0.19. By the Yoneda lemma, the limit of $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{M}$ weighted by Δ^n is just the object X_n.

Example 0.20. By inspection, the limit of $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{M}$ weighted by $\partial\Delta^n$ is the n-th matching object $M_n X$, in other words, the object of boundary data associated to a hypothetical (but possibly non-existent) n-simplex in X.

Some general facts about weighted limits make this second example less surprising. The first observation is that weighted limits are contravariant in the weight. For instance, the matching map $X_n \to M_n X$ is the map between weighted limits induced by the canonical inclusion $\partial\Delta^n \to \Delta^n$. The second, and really the main thing, immediate from the defining universal property that I didn't state, is that weighted limits are cocontinuous in the weight. The simplicial set $\partial\Delta^n$ is built by gluing a collection of $(n-1)$-simplices together along the $(n-2)$-simplices that serve as their pairwise intersections. So the weighted limits is then the limit of the corresponding diagram of objects X_{n-1} and X_{n-2}, which is exactly the usual definition of the matching object. In practice, this means it's easy to define "made-to-order" weights whose weighted limits are whatever you want. The fact that the weight $\partial\Delta^n$ has the "shape" of the thing you're trying to describe in the weighted limit is no coinicidence.

Let me write $\lim^W X$ for these weighted limits. Other common notation (which I secretly prefer) is $\{W, F\}$.

Definition 0.21. Let $\mathcal{M}$ be a model category. Say $X \in \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ is

- **Reedy fibrant** if $\lim^{\Delta^n} X \to \lim^{\partial\Delta^n} X$ is a fibration for all n
- a **Segal space** if it is Reedy fibrant and if $\lim^{\Delta^n} X \to \lim^{\Lambda^n_k} X$ is a trivial fibration for all $0 < k < n$, or equivalently, if $\lim^{\Delta^n} X \to \lim^{\Delta^1\vee\cdots\vee\Delta^1} X$ is a trivial fibration for all n
- a **Rezk space** if it is a Segal space and if $\lim^J X \to \lim^{\Delta^0} X$ is a trivial fibration.

Notes: The first definition is isomorphic to the standard one. The second reduces to the standard one for $\mathcal{M} = \mathbf{sSet}$. Note these maps are automatically fibrations if X is Reedy fibrant because of standard lifting arguments involving adjunctions and the fact that the maps between weights are cofibrations. Here the $\Delta^1 \vee \cdots \vee \Delta^1$ is meant to be the spine of the n-simplex, built by gluing together n 1-simplices along their source and target vertices. By cocontinuity, the corresponding weighted limit of X is exactly the usual $X_1 \times_{X_0} \times \cdots \times_{X_0} X_1$.

Finally, for completeness, note by the example above that $\lim^{\Delta^0} X = X_0$. In the context of quasi-categories or Kan complexes, this $\lim^J X$ is a good candidate for the thing called X^{equiv} before; it's the object of 1-simplices that are equivalences (isomorphisms). Since we already know that this map is a fibration, by the 2-of-3 property we could deduce that it's a weak equivalence iff this is true of the monomorphism $\lim^{\Delta^0} X \to \lim^J X$, which is how the completeness condition (or univalence axiom) is usually stated.

The reason we've stated this in this form is that our goal is to prove the following theorem:

Theorem 0.22 (Verity)**.** *If $\mathcal{M}$ is combinatorial and left proper, then the left Bousfield localization of the Reedy model structure on $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ at the pushout products of generating cofibrations in $\mathcal{M}$ with the generating trivial cofibrations in* **sSet** *gives what we might call the* ***model structure for Rezk objects****. These are exactly the fibrant objects. The result is a tensored, cotensored, and enriched simplicial category that is enriched as a model category over Joyal's model structure for quasi-categories.*

Let's call the axioms analogous to "SM7" with respect to the Joyal model structure "JM7," where we number them so that the only difference is between SM7(iii) and JM7(iii).

