THE UNIVERSITY OF CHICAGO

ALGEBRAIC MODEL STRUCTURES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE 2011

To L, who let me go.

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Acknowledgments

When I was a Part III student at Cambridge, Martin Hyland agreed to advise me in writing an essay on higher category theory. Knowing I was intended for Peter May at Chicago, he suggested I learn about model categories and pointed me towards a recent paper of Richard Garner, a former student, that modified Quillen's small object argument to produce functorial factorizations with better categorical properties, constructing what are now called *algebraic weak factorization systems*. At the time I had absolutely no idea how any of this worked, but this early exposure laid the foundations to pique my interest when, a year and a half later, Mike Shulman, then a graduate student at Chicago, asked whether algebraic weak factorization systems might contribute to a model structure on algebraic Kan complexes. After some thought, I realized I did not know what it meant to have algebraic weak factorizations systems in a model structure, and this project was born.

I would like to thank Martin Hyland for setting me down this road; Peter May for pushing me to justify this use of my time, for countless productive conversations which have made me a better and a broader mathematician, and for his general support and encouragement; Mike Shulman, for adding categorical sophistication to this theory when my own background proved insufficient; Richard Garner, for insights that allowed me to vastly simplify several proofs and for his small object argument which makes this work possible; Dominic Verity, who abetted my escape of a Chicago winter, for teaching me everything that doesn't appear here; Eugenia Cheng and the group at Sheffield, for inviting me speak about these topics during the final writing phase, inspiring a much better presentation of some of the later material; and finally the group at Chicago, who made the past four years pass far too quickly.

I am also grateful for the support of the Winston Churchill Foundation, which allowed me to spend the year at Cambridge where I first became enamored of these topics, and the National Science Foundation, whose Graduate Research Fellowship significantly accelerated this work. Finally, I would like to thank my family and friends for their love, patience, and support.

Abstract

In Part I of this thesis, we introduce *algebraic model structures*, a new context for homotopy theory in which the cofibrations and fibrations are retracts of coalgebras for comonads and algebras for monads and prove "algebraic" analogs of classical results. Using a modified version of Quillen's small object argument, we show that every cofibrantly generated model structure in the usual sense underlies a cofibrantly generated algebraic model structure. We show how to pass a cofibrantly generated algebraic model structure across an adjunction, and we characterize the *algebraic Quillen adjunction* that results. We prove that pointwise algebraic weak factorization systems on diagram categories are cofibrantly generated if the original ones are, and we give an algebraic generalization of the projective model structure. Finally, we prove that certain fundamental comparison maps present in any cofibrantly generated model category are cofibrations when the cofibrations are monomorphisms, a conclusion that does not seem to be provable in the classical, non-algebraic, theory.

In Part II, we define monoidal algebraic model structures and discuss examples. The main structural component requires a new notion: an *algebraic Quillen two-variable adjunction*. The principal technical work is to develop the category theory necessary to define and characterize them. Our investigations reveal an important role played by "cellularity"—loosely, the property of a cofibration being a relative cell complex, not simply a retract of such—which we particularly emphasize. A main result is a simple criterion which shows that algebraic Quillen two-variable adjunctions correspond precisely to cellular structures on the pushout-products of generating (trivial) cofibrations, extending a similar result from Part I.

Preface

Quillen's model structures axiomatize a general framework for homotopy theory, providing tools to study the objects of a category up to some specified notion of weak equivalence. In a model category, the interactions between the weak equivalences and two additional classes of morphisms—the *cofibrations* and the *fibrations*—enable a concrete construction of the *homotopy category*; the category obtained by restricting to the *fibrant-cofibrant* objects and taking homotopy classes of maps is equivalent to the formal localization at the weak equivalences. Furthermore, model structures enable one to perform constructions at the "point-set level"—i.e., in the original category, which typically has more limits and colimits than the homotopy category—that are homotopically meaningful in the sense that they agree with their derived functors.

For these reasons, model categories have proliferated far beyond the original topological setting. But despite their popularity, ordinary model structures only provide a context for homotopy theory and don't contribute directly to calculations. In this thesis we present a new, more rigid, *algebraic* extension of Quillen's notion that has this calculational capacity because certain structures previously only supposed to exist are now specified. This thesis lays the foundations for the theory of *algebraic model structures*, introducing a number of basic definitions, describing conditions that give rise to important examples, and discussing general features.

Algebraic model structures, introduced in Part I of this thesis, provide a setting for homotopy theory in which the trappings of an ordinary model structure—the cofibrations, fibrations, trivial cofibrations, and trivial fibrations—can be thought of algebraically: as maps together with extra data witnessing their membership in the particular class. Each algebraic model structure determines an underlying ordinary model category, in the sense of Quillen [Qui67], but one whose functorial factorizations have richer algebraic structure. Despite the fact that this new definition is much stricter than the original, an algebraic model structure exists whenever an ordinary model structure is cofibrantly generated. The abundance of such examples means that this theory offers a new perspective on classical model categories, which we believe will be of interest to experts, even if they see no immediate need for the algebraic structures we can guarantee.

What does it mean to consider fibrations algebraically? A classical example is provided by the Hurewicz fibrations, which can be characterized as those maps of spaces that admit a path lifting function. Importantly, a path lifting function assigned to some map p is precisely an *algebra structure* for p with respect to a particular monad on the *arrow category*, whose objects are maps of spaces and whose morphisms are commutative squares [May75]. The monad is constructed from the Moore paths monad and describes, among other things, a functorial factorization of each arrow as a cofibration and homotopy equivalence followed by a fibration. The monad on the arrow category is the functor that replaces a map by a fibration, its right factor.

Furthermore, the functor that sends a map to its left factor with respect to this functorial factorization, the trivial cofibration part, is itself a comonad on the arrow category. Its *coalgebras* are necessarily homotopy equivalences and Hurewicz cofibrations. Furthermore, the coalgebra structure assigned to a particular trivial cofibration can be used to construct a canonical solution to any lifting problem against a fibration equipped with a path lifting function. These are the general features of what it means to describe the trivial cofibrations and fibrations as "algebraic": such maps are equipped with specified retractions to their left or right factors which can be used to solve all lifting problems.

In an algebraic model structure, both functorial factorizations have this flavor: the right factors are monads and the left factors are comonads. Algebras are necessarily fibrations and trivial fibrations, and coalgebras are necessarily trivial cofibrations and cofibrations, respectively. Additionally, a natural comparison between the two functorial factorizations induces functors that map algebraic trivial cofibrations to algebraic cofibrations and algebraic trivial fibrations to algebraic trivial fibrations. These functors compatibly assign a canonical solution to any lifting problem between an algebraic trivial cofibration and an algebraic trivial fibration.

The data of an algebraic model structure determines a fibrant replacement monad and cofibrant replacement comonad; algebras and coalgebras are *algebraic fibrant objects* and *algebraic cofibrant objects*. Furthermore, the canonical solution to a particular lifting problem defines a distributive law of the monad over the comonad, which specifies the *algebraic fibrant-cofibrant objects*. Future work will explore the implications of the previously unnoticed fact that any cofibrantly generated model category has a fibrant replacement monad and cofibrant replacement comonad, which, in particular, allows one to construct (co)simplicial resolutions using point-set derived functors.

A paradigmatic cofibrantly generated algebraic model structure is Quillen's original model structure on spaces, generated by the sets $\mathcal{I} = \{S^{n-1} \to D^n\}$ and $\mathcal{J} = \{D^n \to D^n \times I\}$. An algebraic fibration is a Serre fibration equipped with chosen lifts of cylinders in the base space extending specified disks in the total space. An algebraic trivial fibration is a map equipped with lifted contractions filling spheres in the total space which are contractible in the base. Such maps are necessarily both Serre fibrations and weak homotopy equivalences; that is, they are necessarily trivial fibrations in the underlying ordinary model structure. Conversely, all trivial fibrations admit some algebra structure: one need only make a choice among the various fillers for such spheres which necessarily exist. Indeed, whenever the algebraic model structure is cofibrantly generated, all fibrations and all trivial fibrations are algebraic.

Interestingly, the dual statements do not hold: the cofibrations and trivial cofibrations need not admit coalgebra structures, even when the algebraic model structure is cofibrantly generated. Accordingly, we say a cofibration is *cellular* if it admits a coalgebra structure, that is, if there is some algebraic cofibration with this underlying map. The name is motivated by the most familiar example: a relative cell complex is a map of spaces, necessarily a cofibration, that can be described by repeatedly attaching cells to the domain. Such a description is called a *cellular decomposition* for the map and defines a coalgebra structure for the comonad. By convention, whenever two cells can be attached "at the same time" in a particular cellular decomposition, we do so, so that the filtration defined by the cellular decomposition will have fewer, in fact typically only countably many, stages. As is familiar for CW-complexes, a special case, the resulting coalgebra structures are not at all unique, even granting this convention.

Not all cofibrations for this model structure are cellular. The best one can say is that the cofibrations are retracts of relative cell complexes, that is, retracts of cellular cofibrations. Analogously, in any algebraic model category, any cofibration is a retract of a cellular one. More specifically, any cofibration is a retract of a cofibration admitting a free coalgebra structure, namely, its left factor in the cofibration–trivial fibration factorization.

For some algebraic model structures, the cellular cofibrations behave quite differently. For example, in Quillen's model structure on simplicial sets, generated by the sphere and horn inclusions, every cofibration is cellular and furthermore admits a unique coalgebra structure. Here the cofibrations are precisely the monomorphisms and the filtration defined by their cellular decomposition is always countable. The first step attaches all simplices whose boundary appears in the domain. In particular this includes all 0-simplices of the codomain. The second step attaches all simplices whose boundary is present at the end of stage one; in particular, this includes all 1-simplices. In this way, we see that the n-th simplicial set of the resulting filtration contains at least the $(n-1)$-skeleton of the codomain.

When comparing algebraic model structures on different categories, cellularity is more than just a curiosity. A cellularity condition precisely characterizes *algebraic Quillen adjunctions*, a rigid analog of the classical notion. In an ordinary Quillen adjunction, the left adjoint preserves the (trivial) cofibrations and the right adjoint preserves the (trivial) fibrations. In an algebraic Quillen adjunction, a new definition introduced below, the left adjoint lifts to a functor between the algebraic (trivial) cofibrations and the right adjoint lifts to a functor between the algebraic (trivial) fibrations, and furthermore these lifts determine each other, in a sense that is rather delicate to make precise. Modulo a compatibility condition which is not the main point, an adjunction is an algebraic Quillen adjunction if and only if the images of the generating cofibrations and trivial cofibrations under the left adjoint are cellular with respect to the appropriate comonads. Furthermore, a choice of cell structures completely determines the lifted functors and hence the full algebraic Quillen adjunction.

The geometric realization–total singular complex adjunction between simplicial sets and spaces is a member of a large class of examples. The geometric realizations of the generating cofibrations are homeomorphic to the elements of the set $\mathcal{I}$; we assign these maps the simplest possible $\mathcal{I}$-cellular structures. Similarly, the geometric realizations of the horn inclusions are homeomorphic to the elements of $\mathcal{J}$; again, we assign these maps the simplest $\mathcal{J}$-cellular structures. In this case, the naturality condition, glossed over above, demands a good choice of $\mathcal{I}$-cellular structures for the realizations of the horn inclusions: first attach the "missing face" to the geometrically realized horn to make a sphere, and then fill the sphere.

The theme of cellularity, entirely present but not fully apparent to the author in Part I [Rie11], is unmistakable in Part II, which introduces monoidal algebraic model structures, algebraicizing the definition of [Hov99]. Extending the results for the single-variable case, algebraic Quillen two-variable adjunctions exist precisely when the pushout-products of the generating cofibrations and trivial cofibrations are cellular; furthermore, these cellular structures completely determine the algebraic Quillen two-variable adjunctions. The main technical work of Part II is to precisely state the definition of an algebraic Quillen two-variable adjunction and prove these theorems, which are much harder than the single-variable versions, but once this technical work is complete, the definition of a monoidal algebraic model structure is apparent. Monoidal model structures necessarily precede consideration of enriched ones but also inherit nearly all of their complexity, so the definition of an enriched algebraic model structure is also evident. However, we leave this topic for another paper where we have time to more fully consider examples.

In order to state the definitions and prove the theorems appearing in this thesis, we had to develop a fair amount of pure category theory. The precise definition of an algebraic Quillen adjunction proved particularly fruitful, requiring a new notion of morphism between the categorical components of an algebraic model structure. Our discovery of the correct notion was motivated by a desire to replicate the classical situation where a Quillen adjunction can be detected by considering the left adjoint and the (trivial) cofibrations or the right adjoint and the (trivial) fibrations alone. Alternatively, these definitions can be intuited from a purely categorical lens. The components of an algebraic Quillen adjunction are lax morphisms of monads and colax morphisms of comonads related by the calculus of mates [KS74], which governs the interactions between the monads and comonads arising from the same functorial factorization.