For any model category the standard simplicial tensor, cotensor, and enrichment on $\mathcal{M}^{\mathbb{\Delta}^{op}}$ satisfies the common 2/3rds of SM7 and JM7 [Dug01, 4.4-5]. When we localize we change the trivial cofibrations in $\mathcal{M}^{\mathbb{\Delta}^{op}}$ so we have to re-prove SM7(ii), but we have the following simplification:

Lemma 0.23. *Let $\mathcal{K}$ be a tensored, cotensored, and simplicially enriched and a model category.*

- *(i) Given JM7(i), if cotensoring with any simplicial set preserves fibrations between fibrant objects then JM7(ii) holds.*
- *(ii) If $\mathcal{K}$ is left proper, given JM7(i) and JM7(ii), then if for any trivial cofibration $K \to L$ in Joyal's model structure on simplicial sets and any fibrant object $Z \in \mathcal{K}$ the map $Z^L \to Z^K$ is a weak equivalence, then JM7(iii) holds.*

Proof. The proofs of [Dug01, 3.2] for SM7 apply mutatis-mutandis to JM7. □

We use some observations of Hirschhorn, which can be found somewhere in his book. Firstly, if $\mathcal{M}$ is left proper, then so is the Reedy model structure on $\mathcal{M}^{\mathbb{\Delta}^{op}}$ so we can apply Lemma 0.23. If $\mathcal{M}$ is combinatorial, then the Reedy model structure on $\mathcal{M}^{\mathbb{\Delta}^{op}}$ is again so we can localize. By another observation of Hirschhorn, any (Reedy) fibration in the original model structure between fibrant objects in the localized model structure is still a fibration. So to recheck JM7(ii), by Lemma 0.23, we need only check that cotensoring with any simplicial set preserves the new fibrant objects (preservation of the old fibrations being obvious): This follows because taking products with simplicial sets preserve Joyal trivial cofibrations, the Joyal model structure being monoidal with all objects cofibrant. Then JM7(iii) will follow immediately by construction of the localization and Lemma 0.23.

It remains to show that the fibrant objects in the localized model structure are exactly the Rezk objects. It's clear that fibrant objects are complete Segal objects so it remains to show the converse. This is a bit subtle because we have to relate the two variable adjunction define weighted limits to the simplicial model structure but it can be done. The point is, by Reedy fibrancy, the desired lifting thing in simplicial sets is an isofibration between fibrant objects so lifting against an arbitrary trivial cofibration reduces to lifting against the specific ones mentioned above.

Remark. Rezk's model structure for complete Segal spaces (which we've chosen to call **Rezk spaces**) starts with the Quillen's simplicial model structure on **sSet** and then does the localization of Theorem 0.22 — but using a different tensor-cotensor-enrichment structure for bisimplicial sets. The difference between the tensors is that both are defined by restriction the cartesian product to some embedding of **sSet** into $\mathbf{sSet}^{\mathbb{\Delta}^{op}}$ but in one the category of simplicial sets is embedded as constant simplicial objects (Rezk) and in the other as discrete simplicial objects (Verity). If I am understanding this correctly, the conclusion is that the model structure on bisimplicial sets for Rezk objects is enriched in one direction over Quillen's model structure and in the other direction of Joyal's model structure.

Basic category theory of quasi-categories

I should say something about how to do category theory with quasi-categories. Here I'm going to reflect my own personal bias and present things somewhat non-traditionally. This approach is joint work with Verity. I hope our papers will appear soon. All of our definitions of adjunctions, limits, and colimits and so on are the same as those of Joyal/Lurie but we think our approach makes it easier to generalize the proofs from standard category theory to the quasi-categorical context.

If I had to say something general about our strategy it would be that we come as far as possible through enriched category theory, which has the advantage of being already developed and not that hard to use. The philosophy of category theory is that important definitions can be encoded by conditions on maps, i.e., via universal properties, i.e., representably. So now you just have to write these definitions only referring to the hom-spaces (here) between two fixed objects and you've proven a theorem in enriched category theory.

So basically what we do is construct preferred models of things as weighted limits in simplicial sets. There's a general result, quite easy to prove, that says if the weights have a certain form (projectively cofibrant; i.e., built cellularly from representables) then if your diagram is of quasi-categories then the resulting weighted limit is again a quasi-category. Then we translate these simplicially enriched universal properties into (weak) 2-categorical universal properties and do the usual formal category theory.

This is a big story. I guess what I want to do now is tell a part of it I haven't yet talked about locally, which is to explain how to get the spaces to define universal properties representably and tell you some things that are true about them.

The definitions of adjunctions and limits and colimits make use of notion of a slice category so let's start by introducing the quasi-categorical analog. Given $B \xrightarrow{f} A \xleftarrow{g} C$ form

$$\begin{tikzcd} g \downarrow f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta^1} \arrow[d] \\ B \times C \arrow[r, "f \times g"'] & A \times A \end{tikzcd}$$

It's a quasi-category with projections $C \xleftarrow{e_0} g \downarrow f \xrightarrow{e_1} B$ for evaluation at one or other end of the path. Furthermore, there's a canonical representative natural transformation $ge_0 \Rightarrow fe_1$ which I want to represent like this:

$$\begin{tikzcd} g \downarrow f \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

When I draw it in this way I'm actually thinking about just the homotopy class of the path in the homotopy category of the quasi-category $A^{g \downarrow f}$. These things are exactly 2-cells in the (strict) 2-**category of quasi-categories** which is obtained by taking homotopy classes of natural transformations and then forgetting all the higher dimensional cells in the hom-spaces between quasi-categories. It turns out this is a good place to make these definitions.

The point is that these comma quasi-categories are **weak comma objects**, meaning they satisfy a weak universal property. Given any simplicial set X and 2-cell

$$\begin{tikzcd} X \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

there exists some $X \to g \downarrow f$ so that β factors along this map through α. Now these vertices in $g \downarrow f^X$ aren't unique but any two such are isomorphic (i.e., there's an isomorphism between them).

Note that this weak universal property is enough to determine the quasi-category $g \downarrow f$ up to equivalence. In the special case of this construction that will be relevant to the construction of limits and colimits, about more which in a moment, those of you who are more familiar with Lurie's "slicey" or "decalagey" description will be happy to know that those quasi-categories are equivalent to this one, which is the "fat slice" in that case and so satisfy the same universal property. But for definiteness, let us stick with this.

When g or f is an identity, we like to replace it with the name of the object. So for instance, given $f \colon B \rightleftarrows A \colon u$ we could form $f \downarrow A$ and $B \downarrow u$. Again these come with isofibrations to $A \times B$.

Definition 0.24. $f \dashv u$ is an **adjunction of quasi-categories** if and only if there is an equivalence $f \downarrow A \cong B \downarrow u$ over $A \times B$.

Note, because the pullbacks defining these quasi-categories are homotopy pullbacks, we can pull back this equivalence over vertices and get an equivalence $\mathrm{Hom}_A(fb, a) \simeq \mathrm{Hom}_B(b, ua)$ between mapping spaces for any $a \in A$ and $b \in B$.

Note also the right Quillen functor $(-)^X$ preserves everything we're talking about so we can see that adjunctions induce adjunctions between diagram categories. The same is true for precomposition though I'd prove this in a different way.

Using the equivalence and the universal property, the identity 2-cell at u can be used to define the counit, and the identity at f gives the unit, which in turn induce (possibly new) equivalences between the slice quasi-categories. There's a little bit of work here, but the point is we can do it all at once in more generality than I've just described the result.

Example 0.25. Take $A = \Delta^0$ so that f is the unique map and write t for u. What this says it that we have an equivalence $A \downarrow t \cong A$ over A. This $A \downarrow t$, by essentially the same geometry mentioned above but with the domain freed up is equivalent to $A_{/t}$ whose n-simplices are arbitrary $n+1$-simplices with last vertex t. Now the 2-of-3 property says that the projection $A_{/t} \to A$ is a trivial fibration which says that it lifts against any sphere inclusion which says any sphere (bumping up dimensions) in A with last vertex t has a filler. This is to say that $t \in A$ is a **terminal object**. So we've shown that terminal objects are characterized by adjunction, just like in general category theory.

I'd like to say a bit about how general limits and colimits work. We begin with a general definition.

Definition 0.26. In a 2-category, an *absolute right lifting diagram* consists of the data

(0.27)
$$\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[ur, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd}$$

with the following universal property: given any 2-cell χ there exists a unique factorization as displayed below.

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "\Downarrow\chi"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, shift left=3, "\exists!\Downarrow"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] \arrow[ur, "\ell"] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd}$$

Example 0.28. The counit of an adjunction $f \dashv u$ defines an absolute right lifting diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, "\mathrm{id}_A"'] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

and, conversely, this data defines an adjunction.