An alternate approach uses the fact that the (co)algebra structures assigned to a composable pair of (co)fibrations are themselves composable in such a way that the algebraic (co)fibrations become a double category; algebraic Quillen adjunctions consist of compatible lifted double functors. In fact, the main reason the proofs in Part II are much harder than the corresponding arguments in Part I is that the functors between arrow categories arising from a two-variable adjunction do not preserve composability of underlying maps, much less (co)algebras. Thus, in the two-variable case, we are forced to rely upon the mates approach, but even here the necessary category theory does not exist. To describe the correct correspondence between the three lifted functors of an algebraic two-variable adjunction, we introduce *parameterized mates* and prove a few fundamental lemmas. In this paper, we consider only two-variable adjunctions, the case of interest, but the same ideas extend to n-variable adjunctions and can be used to define the 2-cells of a category object in the category of multicategories of a particular sort. Of all the topics developed here, we believe this to be of greatest independent categorical interest.

Part I

Algebraic model structures

I.1 Introduction

Weak factorization systems are familiar in essence if not in name to algebraic topologists. Loosely, they consist of left and right classes of maps in a fixed category that satisfy a dual lifting property and are such that every arrow of the category can be factored as a left map followed by a right one. Neither these factorizations nor the lifts are unique; hence, the adjective "weak." Two weak factorization systems are present in Quillen's definition of a model structure [Qui67] on a category. Indeed, for any weak factorization system, the left class of maps behaves like the cofibrations familiar to topologists while the right class of maps behaves like the dual notion of fibrations.

Category theorists have studied weak factorization systems in their own right, often with other applications in mind. From a categorical point of view, weak factorization systems, even those whose factorizations are described *functorially*, suffer from several defects, the most obvious of which is the failure of the left and right classes to be closed under all colimits and limits, respectively, in the arrow category.

Algebraic weak factorization systems, originally called *natural* weak factorization systems, were introduced in 2006 by Marco Grandis and Walter Tholen [GT06] to provide a remedy. In an algebraic weak factorization system, the functorial factorizations are given by functors that underlie a comonad and a monad, respectively. The left class of maps consists of coalgebras for the comonad and the right class consists of algebras for the monad. The algebraic data accompanying the arrows in each class can be used to construct a canonical solution to any lifting problem that is natural with respect to maps of coalgebras and maps of algebras. A classical construction in the same vein is the path lifting functions which can be chosen to accompany any Hurewicz fibration of spaces [May75].

More recently, Richard Garner adapted Quillen's small object argument so that it produces algebraic weak factorization systems [Gar07, Gar09], while simultaneously simplifying the functorial factorizations so constructed. In practice, this means that whenever a model structure is cofibrantly generated, its weak factorization systems can be "algebraicized" to produce algebraic weak factorization systems, while the underlying model structure remains unchanged.

The consequences of this possibility appear to have been thus far unexplored. This paper begins to do so, although the author hopes this will be the commencement, rather than the culmination, of an investigation into the application of algebraic weak factorization systems to model structures. At the moment, we do not have particular applications in mind to justify this extension of classical model category theory. However, these extensions feel correct from a categorical point of view, and we are confident that suitable applications will be found.

Section I.2 contains the necessary background. We review the definition of a weak factorization system and state precisely what we mean by a functorial factorization. We then introduce algebraic weak factorization systems and describe a few important properties. We explain what it means for a algebraic weak factorization system to be cofibrantly generated and prove a lemma about such factorization systems that will have many applications. More details about Garner's small object argument, including a comparison with Quillen's, are given later, as needed.

Section I.3 is in many ways the heart of this paper. To begin, we define an *algebraic model structure*, that is, a model structure built out of algebraic weak factorization systems instead of ordinary ones. One feature of this definition is that it includes a notion of a natural comparison map between the two functorial factorizations. As an application, one obtains a natural arrow comparing the two fibrant-cofibrant replacements of an object, which can be used to construct a category of algebraically bifibrant objects in our model structure. We prove that cofibrantly generated algebraic model structures can be passed across an adjunction, generalizing a result due to Daniel Kan.

The adjunction between the algebraic model structures in this situation has many interesting properties, consideration of which leads us to define an *algebraic Quillen adjunction*. For such adjunctions, the right adjoint lifts to a functor between the categories of algebras for each pair of algebraic weak factorization systems, which should be thought of as an algebraization of the fact that Quillen right adjoints preserve fibrations and trivial fibrations. Furthermore, the lifts for the fibrations and trivial fibrations are natural, in the sense that they commute with the functors induced by the comparison maps. Dually, the left adjoint lifts to functor between the categories of coalgebras and these lifts are natural. In order to prove that the adjunction described above is an algebraic Quillen adjunction, we must develop a fair bit of theory, a task we defer to later sections.

In Section I.4, we describe the pointwise algebraic weak factorization system on a diagram category and prove that it is cofibrantly generated whenever the inducing one is. This result is only possible because Garner's small object argument allows the "generators" to be a category, rather than simply a set. One place where such algebraic weak factorization systems appear is in Lack's trivial model structure on certain diagram 2-categories [Lac07], and consequently, these algebraic model structures are cofibrantly generated in the new sense, but not in the classical one. We then use the pointwise algebraic weak factorization system together with the work of Section I.3 to obtain a generalization of the projective model structure on a diagram category.

In Section I.5, we showcase some advantages of algebraicizing cofibrantly generated model structures. Using the characterization of cofibrations and fibrations as coalgebras and algebras, we have techniques for recognizing cofibrations constructed as colimits and fibrations constructed as limits that are not available otherwise. We use these techniques to prove the surprising fact that the natural comparison map between the algebraic weak factorization systems of a cofibrantly generated algebraic model category consists of pointwise cofibration coalgebras, at least when the cofibrations in the model structure are monomorphisms. We conclude by applying these techniques to prove that the fibrant replacement monad in this setting preserves certain trivial cofibrations, a fact relevant to the study of categories of algebraically fibrant objects, some of which can be given their own lifted algebraic model structure by recent work of Thomas Nikolaus [Nik10].

In Section I.6, we begin to develop the theory necessary to prove the existence of an important class of algebraic Quillen adjunctions. First, we describe what happens when we have an adjunction between categories with related algebraic weak factorization systems, such that the generators of the one are the image of the generators of the other under the left adjoint, a question that turns out to have a rather complicated answer. In this setting, the right adjoint lifts to a functor between the categories of algebras for the monads of the algebraic weak factorization systems and dually the left adjoint lifts to a functor between the categories of coalgebras, though the proof of this second fact is rather indirect. To provide appropriate context for understanding this result and as a first step towards its proof, we present three general categorical definitions describing comparisons between algebraic weak factorization systems on different categories. The first two definitions, of lax and colax morphisms of algebraic weak factorization systems, combine to give a definition of an adjunction of algebraic weak factorization systems, which is the most important of these notions.

The most expeditious proofs of these results make use of the fact that the categories of algebras and coalgebras accompanying an algebraic weak factorization system each have a canonical composition law that is natural in a suitable double categorical sense; in particular each algebraic weak factorization system gives rise to two double categories, whose vertical morphisms are either algebras or coalgebras and whose squares are morphisms of such. This composition, introduced in Section I.2, provides a recognition principle that identifies an algebraic weak factorization system from either the category of algebras for the monad or the category of coalgebras for the comonad. As a consequence, it suffices in many situations to consider either the comonad or the monad individually, which is particularly useful here.

The existence of adjunctions of cofibrantly generated algebraic weak factorization systems demands an extension of the universal property of Garner's small object argument. We conclude Section I.6 with a statement and proof of the appropriate change-of-base result, which we use to compare the outputs of the small object argument on categories related by adjunctions. This extension is not frivolous; a corollary provides exactly the result we need to prove the naturality statement in the main theorem of the final section.

In Section I.7, we apply the results of the previous section to prove that there is a canonical algebraic Quillen adjunction between the algebraic model structures constructed at the end of Section I.3. The data of this algebraization includes five instances of adjunctions between algebraic weak factorization systems. Two of these are given by the comparison maps for each algebraic model structure. The other three provide an algebraic description of the relationship between the various types of factorizations on the two categories.

For convenience, we'll abbreviate algebraic weak factorization system as *awfs*, which will also be the abbreviation for the plural, with the correct interpretation clear from context. Similarly, we write *wfs* for the singular or plural of weak factorization system. The wfs mentioned in this paper beyond Section I.2.1 underlie some awfs and are therefore functorial.

The author would like to thank Peter May for feedback on innumerable drafts of this paper. The author is also grateful for several conversations with Mike Shulman and Richard Garner, some of the results of which are contained in Theorem I.5.1 and Lemma I.5.3. The latter also conjectured Lemma I.6.9, which enabled a simplification of the initial proof of Theorem I.6.15, while the former also commented on an earlier draft of this paper and suggested the definitions of Section I.6 and the statement and proof of Corollary I.6.17. Anna Marie Bohmann suggested the notation for the natural transformations involved in an awfs.

I.2 Background and recent history

There are many sources that describe the basic properties of weak factorization systems of various stripes (e.g., [KT93] or [RT02]). We choose not to give a full account here and only include the topics that are most essential.

First some notation. We write **n** for the category associated to the ordinal n as a poset, i.e., the category with n objects $0, 1, \ldots, n-1$ and morphisms $i \to j$ just when $i \leq j$. Let $d^0, d^1, d^2 \colon \mathbf{2} \to \mathbf{3}$ denote the three functors which are injective on objects; the superscript indicates which object is not contained in the image. Precomposition induces functors $d_0, d_1, d_2 \colon \mathcal{M}^{\mathbf{3}} \to \mathcal{M}^{\mathbf{2}}$ for any category $\mathcal{M}$; where we write $\mathcal{M}^{\mathcal{A}}$ for the category of functors $\mathcal{A} \to \mathcal{M}$ and natural transformations. We refer to d_1 as the "composition functor" because it composes the two arrows in the image of the generating non-identity morphisms of **3**.

Definition I.2.1. We are particularly interested in the category $\mathcal{M}^{\mathbf{2}}$, sometimes known as the *arrow category* of $\mathcal{M}$. Its objects are arrows of $\mathcal{M}$, which we draw vertically, and its morphisms $(u, v) \colon f \Rightarrow g$ are commutative squares[1]

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

There are canonical forgetful functors $\mathrm{dom}, \mathrm{cod} \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ that project to the top and bottom edges of this square, respectively.

The material in Sections I.2.1 and I.2.2 is well-known to category theorists at least, while the material in Sections I.2.3 - I.2.6 is fairly new. Naturally, we spend more time in the latter sections than in the former.

I.2.1 Weak factorization systems

Colloquially, a *weak factorization system* consists of two classes of arrows, the "left" and the "right", that have a lifting property with respect to each other and satisfy a factorization axiom. The lifting property says that whenever we have a commutative square as in (I.2.2) with l in the left class of arrows and r in the right, there exists an arrow w as indicated so that both triangles commute.

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[1] We depict morphisms of $\mathcal{M}^{\mathbf{2}}$ with a double arrow because (u, v) is secretly a natural transformation between the functors $f, g \colon \mathbf{2} \to \mathcal{M}$, though we do not often think of it as such.

Notation. When every lifting problem of the form posed by the commutative square

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\
\cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot
\end{tikzcd} \tag{I.2.2}$$

has a solution w, we write $l \boxslash r$ and say that l has the *left lifting property* (LLP) with respect to r and, equivalently, that r has the *right lifting property* (RLP) with respect to l. If $\mathcal{A}$ is a class of arrows, we write $\mathcal{A}^{\boxslash}$ for the class of arrows with the RLP with respect to each arrow in $\mathcal{A}$. Similarly, we write ${}^{\boxslash}\mathcal{A}$ for the class of arrows with the LLP with respect to each arrow in $\mathcal{A}$.

In general, $\mathcal{A} \subset {}^{\boxslash}\mathcal{B}$ if and only if $\mathcal{B} \subset \mathcal{A}^{\boxslash}$; in this situation, we write $\mathcal{A} \boxslash \mathcal{B}$ and say that $\mathcal{A}$ has the LLP with respect to $\mathcal{B}$ and, equivalently, that $\mathcal{B}$ has the RLP with respect to $\mathcal{A}$. The operations $(-)^{\boxslash}$ and ${}^{\boxslash}(-)$ form a Galois connection with respect to the posets of classes of arrows of a category, ordered by inclusion.

Definition I.2.3. A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ consists of classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ such that

(i) Every morphism f in $\mathcal{M}$ factors as $r \cdot l$, with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

(ii) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Any class $\mathcal{L}$ that equals ${}^{\boxslash}\mathcal{R}$ for some class $\mathcal{R}$ is *saturated*, which means that $\mathcal{L}$ contains all isomorphisms and is closed under coproducts, pushouts, transfinite composition, and retracts. The class $\mathcal{R} = \mathcal{L}^{\boxslash}$ has dual closure properties, which again has nothing to do with the factorization axiom. The following alternative definition of a weak factorization system is equivalent to the one given above.

Definition I.2.4. A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{M}$ consists of classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ such that

(i) Every morphism f in $\mathcal{M}$ factors as $r \cdot l$, with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

(ii) $\mathcal{L} \boxslash \mathcal{R}$.

(iii) $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

Any model structure provides two examples of weak factorization systems: one for the trivial cofibrations and the fibrations and another for the cofibrations and trivial fibrations. Indeed, a particularly concise definition of a model structure on a complete and cocomplete category $\mathcal{M}$ is the following: a *model structure* consists of three class of maps $\mathcal{C}$, $\mathcal{F}$, $\mathcal{W}$ such that $\mathcal{W}$ satisfies the 2-of-3 property and such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are wfs.[2]

I.2.2 Functorial factorization

Definition I.2.5. A *functorial factorization* is a functor $\vec{E} \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{3}}$ that is a section of the "composition" functor $d_1 \colon \mathcal{M}^{\mathbf{3}} \to \mathcal{M}^{\mathbf{2}}$.

Explicitly, a functorial factorization consists of a pair of functors $L, R \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{2}}$ such that $f = Rf \cdot Lf$ for all morphisms $f \in \mathcal{M}$ and such that the following three conditions hold:

$$\mathrm{cod} L = \mathrm{dom} R, \qquad \mathrm{dom} L = \mathrm{dom}, \qquad \mathrm{cod} R = \mathrm{cod}.$$

Together, $L = d_2 \circ \vec{E}$ and $R = d_0 \circ \vec{E}$ contain all of the data of the functor $\vec{E}$. The fact that L and R arise in this way implies all of the conditions described above.

It will often be convenient to have notation for the functor $\mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ that takes an arrow to the object it factors through, and we typically write E for this, without the arrow decoration. With this notation, the functorial factorization $\vec{E} \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{3}}$ sends a commutative square

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \quad \text{to a commutative rectangle} \quad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ Ef \arrow[r, "E(u{,}v)"] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \tag{I.2.6}$$

We'll refer to $E \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ as the *functor accompanying the functorial factorization* (L, R).

Definition I.2.7. A wfs is called *functorial* if it has a functorial factorization with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all f.

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[2] It is not immediately obvious that $\mathcal{W}$ must be closed under retracts but this does follow by a clever argument the author learned from André Joyal [Joy08, §F].

There is a stronger notion of wfs called an *orthogonal factorization system*, abbreviated *ofs*, in which solutions to a given lifting problem are required to be unique.[3] These are sometimes called *factorization systems* in the literature. It follows from the uniqueness of the lifts that the factorizations of an ofs are always functorial. For this stronger notion, the left class is closed under all colimits and the right under all limits, taken in the arrow category.

Relative to orthogonal factorization systems, wfs with functorial factorizations suffer from two principal defects. The first is that a functorial wfs on $\mathcal{M}$ does not induce a pointwise wfs on a diagram category $\mathcal{M}^{\mathcal{A}}$, where $\mathcal{A}$ is a small category. The functorial factorization does allow us to factor natural transformations pointwise, but in general the resulting left factors will not lift against the right ones, even though their constituent arrows satisfy the required lifting property. This is because the pointwise lifts which necessarily exist are not naturally chosen and so do not fit together to form a natural transformation.

The second defect is that the classes of a functorial wfs, as for a generic wfs, fail in general to be closed under all the limits and colimits that one might expect. Specifically, we might hope that the left class would be closed under all colimits in $\mathcal{M}^{\mathbf{2}}$ and the right class would be closed under all limits. As those who are familiar with working with cofibrations know, this is not true in general.

These failings motivated Grandis and Tholen to define *algebraic weak factorization systems* [GT06], which are functorial wfs with extra structure that addresses both of these issues.

I.2.3 Algebraic weak factorization systems

Any functorial factorization gives rise to two endofunctors $L, R\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{2}}$, which are equipped with natural transformations to and from the identity, respectively. Explicitly, L is equipped with a natural transformation $\vec{\epsilon}\colon L \Rightarrow \mathrm{id}$ whose components consist of the squares

$$\vec{\epsilon}_f = \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"'] & \cdot \end{tikzcd} .$$

We call $\vec{\epsilon}$ the *counit* of the endofunctor L and write $\epsilon_f := Rf$ for the codomain part of the morphism $\vec{\epsilon}_f$. Using the notation of Definition I.2.1, $\vec{\epsilon} = (1, \epsilon)$. The component $\epsilon\colon E \Rightarrow \mathrm{cod}$ is a natural transformation in its own right, where E is as in (I.2.6).

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[3] An example in **Set** takes the epimorphisms as the left class and the monomorphisms as the right class. When we exchange these classes the result is a wfs.

Dually, R is equipped with a natural transformation $\vec{\eta}\colon \mathrm{id} \Rightarrow R$ whose components are squares

$$\vec{\eta}_g = \begin{tikzcd} \cdot \arrow[r, "Lg"] \arrow[d, "g"'] & \cdot \arrow[d, "Rg"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}.$$

We call $\vec{\eta}$ the *unit* of the endofunctor R and write $\eta = \mathrm{dom}\,\vec{\eta}$ for the natural transformation $\mathrm{dom} \Rightarrow E$. We write $\vec{\eta} = (\eta, 1)$ in the notation of Definition I.2.1. We call a functor L equipped with a natural transformation to the identity functor *left pointed* and a functor R equipped with a natural transformation from the identity functor *right pointed*, though the directional adjectives may be dropped when the direction (left vs. right) is clear from context.

Lemma I.2.8. *In a functorial wfs* $(\mathcal{L}, \mathcal{R})$*, the maps in* $\mathcal{R}$ *are precisely those arrows which admit an algebra structure for the pointed endofunctor* $(R, \vec{\eta})$*. Dually, the class* $\mathcal{L}$ *consists of those maps that admit a coalgebra structure for* $(L, \vec{\epsilon})$*.*

Proof. Algebras for a right pointed endofunctor are defined similarly to algebras for a monad, but in the absence of a multiplication natural transformation, the algebra structure maps need only satisfy a unit condition. If $g \in \mathcal{R}$ then it lifts against its left factor as shown

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "Rg"'] \arrow[ur, dashed, "t"] & \cdot \end{tikzcd}.$$

The arrow $(t, 1)\colon Rg \Rightarrow g$ makes g an algebra for $(R, \vec{\eta})$. Conversely, if g has an algebra structure (t, s) then the unit axiom implies that

$$\begin{tikzcd} \cdot \arrow[r, "Lg"] \arrow[d, "g"'] & \cdot \arrow[r, "t"] \arrow[d, "Rg"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

is a retract diagram (hence, $s = 1$). Thus, g is a retract of $Rg \in \mathcal{R}$, which is closed under retracts. □

The notion of a algebraic weak factorization system is an algebraization of the notion of a functorial wfs in which the above pointed endofunctors are replaced with a comonad and a monad respectively.

Definition I.2.9. An *algebraic weak factorization system* (originally, *natural weak factorization system*) on a category $\mathcal{M}$ consists a pair $(\mathbb{L}, \mathbb{R})$, where $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ is a comonad on $\mathcal{M}^{\mathbf{2}}$ and $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ is a monad on $\mathcal{M}^{\mathbf{2}}$, such that $(L, \vec{\epsilon})$ and $(R, \vec{\eta})$ are the pointed endofunctors of some functorial factorization $\vec{E}\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{3}}$. Additionally, the accompanying natural transformation $\Delta\colon LR \Rightarrow RL$ described below is required to be a distributive law of the comonad over the monad.

Because the unit $\vec{\eta}$ arising from the functorial factorization necessarily has the form $\vec{\eta}_f =$

$$\begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

, it follows from the monad axioms that $\vec{\mu}_f =$

$$\begin{tikzcd} \cdot \arrow[r, "\mu_f"] \arrow[d, "R^2 f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

where $\mu \colon ER \Rightarrow E$ is a natural transformation, with E as in (I.2.6). Hence, $\mathbb{R}$ is a *monad over* $\mathrm{cod} \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$, which means that $\mathrm{cod} R = \mathrm{cod}$, $\mathrm{cod}\,\vec{\eta} = \mathrm{id}_{\mathrm{cod}}$ and $\mathrm{cod}\,\vec{\mu} = \mathrm{id}_{\mathrm{cod}}$. This means that Rf has the same codomain as f, and the codomain component of the natural transformations $\vec{\eta}$ and $\vec{\mu}$ is the identity.

Dually, $\mathbb{L}$ is a *comonad over* dom (in the sense that it is a comonad in the 2-category $\mathbf{CAT}/\mathcal{M}$ on the object $\mathrm{dom} \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$). We write $\delta \colon E \Rightarrow EL$ for the natural transformation $\mathrm{cod}\,\vec{\delta}$ analogous to $\mu = \mathrm{dom}\,\vec{\mu}$ defined above. As a consequence of the monad and comonad axioms,

$$\begin{tikzcd} \cdot \arrow[r, "\delta_f"] \arrow[d, "LRf"'] & \cdot \arrow[d, "RLf"] \\ \cdot \arrow[r, "\mu_f"'] & \cdot \end{tikzcd}$$

commutes for all f. (Indeed, the common diagonal composite is the identity.) These squares are the components of a natural transformation $\Delta \colon LR \Rightarrow RL$, which is the distributive law mentioned above. In this context, the requirement that Δ be a distributive law of L over R reduces to a single condition: $\delta \cdot \mu = \mu_L \cdot E(\delta, \mu) \cdot \delta_R$. Because the components of Δ are part of the data of $\mathbb{L}$ and $\mathbb{R}$, this distributive law does not provide any extra structure for the awfs; rather it is a property that we ask that the pair $(\mathbb{L}, \mathbb{R})$ satisfy.[4]

Given an awfs $(\mathbb{L}, \mathbb{R})$, we refer to the $\mathbb{L}$-coalgebras as the left class and the $\mathbb{R}$-algebras as the right class of the awfs. Unraveling the definitions, an $\mathbb{L}$-coalgebra consists of a pair (f, s), where f is an arrow of $\mathcal{M}$ and $(1, s) \colon f \Rightarrow Lf$ is an arrow in $\mathcal{M}^{\mathbf{2}}$ satisfying the usual conditions so that this gives a coalgebra structure with respect to the comonad $\mathbb{L}$. The unit condition says that s solves the canonical lifting problem of f against Rf. Dually, an $\mathbb{R}$-algebra consists of a pair (g, t) such that g is an arrow of $\mathcal{M}$ and $(t, 1) \colon Rg \Rightarrow g$ is an arrow in $\mathcal{M}^{\mathbf{2}}$, where t lifts Lg against g.

The algebra structure of an element g of the right class of an awfs should be thought of as a

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[4] Grandis and Tholen's original definition did not include this condition, but Garner's does. Using Garner's definition, awfs are bialgebras with respect to a two-fold monoidal structure on the category of functorial factorizations (see [Gar07, §3.2]); the distributive law condition says exactly that the monoid and comonoid structures fit together to form a bialgebra. This category provides the setting for the proofs establishing the machinery of Garner's small object argument. We recommend that the first-time reader ignore these details; to repeat a quote the author has seen attributed to Frank Adams, "to operate the machine, it is not necessary to raise the bonnet."

chosen lifting of g against any element of the left class. Given an $\mathbb{L}$-coalgebra (f, s) and a lifting problem $(u, v)\colon f \Rightarrow g$, the arrow $w = t \cdot E(u, v) \cdot s$

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\
\cdot \arrow[r, dashed, "E(u,v)"] \arrow[d, "Rf"'] & \cdot \arrow[u, dashed, shift right=1.5ex, "t"'] \arrow[d, "Rg"] \\
\cdot \arrow[u, dashed, shift right=1.5ex, "s"'] \arrow[r, "v"'] & \cdot
\end{tikzcd} \tag{I.2.10}$$

is a solution to the lifting problem. In particular, all $\mathbb{L}$-coalgebras lift against all $\mathbb{R}$-algebras.

If we let $\mathcal{L}$ and $\mathcal{R}$ denote the arrows in $\mathcal{M}$ that have some $\mathbb{L}$-coalgebra structure or $\mathbb{R}$-algebra structure, respectively, then it is not quite true that $(\mathcal{L}, \mathcal{R})$ is a wfs. This is because retracts of maps in $\mathcal{L}$ will also lift against elements of $\mathcal{R}$, but the categories of coalgebras for a comonad and algebras for a monad are not closed under retracts. We write $\overline{\mathcal{L}}$ for the retract closure of $\mathcal{L}$ and similarly for $\mathcal{R}$ and refer to the wfs $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ as the *underlying wfs* of $(\mathbb{L}, \mathbb{R})$. It is, in particular, functorial.

Remark I.2.11. Because the class of $\mathbb{L}$-algebras is not closed under retracts, not every arrow in the left class of the underlying wfs $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ of the awfs $(\mathbb{L}, \mathbb{R})$ will have an $\mathbb{L}$-coalgebra structure. The same is true for the right class. (But see Lemma I.2.30!)

However, as we saw in Lemma I.2.8, every arrow of $\overline{\mathcal{L}}$ will have a coalgebra structure for the left pointed endofunctor $(L, \vec{\epsilon})$ and conversely every coalgebra will be an element of $\overline{\mathcal{L}}$. It follows that coalgebras for the pointed endofunctors underlying an awfs are closed under retracts; this can also be proved directly. In fact, the coalgebras for the pointed endofunctor underlying a comonad are the retract closure of the coalgebras for the comonad. The proof of this statement uses the fact that the map $(1, s)\colon f \Rightarrow Lf$ makes f a retract of its left factor Lf, which has a free coalgebra structure for the comonad $\mathbb{L}$. Similar results apply to the right class $\overline{\mathcal{R}}$.