Interpreting (0.27) in $\mathbf{qCat}_2$ permits us to form comma objects $C \downarrow \ell$ and $g \downarrow f$ with canonical cones as displayed.

$$\begin{tikzcd} C \downarrow \ell \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\gamma"] & C \\ B \arrow[ur, "\ell"'] & \end{tikzcd} \qquad \begin{tikzcd} g \downarrow f \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd} = \begin{tikzcd} g \downarrow f \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[dr, phantom, shift left=2, "\exists!\Downarrow"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] \arrow[ur, "\ell"] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd}$$

Pasting the canonical cone under $C \downarrow \ell$ onto λ defines a map $C \downarrow \ell \to g \downarrow f$. The universal property of the absolute right lifting diagram applied to α defines a 2-cell under $g \downarrow f$ and over ℓ, displayed on the right above, which induces a map $g \downarrow f \to C \downarrow \ell$.

The following proposition makes two assertions. Firstly, the universal property of the absolute right lifting (ℓ, λ) implies these maps are equivalences. The second assertion is that if the map $C \downarrow \ell \to g \downarrow f$, definable without ascribing any universal property to λ, is an equivalence, then (ℓ, λ) defines an absolute right lifting diagram.

Theorem 0.29. *The data of* (0.27) *defines an absolute right lifting in* $\mathbf{qCat}_2$ *if and only if the induced maps form an equivalence* $C \downarrow \ell \simeq g \downarrow f$. *Conversely, any equivalence induces 2-cells as displayed above which can be used to define maps between comma objects which are again an equivalence.*

Definition 0.30. A **limit** of a diagram $d\colon X \to A$ is an absolute right lifting diagram

(0.31)
$$\begin{tikzcd} & A \arrow[d, "\mathrm{const}"] \\ \Delta^0 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^X \end{tikzcd}$$

and conversely, or equivalently, it's an equivalence $A \downarrow \ell \cong \mathrm{const} \downarrow d$ over A.

Note the thing on the left-hand side has an obvious terminal object, namely the identity at ℓ, which passes across to a terminal object in the quasi-category of cones, which is the Lurie definition (in the equivalent "slicey" version).

A key advantage of this 2-categorical definition of (co)limits in any quasi-category is that it permits us to use standard 2-categorical arguments to give easy proofs of the expected categorical theorems.

Proposition 0.32. *Right adjoints preserve limits.*

Let's briefly recall the classical categorical proof. Given a diagram $X \xrightarrow{d} A$ and a right adjoint $A \xrightarrow{u} B$ to some functor f, a cone with summit b over ud transposes to a cone with summit fu over d, which factors uniquely through the limit cone. This factorization transpose back across the adjunction to show that the image of the limit cone under u defines a limit over ud.

Proof. Given an absolute right lifting diagram (0.31), an adjunction of quasi-categories $f \dashv u$, and hence an adjunction $f^X \dashv u^X$, we must show that

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "c"] \arrow[dl, phantom, "\Downarrow\lambda" near end] & B \arrow[d, "c"] \\ \Delta^0 \arrow[ur, "\ell"] \arrow[r, "d"'] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$

is an absolute right lifting diagram. Given a cone

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$

we first transpose across the adjunction, by composing with f and the counit.

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[r, "f"] \arrow[d] & A \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"] \arrow[rr, bend right, equal, "\Downarrow\epsilon^X"'] & B^X \arrow[r, "f^X"] & A^X \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d] \arrow[dr, phantom, "\exists!\Downarrow\zeta" near start] & B \arrow[r, "f"] & A \arrow[d] \\ \Delta^0 \arrow[urr, "\ell"] \arrow[rr, "d"'] & \arrow[ur, phantom, "\Downarrow\lambda"] & A^X \end{tikzcd}$$

Applying the universal property of the limit cone λ produces a factorization ζ, which may then be transposed back across the adjunction by composing with u and the counit.

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d] \arrow[dr, phantom, "\exists!\Downarrow\zeta" near start] & B \arrow[r, "f"] \arrow[rr, bend left, equal, "\Downarrow\eta"] & A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ \Delta^0 \arrow[urr, "\ell"] \arrow[rr, "d"'] & \arrow[ur, phantom, "\Downarrow\lambda"] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[r, "f"'] \arrow[d] \arrow[rr, bend left, equal, "\Downarrow\eta"] & A \arrow[r, "u"'] \arrow[d] & B \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"] \arrow[rr, bend right, equal, "\Downarrow\epsilon^X"'] & B^X \arrow[r, "f^X"] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$

$$= \begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[rr, bend left, equal] \arrow[d] & & B \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"] \arrow[rr, bend right, equal, "\Downarrow\epsilon^X"'] & B^X \arrow[r, "f^X"] \arrow[rr, bend left, equal, "\Downarrow\eta^X"] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & \arrow[d, phantom, "\Downarrow\chi"] & B \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$