Example I.2.12. Any orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ is an awfs. Orthogonal factorization systems are always functorial, with all possible choices of functorial factorizations canonically isomorphic. The comultiplication and multiplication natural transformations for the functors L and R are defined to be the unique solutions to the lifting problems

$$\begin{tikzcd}
\cdot \arrow[r, "LL"] \arrow[d, "L"'] & \cdot \arrow[d, "RL"] \\
\cdot \arrow[r, equal] \arrow[ur, dashed, "\delta"'] & \cdot
\end{tikzcd}$$

and

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "LR"'] & \cdot \arrow[d, "R"] \\ \cdot \arrow[r, "RR"'] \arrow[ur, dashed, "\mu"] & \cdot \end{tikzcd}$$

. Every element of $\mathcal{R}$ has a unique $\mathbb{R}$-algebra structure and the structure map is an isomorphism. Similarly, every element of $\mathcal{L}$ has a unique $\mathbb{L}$-coalgebra structure, with structure map an isomorphism. It follows that the classes of $\mathbb{R}$-algebras and $\mathbb{L}$-coalgebras are closed under retracts. The remaining details are left as an exercise.

In light of Remark I.2.11, why does it make sense to use a definition of awfs that privileges coalgebra structures for the comonad $\mathbb{L}$ over coalgebras for the left pointed endofunctor $(L, \vec{\epsilon})$, and similarly on the right? We suggest three justifications. The first is that coalgebras for the comonad are often "nicer" than coalgebras for the pointed endofunctor. In examples, the former are analogous to "relative cell complexes" while the latter are the "retracts of relative cell complexes." A second reason is that we can compose coalgebras for the comonad in an awfs, meaning we can give the composite arrow a canonical coalgebra structure. This definition, which will be given in Section I.2.5, uses the multiplication for the monad explicitly, so is not possible without this extra algebraic structure. Finally, and perhaps most importantly, coalgebras for a comonad are closed under colimits, as we will prove in Theorem I.2.16. There is no analogous result for $(L, \vec{\epsilon})$-coalgebras. The upshot is that when examining colimits, the extra effort to check that a diagram lands in $\mathbb{L}$-**coalg** is often worth it.

Remark I.2.13. The original name *natural* weak factorization system is in some sense a misnomer. In most cases, the lift of a map r in the right class against its left factor is not *natural*; it's simply *chosen* and recorded in the fact that associate to the arrow r a piece of *algebraic* data. Solutions to lifting problems of the form (I.2.2) are constructed by combining the coalgebraic and algebraic data of l and r with a functorial factorization of the square. These lifts are not natural with respect to all morphisms in the arrow category. They are however natural with respect to morphisms of $\mathbb{L}$-**coalg** and $\mathbb{R}$-**alg**, but that is true precisely because morphisms in a category of algebras are required to preserve the algebraic structure.

In an important special case, however, there are natural lifts; namely, for the free morphisms that arise as left and right factors of arrows. Hence, the adjective "natural" appropriately describes these factorizations. The multiplication of the monad $\mathbb{R}$ gives any arrow of the form Rf a natural $\mathbb{R}$-algebra structure μ_f. Similarly, the arrows Lf have a natural $\mathbb{L}$-coalgebra structure δ_f using the comultiplication of the comonad. Of course, it may be that there are other ways to choose lifting data for these arrows, but the natural choices provided by the comultiplication and multiplication have the property that the map from Lf to Lg or Rf to Rg arising from any map $(u,v)\colon f \Rightarrow g$ preserves the lifting data.

We conclude this section with one final definition that will prove very important in Section I.3 and beyond.

Definition I.2.14. A *morphism of awfs* $\xi\colon (\mathbb{L},\mathbb{R}) \to (\mathbb{L}',\mathbb{R}')$ is a natural transformation $\xi\colon E \Rightarrow E'$ that is a *morphism of functorial factorizations*, i.e., such that

$$\begin{tikzcd}
 & \cdot \arrow[dl, "Lf"'] \arrow[dr, "L'f"] & \\
Ef \arrow[rr, "\xi_f"] \arrow[dr, "Rf"'] & & E'f \arrow[dl, "R'f"] \\
 & \cdot &
\end{tikzcd} \tag{I.2.15}$$

commutes, and such that the natural transformations $(1,\xi)\colon L \Rightarrow L'$ and $(\xi,1)\colon R \Rightarrow R'$ are comonad and monad morphisms, respectively, which means that these natural transformations satisfy unit and associativity conditions. It follows that a morphism of awfs ξ induces functors $\xi_*\colon \mathbb{L}\text{-}\mathbf{coalg} \to \mathbb{L}'\text{-}\mathbf{coalg}$ and $\xi^*\colon \mathbb{R}'\text{-}\mathbf{alg} \to \mathbb{R}\text{-}\mathbf{alg}$ between the Eilenberg-Moore categories of coalgebras and algebras.

I.2.4 Limit and colimit closure

It remains to explain how an awfs rectifies the defects mentioned at the end of I.2.2. We will speak at length about induced pointwise awfs later in Section I.4, but we can deal with colimit and limit closure right now.

Let $\mathbb{R}$-**alg** denote the Eilenberg-Moore category of algebras for the monad $\mathbb{R}$ and let $\mathbb{L}$-**coalg** similarly denote the category of coalgebras for $\mathbb{L}$. It is a well-known categorical fact that the forgetful functors $U\colon \mathbb{R}\text{-}\mathbf{alg} \to \mathcal{M}^{\mathbf{2}}$, $U\colon \mathbb{L}\text{-}\mathbf{coalg} \to \mathcal{M}^{\mathbf{2}}$ create all limits and colimits, respectively, that exist in $\mathcal{M}^{\mathbf{2}}$. It follows that the right and left classes of the awfs $(\mathbb{L},\mathbb{R})$ are closed under limits and colimits, respectively. We have proven the following result of [GT06].

Proposition I.2.16 (Grandis-Tholen)**.** *If $\mathcal{M}$ has colimits (respectively limits) of a given type, then* $\mathbb{L}$-**coalg** *(respectively* $\mathbb{R}$-**alg***) has them, formed as in* $\mathcal{M}^{\mathbf{2}}$.

Remark I.2.17. It is possible to interpret I.2.16 too broadly. This does not say that for any diagram in $\mathcal{M}^{\mathbf{2}}$ such that the objects have a coalgebra structure, the colimit will have a coalgebra structure. This conclusion will only follow if the maps of the colimit diagram are arrows in $\mathbb{L}$-**coalg** and not just in $\mathcal{M}^{\mathbf{2}}$.

However, we do now have a method for proving that a particular colimit is a coalgebra: namely checking that the maps in the relevant colimiting diagram are maps of coalgebras. While this can be tedious, it will allow us to prove surprising results about cofibrations, which the author suspects are intractable by other methods. (See, e.g., Theorem I.5.1. It is also possible to prove Corollary I.6.16 directly in this manner.)

Example I.2.18. An example will illustrate this important point, though we have to jump ahead a bit. As a consequence of Garner's small object argument (see I.2.28), there is an awfs on **Top** such that the left class of its underlying wfs consists of the cofibrations for the Quillen model structure. It is well-known that the pushout of cofibrations is not always a cofibration. For example, the vertical maps of

$$\begin{tikzcd}
D^{n+1} \arrow[d, equal] & * \arrow[l] \arrow[r, equal] \arrow[d, "j"] & * \arrow[d, equal] \\
D^{n+1} & S^n \arrow[l, "j_{n+1}"'] \arrow[r] & *
\end{tikzcd} \tag{I.2.19}$$

are all cofibrations and coalgebras in the Quillen model structure,[5] but the pushout $D^{n+1} \twoheadrightarrow S^{n+1}$ is not. This tells us that one of the squares of (I.2.19) is not a map of coalgebras, and furthermore there are no coalgebra structures for the vertical arrows such that both squares are maps of coalgebras.

By contrast, the pushout of

$$\begin{tikzcd}
D^n \arrow[d, "i_N"'] & S^{n-1} \arrow[l, "j_n"'] \arrow[r] \arrow[d, "j_n"] & * \arrow[d, "j"] \\
S^n & D^n \arrow[l, "i_S"'] \arrow[r, two heads] & S^n
\end{tikzcd} \tag{I.2.20}$$

---

[5] The arrow j inherits its cofibration structure as a pushout of the generating cofibration j_n as shown

$$\begin{tikzcd}
S^{n-1} \arrow[r, "u"] \arrow[d, "j_n"'] & * \arrow[d, "j"] \\
D^n \arrow[r, "v"'] & S^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

. Explicitly, if $c_n \colon D^n \to Qj_n$ gives j_n its coalgebra structure, then the cone $(Cj, Q(u, v) \cdot c_n)$ gives j its coalgebra structure, where Q is the functor accompanying the functorial factorization of this awfs.

is a cofibration and a coalgebra because all three vertical arrows have a coalgebra structure and the squares of (I.2.20) preserve them. (The maps $i_N, i_S : D^n \to S^n$ include the disk as the northern or southern hemisphere of the sphere.) Of course, this fact could be deduced directly because the pushout $S^n \to S^n \vee S^n$ is an inclusion of a sub-CW-complex, but in more complicated examples this technique for detecting cofibrations will prove useful.

I.2.5 Composing algebras and coalgebras

Unlike the situation for ordinary monads on arrow categories, the category of algebras for the monad of an awfs $(\mathbb{L}, \mathbb{R})$ can be equipped with a canonical composition law, which is natural in a suitable "double categorical" sense, described below. Furthermore, the comultiplication for the comonad $\mathbb{L}$ can be recovered from this composition, so one can recognize an awfs by considering only the category $\mathbb{R}$-**alg** together with its natural composition law. Later, in Section I.6.2, we will extend this recognition principle to morphisms between awfs. In concrete applications, this allows us to ignore the category $\mathbb{L}$-**coalg**, which we'll see can be a bit of a pain.

In this section, we give precise statements of these facts and describe their proofs. Their most explicit appearance in the literature is [Gar10, §2], but see also [Gar09, §A] or [Gar07, §6.3]. The dual statements also hold.

Recall that when $\mathbb{R}$ is a monad from an awfs $(\mathbb{L}, \mathbb{R})$, an $\mathbb{R}$-algebra structure for an arrow f has the form $(s, 1)\colon Rf \Rightarrow f$; accordingly, we write (f, s) for the corresponding object of $\mathbb{R}$-**alg**. Let $(f, s), (f', s') \in \mathbb{R}$-**alg**. We say a morphism $(u, v)\colon f \Rightarrow f'$ in $\mathcal{M}^{\mathbf{2}}$ is a map of algebras (with the particular algebra structures s and s' already in mind) when (u, v) lifts to a morphism $(u, v)\colon (f, s) \Rightarrow (f', s')$ in $\mathbb{R}$-**alg**. It follows from the definition that this holds exactly when $s' \cdot E(u, v) = u \cdot s$, where $E\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ is the functor accompanying the functorial factorization of $(\mathbb{L}, \mathbb{R})$. This condition says that the top face of the following cube, which should be interpreted as a map from the algebra depicted on the left face to the algebra on the right face, commutes.

$$\begin{tikzcd}
\cdot \arrow[rr, "E(u{,}v)"] \arrow[dr, "s"] \arrow[dd, "Rf"'] & & \cdot \arrow[dr, "s'"] \arrow[dd, "Rf'"] & \\
& \cdot \arrow[rr, "u"] \arrow[dd, "f"] & & \cdot \arrow[dd, "f'"] \\
\cdot \arrow[rr, "v"'] \arrow[dr, equal] & & \cdot \arrow[dr, equal] & \\
& \cdot \arrow[rr, "v"'] & & \cdot
\end{tikzcd}$$

Definition I.2.21. Let $(f, s), (g, t) \in \mathbb{R}$-**alg** with $\mathrm{cod} f = \mathrm{dom} g$. Then gf has a canonical $\mathbb{R}$-algebra structure

$$E(gf) \xrightarrow{\delta_{gf}} EL(gf) \xrightarrow{E(1,t\cdot E(f,1))} Ef \xrightarrow{s} \mathrm{dom} f$$

where $\delta \colon E \Rightarrow EL$ is the natural transformation arising from the comultiplication of the comonad $\mathbb{L}$.

Write $(g, t) \bullet (f, s) = (gf, t \bullet s)$ for this composition operation. It is natural in the following sense.

Lemma I.2.22. *Let* $(u, v) \colon (f, s) \Rightarrow (h, s')$ *and* $(v, w) \colon (g, t) \Rightarrow (k, t')$ *be morphisms in* $\mathbb{R}$-**alg**. *Then* $(u, w) \colon (gf, t \bullet s) \Rightarrow (kh, t' \bullet s')$ *is a map of* $\mathbb{R}$*-algebras.*

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "v"'] \arrow[d, "g"'] & \cdot \arrow[d, "k"] \\
\cdot \arrow[r, "w"'] & \cdot
\end{tikzcd}$$

Proof. The proof is an easy diagram chase. □

Remark I.2.23. It follows from Lemma I.2.22 that algebras for a monad arising from an awfs $(\mathbb{L}, \mathbb{R})$ form a (strict) double category **Alg**$\mathbb{R}$: objects are objects of $\mathcal{M}$, horizontal arrows are morphisms in $\mathcal{M}$, vertical arrows are $\mathbb{R}$-algebras, and squares are morphisms of algebras. The content of Lemma I.2.22 is that morphisms of algebras can be composed vertically as well as horizontally. It remains to check that composition of algebras is strictly associative, but this is a straightforward exercise.