Here the second equality is immediate from the definition of η^X and the third is by the triangle identity defining the adjunction $f^X \dashv u^X$. The pasted composite of ζ and η is the desired factorization of χ through λ. The proof that this factorization is unique is left to the reader. It again parallels the classical argument: the essential point is that the transposes are unique. □

Fibrational perspective

To the best of my understand, the real innovation of the Lurie approach to quasi-category theory, extending the Joyal one, is his use of what might be called the "fibrational perspective," which involves a quasi-categorical generalization of the so-called "Grothendieck construction." This is a big story that I hope someone will talk about in more detail. Here let me just pave the way with some very elementary observations about the use of fibrations in model category theory and in quasi-category theory.

The following result implies that right derived functors of right Quillen functors can be constructed by precomposing with fibrant replacement.

Lemma 0.33 (Ken Brown's lemma)**.** *Any functor that preserves trivial fibrations between fibrant objects preserves weak equivalences between fibrant objects.*

Proof. In any model category, given any map $f\colon X \to Y$ between fibrant objects, it is possible to construct a fibrant object Z, fibrations

$$\begin{tikzcd} & Z \arrow[dl, two heads, "p"] \arrow[dr, two heads, "q"] & \\ X \arrow[ur, bend left, "j"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

and a section j to p that factors f as qj. When f is a weak equivalence, these maps are all weak equivalences, and the conclusion follows from the hypothesis by a straightforward application of the 2-of-3 property. ◻

Another way to think about fibrations is that they allow one to "avoid making choices."

Construction 0.34 (composition in a quasi-category)**.** We have a pushout in simplicial sets:[14]

$$\begin{tikzcd} \Delta^0 \arrow[r, "d^1"] \arrow[d, "d^0"'] & \Delta^1 \arrow[d] \\ \Delta^1 \arrow[r] & \Lambda^2_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Homing into a quasi-category X, by adjunction, turns this pushout into a pullback

$$\begin{tikzcd} & & X^{\Lambda^2_1} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & X^{\Delta^1} \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "t"] & & X^{\Delta^1} \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "t"] & \\ X & & X & & X \end{tikzcd}$$

all of whose maps are fibrations. We've relabeled the maps induced by $d^1, d^0\colon [0] \rightrightarrows [1]$ as the source and target projections respectively. We think of $X^{\Lambda^2_1}$ as the quasi-category of composable arrows in X and the composite fibrations displayed as the various projections to the source, middle object, target, first factor, and last factor.

Now the combinatorics discussed above implies that the Segal map $X^{\Delta^2} \xrightarrow{\sim} X^{\Lambda^2_1}$ is a trivial fibration. In particular, there exists a non-canonically defined section which can be used to construct a composition map $X^{\Lambda^2_1} \xrightarrow{\circ} X^{\Delta^1}$ compatible with the source and target projections.

$$\begin{tikzcd} & X^{\Delta^2} \arrow[dl, two heads, "\sim"] \arrow[dr, two heads, "d^1"] & \\ X^{\Lambda^2_1} \arrow[ur, dashed, bend left] \arrow[d, two heads, "s"'] \arrow[drr, two heads, "t"' near end] & & X^{\Delta^1} \arrow[d, two heads, "t"] \arrow[dll, two heads, "s" near end] \\ X & & X \end{tikzcd}$$

[14]This example shows that quasi-categories are not closed under colimits.

References

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy Limits, Completions and Localizations.* Lecture Notes in Mathematics, 1972.

[Dug01] D. Dugger. *Replacing model categories with simplicial ones.* Trans. Amer. Math. Soc. 353 (2001), no. 12, 5003–5027.

[DS11] D. Dugger and D. Spivak. Mapping spaces in quasi-categories. Algebr. Geom. Topol. 11 (2011), no. 1, 263–325.

[Joy02] A. Joyal. *Quasi-categories and Kan complexes.* J. Pure Appl. Algebra 175 (2002), no. 1-3, 207–222.

[Lur09] J. Lurie. *Higher Topos Theory*. Annals of Mathematics Studies, 170, Princeton University Press, 2009.

[Rie13] E. Riehl. *Categorical homotopy theory.* Available at www.math.harvard.edu/~eriehl/cathtpy.pdf.

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`