Lemma I.2.22 has a converse, which provides a means for recognizing awfs from categories of algebras.

Theorem I.2.24 (Garner)**.** *Suppose* $\mathbb{R}$ *is a monad on* $\mathcal{M}^{\mathbf{2}}$ *over* $\operatorname{cod}\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$. *Specifying a natural composition law on* $\mathbb{R}$-**alg** *is equivalent to specifying an awfs* $(\mathbb{L}, \mathbb{R})$ *on* $\mathcal{M}$.

Proof. Because $\mathbb{R}$ is a monad over cod, the components of its unit define a functorial factorization on $\mathcal{M}$ (see the beginning of Section I.2.3). In particular, the functor L and counit $\vec{\epsilon}$ have already been determined. It remains to define $\delta\colon E \Rightarrow EL$ so that $\vec{\delta} = (1, \delta)\colon L \Rightarrow L^2$ makes $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ into a comonad satisfying the distributive law with respect to $\mathbb{R}$.

Given a natural composition law on the category of $\mathbb{R}$-algebras and a morphism $f \in \mathcal{M}$, we define $\delta_f\colon Ef \to ELf$ to be

$$\delta_f := \; Ef \xrightarrow{E(L^2f,1)} E(Rf \cdot RLf) \xrightarrow{\mu_f \bullet \mu_{Lf}} ELf \;,$$

where $\mu_f \bullet \mu_{Lf}$ is the algebra structure for the composite of the free algebras (RLf, μ_{Lf}) and (Rf, μ_f). Equivalently, δ_f is defined to be the domain component of the adjunct to the morphism

$$\begin{tikzcd} \cdot \arrow[r, "L^2f"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf \cdot RLf = U(Rf \cdot RLf, \mu_f \bullet \mu_{Lf})"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

with respect to the (monadic) adjunction $\begin{tikzcd} \mathbb{R}\text{-}\mathbf{alg} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{M}^{\mathbf{2}} \arrow[l, shift right, "F"'] \end{tikzcd}$.

By taking adjuncts of the unit and associativity conditions for a comonad, it is easy to check that such δ makes $\mathbb{L}$ a comonad. The distributive law can be verified using the fact that $\mu_f \bullet \mu_{Lf}$ is, as an algebra structure, compatible with the multiplication for the monad $\mathbb{R}$. We leave the verification of these diagram chases to the reader; see also [Gar10, Proposition 2.8]. □

I.2.6 Cofibrantly generated awfs

There are a few naturally occurring examples of awfs where the familiar functorial factorizations for some wfs underlie a comonad and a monad. One toy example is the so-called "graph" factorization of an arrow through the product of its domain and codomain. There are more serious examples, including the wfs from the Quillen model structure on $\mathbf{Ch}_R$ and the folk model structure on **Cat**. However, the examples topologists find in nature are less obviously "algebraic," and consequently awfs have not generated a lot of interest among topologists. Recently, Garner has developed a variant of Quillen's small object argument, modeled upon a familiar transfinite construction from category theory, that produces *cofibrantly generated* awfs. In any cocomplete category satisfying an appropriate smallness condition, general enough to include the desired examples, Garner's small object argument can be applied in place of Quillen's, and the resulting awfs have the same underlying wfs as those produced by the usual small object argument. The functorial factorizations are different but also arguably better than Quillen's in that the objects constructed are somehow "smaller" (in the sense that superfluous "cells" are not multiply attached) and also the transfinite process by which they are constructed actually converges, rather than terminating arbitrarily at some chosen ordinal. Furthermore, Garner's small object argument can be run for a generating small category, not merely for generating sets, a generalization whose power will become apparent in Section I.4.

In this section, we explain in detail the defining properties of *cofibrantly generated* awfs, produced by Garner's small object argument. A more detailed overview of his construction is given in Section I.4, where it will first be needed. See also [Gar07] or [Gar09].

First, we extend the notation $(-)^{\boxslash}$ to categories over $\mathcal{M}^{\mathbf{2}}$, as opposed to mere sets of arrows.

Definition I.2.25. We define a pair of functors

$$\begin{tikzcd} (-)^{\boxslash} \colon \mathbf{CAT}/\mathcal{M}^{\mathbf{2}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (\mathbf{CAT}/\mathcal{M}^{\mathbf{2}})^{\mathrm{op}} \colon {}^{\boxslash}(-) \arrow[l, shift left] \end{tikzcd}$$

that are mutually right adjoint. If $\mathcal{J}$ is a category over $\mathcal{M}^{\mathbf{2}}$, the objects of $\mathcal{J}^{\boxslash}$ are pairs (g, ϕ), where g is an arrow of $\mathcal{M}$ and ϕ is a *lifting function* that assigns each square

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "j"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

with $j \in \mathcal{J}$ a lift $\phi(j, u, v)$ that makes the usual triangles commute. We also require that ϕ is coherent with respect to morphisms in $\mathcal{J}$. Explicitly, given $(a, b) \colon j' \Rightarrow j$ in $\mathcal{J}$, we require that $\phi(j', ua, vb) = \phi(j, u, v) \cdot b$, which says that the triangle of lifts in the diagram below commutes.

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "j'"'] & \cdot \arrow[r, "u"] \arrow[d, "j"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "b"'] \arrow[urr, dashed] & \cdot \arrow[r, "v"'] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Morphisms $(g,\phi) \to (g',\phi')$ of $\mathcal{J}^{\boxslash}$ are arrows in $\mathcal{M}^{\mathbf{2}}$ that preserve the lifting functions. The category $\mathcal{J}^{\boxslash}$ is equipped with an obvious forgetful functor to $\mathcal{M}^{\mathbf{2}}$ that ignores the lifting data. When $\mathcal{J}$ is a set, the image of $\mathcal{J}^{\boxslash}$ under this forgetful functor is the set $\mathcal{J}^{\boxslash}$ defined in Section I.2.1.

Garner provides two definitions of a cofibrantly generated awfs [Gar09], though his terminology more closely parallels the theory of monads. An awfs $(\mathbb{L},\mathbb{R})$ is *free* on a small category $J\colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ if there is a functor

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "\gamma"] \arrow[dr, "J"'] & & \mathbb{L}\text{-}\mathbf{coalg} \arrow[dl, "U"] \\ & \mathcal{M}^{\mathbf{2}} & \end{tikzcd} \tag{I.2.26}$$

that is initial with respect to morphisms of awfs among functors from $\mathcal{J}$ to categories of coalgebras of awfs. A stronger notion is of an *algebraically-free* awfs, for which we require that the composite functor

$$\mathbb{R}\text{-}\mathbf{alg} \xrightarrow{\text{lift}} (\mathbb{L}\text{-}\mathbf{coalg})^{\boxslash} \xrightarrow{\gamma^{\boxslash}} \mathcal{J}^{\boxslash} \tag{I.2.27}$$

is an isomorphism of categories. The functor "lift" uses the algebra and coalgebra structures of $\mathbb{R}$-algebras and $\mathbb{L}$-coalgebras to define lifting functions via the construction of I.2.10. The isomorphism (I.2.27) should be compared with the isomorphism of sets $\mathcal{R} \cong \mathcal{J}^{\boxslash}$, which is the usual notion of a cofibrantly generated wfs $(\mathcal{L},\mathcal{R})$.

We will say that the awfs produced by Garner's small object argument are *cofibrantly generated*. Garner proves that these awfs are both free and algebraically-free; we will find occasion to use both defining properties.

Theorem I.2.28 (Garner)**.** *Let $\mathcal{M}$ be a cocomplete category satisfying either of the following conditions.*

- $(*)$ *Every $X \in \mathcal{M}$ is α_X-presentable for some regular cardinal α_X.*
- $(\dagger)$ *Every $X \in \mathcal{M}$ is α_X-bounded with respect to some proper, well-copowered orthogonal factorization system on $\mathcal{M}$, for some regular cardinal α_X.*

Let $J\colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ be a category over $\mathcal{M}^{\mathbf{2}}$, with $\mathcal{J}$ small. Then the free awfs on $\mathcal{J}$ exists and is algebraically-free on $\mathcal{J}$.

We won't define all these terms here. What's important is to know that the categories of interest satisfy one of these two conditions. Locally presentable categories, such as **sSet**, satisfy

(∗). **Top**, **Haus**, and **TopGp** all satisfy (†). We say a category $\mathcal{M}$ *permits the small object argument* if it is cocomplete and satisfies either (∗) or (†).

Remark I.2.29. This notion of cofibrantly generated is broader than the usual one — see Example I.4.4 for a concrete example — as ordinary cofibrantly generated wfs are generated by a set of maps, rather than a category. We will refer to this as the "discrete case", discrete small categories being simply sets.

As is the case for ordinary wfs, cofibrantly generated awfs behave better than generic ones. We conclude this introduction with an easy lemma, which will prove vital to proofs in later sections.

Lemma I.2.30. *If an awfs* $(\mathbb{L}, \mathbb{R})$ *on* $\mathcal{M}$ *is cofibrantly generated, then the class* $\mathcal{R}$ *of arrows that admit an* $\mathbb{R}$*-algebra structure is closed under retracts.*

Proof. When $(\mathbb{L}, \mathbb{R})$ is generated by $\mathcal{J}$, we have an isomorphism of categories $\mathbb{R}$-**alg** $\cong \mathcal{J}^{\boxslash}$ over $\mathcal{M}^{\mathbf{2}}$. The forgetful functor $U\colon \mathbb{R}\text{-}\mathbf{alg} \to \mathcal{M}^{\mathbf{2}}$ sends $(g, \phi) \in \mathcal{J}^{\boxslash}$ to g. We wish to show that its image is closed under retracts. Suppose h is a retract of g as shown

$$\begin{tikzcd}
\cdot \arrow[r, "i_1"] \arrow[d, "h"'] & \cdot \arrow[r, "r_1"] \arrow[d, "g"'] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "i_2"'] & \cdot \arrow[r, "r_2"'] & \cdot
\end{tikzcd}$$

Define a lifting function ψ for h by

$$\psi(j, u, v) := r_1 \cdot \phi(j, i_1 \cdot u, i_2 \cdot v).$$

The equations from the retract diagram show that ψ is indeed a lifting function. It remains to check that ψ is coherent with respect to morphisms $(a, b)\colon j' \Rightarrow j$ of $\mathcal{J}$. We compute

$$\psi(j', u \cdot a, v \cdot b) = r_1 \cdot \phi(j', i_1 \cdot u \cdot a, i_2 \cdot v \cdot b) = r_1 \cdot \phi(j, i_1 \cdot u, i_2 \cdot v) \cdot b = \psi(j, u, v) \cdot b,$$

as required. □

The upshot of Lemma I.2.30 is that every arrow in the right class of the ordinary wfs $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ underlying a cofibrantly generated awfs $(\mathbb{L}, \mathbb{R})$ has an $\mathbb{R}$-algebra structure. When our awfs is cofibrantly generated, we emphasize this result by writing $(\overline{\mathcal{L}}, \mathcal{R})$ for the underlying wfs. We will also refer to a lifting function ϕ associated to an element of $g \in \mathcal{J}^{\boxslash}$ as an algebra structure for g, in light of (I.2.27) and this result.

Remark I.2.31. Garner proves the discrete version of Lemma I.2.30 in [Gar09]: when the generating category $\mathcal{J}$ is discrete, $\mathcal{R}$ is closed under retracts and the wfs $(\overline{\mathcal{L}}, \mathcal{R})$ is cofibrantly generated in the usual sense by this set of maps. As a consequence, the new notion of "cofibrantly generated" agrees with the usual one, in the case where they ought to overlap.

As a final note, the composition law for the algebras of a cofibrantly generated awfs is particularly easy to describe using the isomorphism (I.2.27).

Example I.2.32. Suppose $(\mathbb{L}, \mathbb{R})$ is an awfs on $\mathcal{M}$ generated by a category $\mathcal{J}$. Consider composable objects $(f, \phi), (g, \psi) \in \mathcal{J}^{\boxslash} \cong \mathbb{R}$-**alg**, i.e., suppose $\operatorname{cod} f = \operatorname{dom} g$. Their canonical composite is $(gf, \psi \bullet \phi)$ where

$$\psi \bullet \phi(j, a, b) := \phi(j, a, \psi(j, f \cdot a, b)),$$

and this is natural in the sense described by Lemma I.2.22.

In the remaining sections, we will present new results relating awfs to model structures, taking frequent advantage of the machinery provided by Garner's small object argument.

I.3 Algebraic model structures

The reasons that most topologists care (or should care) about weak factorizations systems is because they figure prominently in model categories, which are equipped with an interacting pair of them. Using Garner's small object argument, whenever these wfs are cofibrantly generated, they can be algebraicized to produce awfs. This leads to the question: is there a good notion of an *algebraic* model structure? What is the appropriate definition?

Historically, model categories arose to enable computations in the homotopy category defined for a pair $(\mathcal{M}, \mathcal{W})$, where $\mathcal{W}$ is a class of arrows of $\mathcal{M}$ called the *weak equivalences* that one would like to manipulate as if they were isomorphisms. But with all of the subsequent development of the theory of model categories, this philosophy that the weak equivalences should be of primary importance is occasionally lost. With this principle in mind, the author has decided that an algebraic model structure is something one should give a *pair* $(\mathcal{M}, \mathcal{W})$, rather than a category $\mathcal{M}$;

that is to say, one ought to have a particular class of weak equivalences in mind already. This suggests the following "minimalist" definition.

Definition I.3.1. An *algebraic model structure* on a pair $(\mathcal{M}, \mathcal{W})$, where $\mathcal{M}$ is a complete and cocomplete category and $\mathcal{W}$ is a class of morphisms satisfying the 2-of-3 property, consists of a pair of awfs $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ on $\mathcal{M}$ together with a morphism of awfs

$$\xi \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$$

such that the underlying wfs of $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ give the trivial cofibrations, fibrations, cofibrations, and trivial fibrations, respectively, of a model structure on $\mathcal{M}$, with weak equivalences $\mathcal{W}$. We call ξ the *comparison map*.

The comparison map ξ gives an algebraic way to regard a trivial cofibration as a cofibration and a trivial fibration as a fibration. We will say considerably more about this in a moment.

Let $\mathcal{C}_t$ denote the underlying class of maps with a $\mathbb{C}_t$-coalgebra structure and define $\mathcal{C}$, $\mathcal{F}_t$, and $\mathcal{F}$ likewise. By definition $(\overline{\mathcal{C}_t}, \overline{\mathcal{F}})$ and $(\overline{\mathcal{C}}, \overline{\mathcal{F}_t})$ are the underlying wfs of $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$, respectively, where the bar denotes retract closure. The triple $(\overline{\mathcal{C}}, \overline{\mathcal{F}}, \mathcal{W})$ arising from an algebraic model structure gives a model structure on $\mathcal{M}$ in the ordinary sense; we call this the *underlying ordinary model structure* on $\mathcal{M}$.

We say that an algebraic model structure is *cofibrantly generated* if the two awfs are cofibrantly generated, in the sense described in Section I.2.6. In this case, $\mathcal{F} = \overline{\mathcal{F}}$ and $\mathcal{F}_t = \overline{\mathcal{F}_t}$ by Lemma I.2.30.

It is convenient to have notation for the two functorial factorizations. Let $Q = \operatorname{cod} C = \operatorname{dom} F_t$ be the functor $\mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ accompanying the functorial factorization of $(\mathbb{C}, \mathbb{F}_t)$, i.e., the functor that picks out the object that an arrow factors through. Let R be the analogous functor for $(\mathbb{C}_t, \mathbb{F})$. This notation is meant to suggest cofibrant and fibrant replacement, respectively.

With this notation, the comparison map $\xi \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ consists of natural arrows ξ_f for each $f \in \mathcal{M}^{\mathbf{2}}$ such that

$$\begin{tikzcd} & \operatorname{dom} f \arrow[dl, "C_t f"'] \arrow[dr, "Cf"] & \\ Rf \arrow[rr, "\xi_f"] \arrow[dr, "Ff"'] & & Qf \arrow[dl, "F_t f"] \\ & \operatorname{cod} f & \end{tikzcd} \tag{I.3.2}$$

commutes. Because ξ is a morphism of awfs, it induces functors

$$\xi_* \colon \mathbb{C}_t\textbf{-coalg} \to \mathbb{C}\textbf{-coalg} \quad \text{and} \quad \xi^* \colon \mathbb{F}_t\textbf{-alg} \to \mathbb{F}\textbf{-alg},$$

which provide an algebraic way to regard a trivial cofibration as cofibration and a trivial fibration as a fibration. These maps have the following property. Given a lifting problem between a trivial cofibration j and a trivial fibration q, there are two natural ways to solve it: regard the trivial cofibration as a cofibration and use the awfs $(\mathbb{C}, \mathbb{F}_t)$ or regard the trivial fibration as a fibration and use the awfs $(\mathbb{C}_t, \mathbb{F})$.[6] In figure (I.3.3) below, the former option solves the lifting problem $(u,v)\colon\ j \Rightarrow q$ by tracing the path around the middle of the back of the cube, while the latter option traces along the front of the cube. Naturality of ξ says that the lifts constructed by each method are the same!

$$\begin{tikzcd}
 & & & \cdot \arrow[d, "Cj"'] \arrow[ddll, equal] \arrow[ddrr, "u"] & & & \\
 & & & \cdot \arrow[d, "F_t j"'] \arrow[ddrr, "Q(u{,}v)"] & & & \\
\cdot \arrow[r, equal] \arrow[dd, "j"', "\sim"] & \cdot \arrow[d, "C_t j"'] \arrow[ddrr, "u"] & & \cdot \arrow[ddll, equal] \arrow[ddrr, "v"] & & \cdot \arrow[r, equal] \arrow[d, "Cq"] & \cdot \arrow[dd, "q", "\sim"'] \\
 & \cdot \arrow[d, "Fj"'] \arrow[uurr, "\xi_j"] \arrow[ddrr, "R(u{,}v)"'] & & & & \cdot \arrow[d, "F_t q"] \arrow[ur, "t"'] & \\
\cdot \arrow[r, equal] \arrow[ur, "s"] & \cdot \arrow[ddrr, "v"'] & & \cdot \arrow[d, "C_t q"] \arrow[uurr, equal] & & \cdot \arrow[r, equal] & \cdot \\
 & & & \cdot \arrow[d, "Fq"] \arrow[uurr, "\xi_q"'] & & & \\
 & & & \cdot \arrow[uurr, equal] & & &
\end{tikzcd} \tag{I.3.3}$$

I.3.1 Comparing fibrant-cofibrant replacements

Any algebraic model structure induces a *fibrant replacement monad* $\mathbb{R}$ and a *cofibrant replacement comonad* $\mathbb{Q}$ on the category $\mathcal{M}$ (as opposed to the arrow category $\mathcal{M}^{\mathbf{2}}$ on which the monads and comonads of the awfs act). The monad $\mathbb{R}$ arises as follows. The category $\mathcal{M}$ includes into

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[6] The map ξ assigns $\mathbb{C}$-coalgebra structures to $\mathbb{C}_t$-coalgebras. Similarly, ξ maps the trivial cofibrations which are merely coalgebras for the pointed endofunctor underlying $\mathbb{C}_t$ to coalgebras for the pointed endofunctor of $\mathbb{C}$, which we saw in Remark I.2.11 suffices to construct lifts (I.2.10), which have the naturality property of (I.3.3). Similar remarks apply to the fibrations.

$\mathcal{M}^{\mathbf{2}}$ by sending an object X to the unique arrow from X to the terminal object. This inclusion is a section to the functor $\mathrm{dom}\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$. Because the monad $\mathbb{F}$ is a monad over cod, it induces a monad $\mathbb{R} = (R, \eta, \mu)$ on $\mathcal{M}$ which we call the *fibrant replacement monad*. The functor R is obtained from the previous functor $R\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ accompanying the functorial factorization of $(\mathbb{C}_t, \mathbb{F})$ by precomposing R by this inclusion. We regret that our notation is somewhat ambiguous. The domain of R should be apparent from whether an object in the image of R is the image of an object or arrow of $\mathcal{M}$. The arrows in the image of the two functors are related as follows: $Rf = R(f, 1_*)$, where 1_* denotes the identity at the terminal object.

Dually, we can include $\mathcal{M}$ into $\mathcal{M}^{\mathbf{2}}$ by slicing under the initial object. Using this inclusion, the comonad $\mathbb{C}$ induces a comonad $\mathbb{Q} = (Q, \epsilon, \delta)$ on $\mathcal{M}$ which we call the *cofibrant replacement comonad*. Once again, the functor $Q\colon \mathcal{M} \to \mathcal{M}$ is obtained from the previous functor $Q\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ by precomposing Q by this inclusion. Algebras for $\mathbb{R}$ are called *algebraically fibrant objects* and coalgebras for $\mathbb{Q}$ are called *algebraically cofibrant objects*.

Another application of the natural lift illustrated in (I.3.3) is in comparing fibrant-cofibrant replacements of an object. Let $\mathcal{M}$ be a category with an algebraic model structure and let $X \in \mathcal{M}$. We can define its fibrant-cofibrant replacement to be either RQX or QRX, both of which are weakly equivalent to X. Classically, there is no natural comparison between these choices, but in any algebraic model structure there is a natural arrow $RQX \to QRX$ built out of the comparison map together with the components of the awfs.

Lemma I.3.4. *Let $\mathcal{M}$ be a category with an algebraic model structure and let R and Q be the induced fibrant and cofibrant replacement on $\mathcal{M}$. Then there is a canonical natural transformation $\chi\colon RQ \Rightarrow QR$.*

Proof. Classically, one obtains a map $RQX \to QRX$ by first lifting i against q and j against p, as in the figure on the left below. Because the maps i, j, p, and q are all obtained by factoring, they have free coalgebra or algebra structures for the awfs $(\mathbb{C}_t, \mathbb{F})$ or $(\mathbb{C}, \mathbb{F}_t)$. Thus, each of these lifting problems has a natural solution (see Remark I.2.13). After a diagram chase, we can write the solution to the first lifting problem as $Q\eta_X$ and the second as $R\epsilon_X$, using the unit and counit of the monad $\mathbb{R}$ and the comonad $\mathbb{Q}$.

$$\begin{tikzcd}
& \emptyset \arrow[dl, tail, "i"'] \arrow[dr] & \\
QX \arrow[rr, dashed, "Q\eta_X"] \arrow[dr, two heads, "\epsilon_X", "\sim"'] \arrow[dd, tail, "\eta_{QX}=j"', "\sim"] & & QRX \arrow[dd, two heads, "q=\epsilon_{RX}", "\sim"'] \\
& X \arrow[dr, tail, "\eta_X", "\sim"'] & \\
RQX \arrow[rr, dashed, "R\epsilon_X"'] \arrow[dr, two heads] & & RX \arrow[dl, two heads, "p"] \\
& * &
\end{tikzcd}
\qquad
\begin{tikzcd}
QX \arrow[r, "Q\eta_X"] \arrow[d, tail, "\eta_{QX}=j"', "\sim"] & QRX \arrow[d, two heads, "q=\epsilon_{RX}", "\sim"'] \\
RQX \arrow[r, "R\epsilon_X"'] \arrow[ur, dashed, "\chi_X"] & RX
\end{tikzcd}$$

The arrows $Q\eta_X$ and $R\epsilon_X$ present a lifting problem between j and q that can be solved naturally using either awfs, as depicted in figure (I.3.3). The solutions to these lifting problems are the components of a natural transformation $RQ \Rightarrow QR$ comparing the two fibrant-cofibrant replacements. □

This natural map is particularly well-behaved; hence the following theorem.

Theorem I.3.5. *The functor Q lifts to a cofibrant replacement comonad on the category $\mathbb{R}$-**alg** of algebraically fibrant objects. Dually, the functor R lifts to a fibrant replacement monad on the category $\mathbb{Q}$-**coalg** of algebraically cofibrant objects. The Eilenberg-Moore categories for this lifted comonad and lifted monad are isomorphic and give a notion of "algebraically bifibrant objects."*

Proof. By a well-known categorical result [PW02], it suffices to find a natural transformation $\chi\colon RQ \Rightarrow QR$ that is a distributive law of the monad over the comonad, i.e., a lax morphisms of monads and a colax morphism of comonads (see Section I.6.1). The natural map of Lemma I.3.4 satisfies the desired property: The defining lifting problem shows that χ is compatible with the unit and counit for R and Q. It remains to show that χ is compatible with the multiplication, i.e., that

$$\begin{tikzcd}
& RQR \arrow[dr, "\chi R"] & \\
R^2Q \arrow[ur, "R\chi"] \arrow[d, "\mu Q"'] & & QR^2 \arrow[d, "Q\mu"] \\
RQ \arrow[rr, "\chi"] & & QR
\end{tikzcd}$$

commutes, and a dual condition for the comultiplication. The necessary diagram chase uses the fact that the awfs $(\mathbb{C}_t, \mathbb{F})$ satisfies the distributive law and the fact that the comparison map ξ induces a monad morphism. We leave the remaining details to the reader. □

Unless we are talking about fibrant or cofibrant replacement specifically, R and Q will be functors $\mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ accompanying the functorial factorizations of an algebraic model structure.

I.3.2 The comparison map

The least familiar component of the definition of an algebraic model structure given above is the comparison map. In figure (I.3.3), Lemma I.3.4, and Theorem I.3.5, we saw some of its useful properties, but the question remains: in what circumstances might one expect a comparison map to exist? We discuss several answers to this question in this section.

Remark I.3.6. Let $\mathcal{J}$ be the generating category for the awfs $(\mathbb{C}_t, \mathbb{F})$ and let $(\mathbb{C}, \mathbb{F}_t)$ be an awfs on the same category. A comparison map between $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ exists if and only if there is a functor $\zeta \colon \mathcal{J} \to \mathbb{C}$-**coalg** over $\mathcal{M}^{\mathbf{2}}$. This is because Garner's small object argument produces a canonical map $\gamma \colon \mathcal{J} \to \mathbb{C}_t$-**coalg** in $\mathbf{CAT}/\mathcal{M}^{\mathbf{2}}$ that is universal among arrows from $\mathcal{J}$ to categories of coalgebras for the left half of an awfs on $\mathcal{M}$, in the sense that every such morphism ζ factors uniquely as $\xi_* \circ \gamma$, where ξ is a morphism of awfs. See [Gar09, §3].

As far as the author is aware, model category theorists have not written about the issue of comparing the two wfs provided by an ordinary model structure, a fact that first came to her attention through discussions with Martin Hyland. But the existence of such a comparison map is more reasonable than one might expect: Peter May notes [MP11] that the universal property of the colimits in Quillen's small object argument gives such a natural transformation, provided we assume that the generating trivial cofibrations $\mathcal{J}$ are contained in the generating cofibrations $\mathcal{I}$.

In many cases, this admittedly untraditional assumption is quite reasonable: the generating trivial cofibrations are of course cofibrations, so including them with the generators does not change the resulting model structure. In the setting of algebraic model structures, this inclusion takes the form of a functor $\mathcal{J} \to \mathcal{I}$ over $\mathcal{M}^{\mathbf{2}}$, which induces a comparison map by the universal property of the "free" awfs generated by $\mathcal{J}$ (see Section I.2.6). When such a functor exists, "the" comparison map always refers to this one, though a priori some other might exist. In some of the results that follow, we require that there be a functor $\mathcal{J} \to \mathcal{I}$ between the generating categories of a cofibrantly generated algebraic model structure without feeling too badly about it.

The following remark supports our Definition I.3.1.

Remark I.3.7. Any ordinary cofibrantly generated model structure on a category that permits the small object argument can be made into an algebraic model structure by replacing the generating cofibrations $\mathcal{I}$ by $\mathcal{I} \cup \mathcal{J}$ and applying Garner's small object argument in place of Quillen's. The underlying ordinary model structure of the resulting algebraic model structure is the same as before, by which we mean that the classes $\overline{\mathcal{C}}$, $\mathcal{F}$, and $\mathcal{W}$ are unchanged. Thus, the abundance of cofibrantly generated model structures (in the ordinary sense) gives rise to an abundance of examples of algebraic model structures, which are then of course cofibrantly generated.

While altering the generating cofibrations does not change the underlying model structure, it does change the cofibration-trivial fibration factorization. Given that the generating cofibrations are often more natural than the generating trivial cofibrations,[7] we provide the following alternative method for obtaining a comparison map for a cofibrantly generated algebraic model structure by altering $\mathcal{J}$ as opposed to $\mathcal{I}$, vis-à-vis a theorem inspired by [Hir03, 11.2.9]. As will become clear in the proof below, this method only applies in the case where the trivial cofibrations are generated by a set, as opposed to a category.

In the following proof even though $\mathcal{J}$ is discrete, we regard it as a category over $\mathcal{M}^{\mathbf{2}}$ with an injection $\mathcal{J} \xrightarrow{J} \mathcal{M}^{\mathbf{2}}$. With this perspective, we need a technical note. While Garner's small object argument works for any small category $J \colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ above the arrow category, in practice, the functor J is injective, and we can identify $\mathcal{J}$ with its image and think of it as a set of arrows together with some coherence conditions in the form of morphisms between these arrows. As stated, Theorem I.3.8 requires that J be injective, though one could imagine that more careful wording would allow us to drop this assumption. We chose this simplification because we cannot think of any applications where this restriction is prohibitive.

Theorem I.3.8. *Suppose $\mathcal{J}$ is a set and $\mathcal{I}$ is a category over $\mathcal{M}^{\mathbf{2}}$ such that the underlying wfs $(\overline{\mathcal{C}_t}, \mathcal{F})$ and $(\overline{\mathcal{C}}, \mathcal{F}_t)$ of the awfs $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ that they generate give a model structure on $(\mathcal{M}, \mathcal{W})$, in the ordinary sense. Then $\mathcal{J}$ can be replaced by a set $\mathcal{J}'$ over $\mathcal{M}^{\mathbf{2}}$ such that*

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7. Indeed, in many Bousfield localizations, the generating trivial cofibrations are not known explicitly.

(i) *There is a functor* $\mathcal{J}'^{\boxslash} \to \mathcal{J}^{\boxslash}$ *over* $\mathcal{M}^{\mathbf{2}}$*, necessarily faithful, that is bijective on the underlying classes of arrows. It follows that* $\mathcal{J}'$ *and* $\mathcal{J}$ *generate the same underlying wfs.*

(ii) *There is a functor* $\zeta \colon \mathcal{J}' \to \mathbb{C}$-**coalg** *over* $\mathcal{M}^{\mathbf{2}}$*.*

The set $\mathcal{J}'$ *generates an awfs* $(\mathbb{C}'_t, \mathbb{F}')$*. It follows from the universal property of the functor* $\mathcal{J}' \to \mathbb{C}'_t$-**coalg** *that* $\mathcal{I}$ *and* $\mathcal{J}'$ *generate an algebraic model structure on* $\mathcal{M}^{\mathbf{2}}$ *with the same underlying model structure* $(\overline{\mathcal{C}}, \mathcal{F}, \mathcal{W})$*.*

Proof. Define $\mathcal{J}' \xrightarrow{J'} \mathcal{M}^{\mathbf{2}}$ to be the composite $\mathcal{J} \xrightarrow{J} \mathcal{M}^{\mathbf{2}} \xrightarrow{C} \mathcal{M}^{\mathbf{2}}$ where C is the comonad generated by $\mathcal{I}$. For each $j \in \mathcal{J}$, the corresponding element of $\mathcal{J}'$ is its left factor Cj. We claim that $\mathcal{J}' = \{Cj \mid j \in \mathcal{J}\}$ satisfies conditions (i) and (ii) above. For (ii), define ζ to be the map that assigns each Cj its canonical free coalgebra structure (Cj, δ_j).

For (i), note that for each $j \in \mathcal{J}$ the lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "Cj"] \arrow[d, "j"'] & \cdot \arrow[d, "F_t j"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed, "s"] & \cdot \end{tikzcd}$$

has a solution s, which gives us a retract diagram

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "j"'] & \cdot \arrow[r, equal] \arrow[d, "Cj"] & \cdot \arrow[d, "j"] \\ \cdot \arrow[r, "s"'] & \cdot \arrow[r, "F_t j"'] & \cdot \end{tikzcd}$$

. We use this to define a functor $\mathcal{J}'^{\boxslash} \to \mathcal{J}^{\boxslash}$ as in the proof of Lemma I.2.30. On objects, define

$$\mathcal{J}'^{\boxslash} \ni (g, \psi) \mapsto (g, \phi) \in \mathcal{J}^{\boxslash},$$

where $\phi(j, u, v) := \psi(Cj, u, v \cdot F_t j) \cdot s$ for all lifting problems $(u, v) \colon j \Rightarrow g$. Because $\mathcal{J}$ is discrete, the lifting function ϕ need not satisfy any coherence conditions. Given a morphism $(h, k) \colon (g, \psi) \to (g', \psi')$ in $\mathcal{J}'^{\boxslash}$, it follows that

$$\begin{aligned} \phi'(j, h \cdot u, k \cdot v) &= \psi'(Cj, h \cdot u, k \cdot v \cdot F_t j) \cdot s \\ &= h \cdot \psi(Cj, u, v \cdot F_t j) \cdot s \\ &= h \cdot \phi(j, u, v), \end{aligned}$$

which says precisely that $(h, k) \colon (g, \phi) \to (g', \phi')$ is a morphism in $\mathcal{J}^{\boxslash}$. So $\mathcal{J}'^{\boxslash} \to \mathcal{J}^{\boxslash}$ is a functor over $\mathcal{M}^{\mathbf{2}}$.

It remains to show that this functor is surjective on the underlying arrows of $\mathcal{J}'^{\boxslash}$ and $\mathcal{J}^{\boxslash}$. Let $j \in \mathcal{J}$ and $(g, \phi) \in \mathcal{J}^{\boxslash}$; by definition $g \in \mathcal{F}$. By the 2-of-3 property, $Cj \in \mathcal{C} \cap \mathcal{W} \subset \overline{\mathcal{C}_t}$, so $Cj \boxslash g$.

As $\mathcal{J}'$ is discrete, any choice of lifts against the Cj can be used to define a lifting function ψ so that $(g, \psi) \in \mathcal{J}'^{\boxslash}$. Of course the functor defined above need not map (g, ψ) to (g, ϕ) but it does mean that g is in the image when we forget down to $\mathcal{M}^{\mathbf{2}}$, which is all that we claimed. $\square$

Remark I.3.9. We also have a faithful functor $\mathcal{J}^{\boxslash} \to \mathcal{J}'^{\boxslash}$ over $\mathcal{M}^{\mathbf{2}}$ that is bijective on the underlying classes of arrows; this one, however, takes a bit more effort to define. Define $\mathcal{J}'' \xrightarrow{J''} \mathcal{M}^{\mathbf{2}}$ to be the composite $\mathcal{J}' \xrightarrow{J'} \mathcal{M}^{\mathbf{2}} \xrightarrow{C_t} \mathcal{M}^{\mathbf{2}}$. We have a functor $\gamma\colon \mathcal{J}'' \to \mathbb{C}_t$-**coalg** over $\mathcal{M}^{\mathbf{2}}$ that assigns each arrow its free coalgebra structure. Mirroring the argument above, elements of $\mathcal{J}'$ are retracts of elements of $\mathcal{J}''$, so we have $\mathcal{J}''^{\boxslash} \to \mathcal{J}'^{\boxslash}$ over $\mathcal{M}^{\mathbf{2}}$. Our desired functor is the composite

$$\mathcal{J}^{\boxslash} \cong \mathbb{F}\text{-}\mathbf{alg} \xrightarrow{\text{lift}} (\mathbb{C}_t\text{-}\mathbf{coalg})^{\boxslash} \xrightarrow{\gamma^{\boxslash}} \mathcal{J}''^{\boxslash} \longrightarrow \mathcal{J}'^{\boxslash}$$

defined with help from the functor of I.2.25 and the isomorphism (I.2.27). Note that these functors are not inverse equivalences.

The upshot is that we can get an algebraic model structure from an ordinary cofibrantly generated model structure without changing the generating cofibrations. This argument does not appear to extend to non-discrete categories $\mathcal{J}$ because, in absence of a comparison map, the $\mathbb{F}$-algebra structures of the $F_t j$ are *chosen* and not *natural* with respect to morphisms in $\mathcal{J}$; see Remark I.2.13. Note, the proof of Theorem I.3.8 did not require that the awfs $(\mathbb{C}, \mathbb{F}_t)$ is cofibrantly generated, though in examples this is typically the case.

In Section I.5, we will show that the components of the comparison map in a cofibrantly generated algebraic model structure satisfying additional, relatively mild, hypotheses are themselves $\mathbb{C}$-coalgebras and hence cofibrations.

I.3.3 Algebraic model structures and adjunctions

Many cofibrantly generated model structures are produced from previously known ones by passing the generating sets across an adjunction. We repeat this trick for cofibrantly generated algebraic model structures, extending a well-known theorem due to Kan [Hir03, 11.3.2].

An adjunction $T\colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{K}\colon S$ lifts to an adjunction on the arrow categories $\mathcal{M}^{\mathbf{2}}$ and $\mathcal{K}^{\mathbf{2}}$, which we also denote by $T \dashv S$. In particular, a small category $J\colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ over $\mathcal{M}^{\mathbf{2}}$ becomes a small category $TJ\colon \mathcal{J} \to \mathcal{K}^{\mathbf{2}}$ over $\mathcal{K}^{\mathbf{2}}$. Because our notation has usually described the generating category as opposed to its functor to $\mathcal{M}^{\mathbf{2}}$, we write $T\mathcal{J}$ to mean the category $\mathcal{J}$ that maps to $\mathcal{K}^{\mathbf{2}}$ via the functor TJ.

Theorem I.3.10. *Let $\mathcal{M}$ have an algebraic model structure, generated by $\mathcal{I}$ and $\mathcal{J}$ and with weak equivalences $\mathcal{W}_{\mathcal{M}}$. Let $T\colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}\colon S$ be an adjunction. Suppose $\mathcal{K}$ permits the small object argument and also that*

($\star$) *S maps arrows underlying the left class of the awfs cofibrantly generated by $T\mathcal{J}$ into $\mathcal{W}_{\mathcal{M}}$.*

Then $T\mathcal{J}$ and $T\mathcal{I}$ generate an algebraic model structure on $\mathcal{K}$ with $\mathcal{W}_{\mathcal{K}} = S^{-1}(\mathcal{W}_{\mathcal{M}})$. Furthermore, $T \dashv S$ is a Quillen adjunction for the underlying ordinary model structures.

In the literature, ($\star$) is known as the *acyclicity condition* because the arrows underlying the left class of the awfs generated by $T\mathcal{J}$ are the proposed *acyclic* (trivial) cofibrations for the model structure on $\mathcal{K}$.

Proof of Theorem I.3.10. By the small object argument, $T\mathcal{J}$ and $T\mathcal{I}$ generate awfs $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ with underlying wfs $(\overline{\mathcal{C}_t}, \mathcal{F})$ and $(\overline{\mathcal{C}}, \mathcal{F}_t)$. With this notation, the condition ($\star$) says that $S(\mathcal{C}_t) \subset \mathcal{W}_{\mathcal{M}}$.

When the comparison map of the algebraic model structure on $\mathcal{M}$ arises from a functor $\mathcal{J} \to \mathcal{I}$ over $\mathcal{M}^{\mathbf{2}}$, composing with T induces a functor $\mathcal{J} \to \mathcal{I}$ over $\mathcal{K}^{\mathbf{2}}$, which gives the comparison map between the resulting awfs. In the general case, the comparison map on $\mathcal{M}$ specifies a functor $\mathcal{J} \to \mathbb{C}^{\mathcal{M}}$-**coalg** to the category of coalgebras for the awfs on $\mathcal{M}$ generated by $\mathcal{I}$. In Corollary I.6.16, a significant result whose proof is deferred to Section I.6, we will prove that there is a functor $\mathbb{C}^{\mathcal{M}}$-**coalg** $\to$ $\mathbb{C}$-**coalg** lifting T. This gives rise to a functor $\mathcal{J} \to \mathbb{C}$-**coalg** lifting T, or equivalently a functor $T\mathcal{J} \to \mathbb{C}$-**coalg** over $\mathcal{K}^{\mathbf{2}}$. The comparison map $(\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ for $\mathcal{K}$ is then induced by the universal property of the functor $T\mathcal{J} \to \mathbb{C}_t$-**coalg** produced by Garner's small object argument.

The class $\mathcal{W}_{\mathcal{K}}$ is retract closed by functoriality of S. It remains to show that $\overline{\mathcal{C}_t} = \overline{\mathcal{C}} \cap \mathcal{W}_{\mathcal{K}}$ and $\mathcal{F}_t = \mathcal{F} \cap \mathcal{W}_{\mathcal{K}}$. In fact, by [Hir03, 11.3.1] we need only verify three of the four relevant inclusions.

The inclusion $\mathcal{C}_t \subset \mathcal{C}$ is immediate, since the comparison map explicitly provides each trivial cofibration with a cofibration structure; taking retract closures $\overline{\mathcal{C}_t} \subset \overline{\mathcal{C}}$. The hypothesis ($\star$) says that $\mathcal{C}_t \subset \mathcal{W}_{\mathcal{K}}$ and $\mathcal{W}_{\mathcal{K}}$ is retract closed by functoriality of S, so $\overline{\mathcal{C}_t} \subset \mathcal{W}_{\mathcal{K}}$. Hence, $\overline{\mathcal{C}_t} \subset \overline{\mathcal{C}} \cap \mathcal{W}_{\mathcal{K}}$.

Similarly, the comparison map guarantees that $\mathcal{F}_t \subset \mathcal{F}$. If $g \in \mathcal{F}_t$ then it has some algebra structure $(g, \phi) \in T\mathcal{I}^{\boxslash}$ by Lemma I.2.30. By adjunction $(Sg, \phi^\sharp) \in \mathcal{I}^{\boxslash}$, where the arrows of $\phi^\sharp$ are the adjuncts of the corresponding arrows of ϕ. So Sg is a trivial fibration for the model structure on $\mathcal{M}$. In particular, $Sg \in \mathcal{W}_{\mathcal{M}}$, which says that $g \in \mathcal{W}_{\mathcal{K}}$. So $\mathcal{F}_t \subset \mathcal{F} \cap \mathcal{W}_{\mathcal{K}}$.

It remains to show that $\mathcal{F} \cap \mathcal{W} \subset \mathcal{F}_t$; we will appeal to Lemma I.2.30 on two occasions. Suppose $f \in \mathcal{F} \cap \mathcal{W}_{\mathcal{K}}$. By Lemma I.2.30, f has some algebra structure $(f, \psi) \in T\mathcal{J}^{\boxslash}$ and by adjunction $(Sf, \psi^\sharp) \in \mathcal{J}^{\boxslash}$. As $f \in \mathcal{W}_{\mathcal{K}}$, Sf is a trivial fibration in the algebraic model structure on $\mathcal{M}$; by Lemma I.2.30, it follows that there is some algebra structure ζ such that $(Sf, \zeta) \in \mathcal{I}^{\boxslash}$. By adjunction, $(f, \zeta^\flat) \in T\mathcal{I}^{\boxslash}$, where $\zeta^\flat$ denotes the adjunct of ζ, which says that $f \in \mathcal{F}_t$, as desired.

The above argument showed that S preserves fibrations and trivial fibrations. Hence, $T \dashv S$ is a Quillen adjunction. □

I.3.4 Algebraic Quillen adjunctions

Given the close connection between the algebraic model structures of Theorem I.3.10, it is not surprising that quite a lot more can be said about the nature of the Quillen adjunction between them. This leads to the notion of an *algebraic Quillen adjunction*, of which the adjunction of Theorem I.3.10 will be an example. We preview the definition and corresponding theorem below, but postpone the proofs, which are categorically intensive, to Sections I.6 and I.7. These sections are not dependent on the intermediate material, so a categorically inclined reader may wish to skip there directly.

Morphisms of awfs provide a means of comparing awfs on the same category, but as far as the author is aware, there are no such comparisons for awfs on different categories in the literature. We define three useful types of morphisms precisely in Section I.6, but here are the main ideas.

Let $(\mathbb{C}, \mathbb{F})$ and $(\mathbb{L}, \mathbb{R})$ be awfs on $\mathcal{M}$ and $\mathcal{K}$ respectively. A *colax morphism of awfs* $(\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ is a functor $T\colon \mathcal{M} \to \mathcal{K}$ together with a specified lifting of T to a functor $\tilde{T}\colon \mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg** satisfying one additional requirement. By a categorical result [Joh75], the lift $\tilde{T}$ is determined by a characterizing natural transformation; together T and this natural transformation is called a *colax morphism of comonads* or simply a *comonad morphism*. We ask that the natural transformation characterizing $\tilde{T}$ also determines an extension of T to a functor $\hat{T}\colon \mathbf{Kl}(\mathbb{F}) \to \mathbf{Kl}(\mathbb{R})$ between the Kleisli categories of the monads.

The Kleisli category of a monad $\mathbb{R}$ is the full subcategory of $\mathbb{R}$-**alg** on the free algebras, which are the objects in the image of the (monadic) free-forgetful adjunction. **Kl**($\mathbb{R}$) is initial in the category of adjunctions determining that monad; the Eilenberg-Moore category $\mathbb{R}$-**alg** is terminal. At the moment, the only justification we can give for this additional requirement, beyond the fact that it holds in important examples, is that colax morphisms of awfs should interact with both sides of the awfs. A more convincing justification is Lemma I.6.9.

Dually, a *lax morphism of awfs* $(\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ is a functor $S : \mathcal{K} \to \mathcal{M}$ together with a specified lift $\tilde{S} : \mathbb{R}$-**alg** $\to \mathbb{F}$-**alg** such that the natural transformation characterizing $\tilde{S}$ must determine an extension of S to a functor between the coKleisli categories of $\mathbb{L}$ and $\mathbb{C}$. When the functor S or T is the identity, both lax and colax morphisms of awfs are exactly morphisms of awfs, which is another clue that these are reasonable notions.

Combining these, we arrive at the notion of *adjunction of awfs*, which is the most relevant to this context. An *adjunction of awfs* $(T, S) \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ consists of an adjoint pair of functors $T \colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ such that $T \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ is a colax morphism of awfs, $S : (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ is a lax morphism of awfs, and the characterizing natural transformations for these morphisms are related in a suitable fashion.

Adjunctions of awfs over identity adjunctions are exactly morphisms of awfs, with both characterizing natural transformations equal to the natural transformation of Definition I.2.14. Note, adjunctions of awfs can be canonically composed. We can now define *algebraic Quillen adjunctions*.

Definition I.3.11. Let $\mathcal{M}$ have an algebraic model structure $\xi^{\mathcal{M}} \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ and let $\mathcal{K}$ have an algebraic model structure $\xi^{\mathcal{K}} \colon (\mathbb{L}_t, \mathbb{R}) \to (\mathbb{L}, \mathbb{R}_t)$. An adjunction $T \colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ is an *algebraic Quillen adjunction* if there exist adjunctions of awfs

$$\begin{tikzcd}
(\mathbb{C}_t, \mathbb{F}) \arrow[r, "{(T,S)}"] \arrow[d, "\xi^{\mathcal{M}}"'] \arrow[dr, "{(T,S)}"] & (\mathbb{L}_t, \mathbb{R}) \arrow[d, "\xi^{\mathcal{K}}"] \\
(\mathbb{C}, \mathbb{F}_t) \arrow[r, "{(T,S)}"'] & (\mathbb{L}, \mathbb{R}_t)
\end{tikzcd}$$

such that both triangles commute.

Note the left adjoint of an adjunction of awfs preserves coalgebras and hence (trivial) cofibrations and dually the right adjoint preserves algebras and hence (trivial) fibrations. In particular, an algebraic Quillen adjunction is a Quillen adjunction, in the ordinary sense. As we shall prove in Section I.7, the naturality condition of the definition of algebraic Quillen adjunction is equivalent to the condition that the lifts depicted below commute.

$$\begin{tikzcd}[column sep=small]
\mathbb{R}_t\text{-}\mathbf{alg} \arrow[rr, "\tilde{S}_t"] \arrow[dr, "(\xi^{\mathcal{K}})^*"'] \arrow[dd] & & \mathbb{F}_t\text{-}\mathbf{alg} \arrow[dr, "(\xi^{\mathcal{M}})^*"] \arrow[dd] & \\
& \mathbb{R}\text{-}\mathbf{alg} \arrow[rr, "\tilde{S}"] \arrow[dl] & & \mathbb{F}\text{-}\mathbf{alg} \arrow[dl] \\
\mathcal{K}^{\mathbf{2}} \arrow[rr, "S"'] & & \mathcal{M}^{\mathbf{2}} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
\mathbb{C}_t\text{-}\mathbf{coalg} \arrow[rr, "\tilde{T}_t"] \arrow[dr, "(\xi^{\mathcal{M}})_*"'] \arrow[dd] & & \mathbb{L}_t\text{-}\mathbf{coalg} \arrow[dr, "(\xi^{\mathcal{K}})_*"] \arrow[dd] & \\
& \mathbb{C}\text{-}\mathbf{coalg} \arrow[rr, "\tilde{T}"] \arrow[dl] & & \mathbb{L}\text{-}\mathbf{coalg} \arrow[dl] \\
\mathcal{M}^{\mathbf{2}} \arrow[rr, "T"'] & & \mathcal{K}^{\mathbf{2}} &
\end{tikzcd}
\tag{I.3.12}$$

Similarly, the corresponding extensions to Kleisli and coKleisli categories commute.

Somewhat surprisingly due to the numerous conditions required by their components, algebraic Quillen adjunctions exist in familiar situations.

Theorem I.3.13. *Let* $T\colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{K}\colon S$ *be an adjunction. Suppose* $\mathcal{M}$ *has an algebraic model structure, generated by* $\mathcal{I}$ *and* $\mathcal{J}$*, with comparison map* $\xi^{\mathcal{M}}$*. Suppose* $\mathcal{K}$ *has the algebraic model structure, generated by* $T\mathcal{I}$ *and* $T\mathcal{J}$*, with canonical comparison map* $\xi^{\mathcal{K}}$*. Then* $T \dashv S$ *is canonically an algebraic Quillen adjunction.*

The proof is deferred to Section I.7.

I.4 Pointwise awfs and the projective model structure

One of the features of an awfs that is not true of an ordinary wfs or even of a functorial wfs is that an awfs on a category $\mathcal{M}$ induces an awfs on the diagram category $\mathcal{M}^{\mathcal{A}}$ for any small category $\mathcal{A}$, where the factorizations are defined pointwise. The comultiplication and multiplication maps are precisely what is needed to define natural transformations that ensure that the left and right factors have the desired lifting properties. Furthermore, and completely unlike the non-algebraic situation, such *pointwise awfs* are cofibrantly generated if the original awfs is. After proving this result, we will give an example of a class of cofibrantly generated algebraic model structures whose underlying ordinary model structures are not cofibrantly generated in the classical sense.

We then construct a projective algebraic model structure on $\mathcal{M}^{\mathcal{A}}$ from a cofibrantly generated algebraic model structure on $\mathcal{M}$. The awfs in the projective model structure will not be the pointwise awfs, though these awfs will make an appearance in the proof establishing this model structure.

We first take a detour to describe Garner's small object argument in more detail, as these details will be used in the proofs in this section and the next.

I.4.1 Garner's small object argument

Like Quillen's, Garner's small object argument produces a functorial factorization through a colimiting process that takes many steps, a key difference being that the resulting functorial factorization canonically underlies an awfs. Each step gives rise to a functorial factorization in which the left functor is a comonad. At the final step, the right functor is also a monad.

At step zero, Quillen's small object argument forms a coproduct over all squares from the generating cofibrations to the arrow f. In Garner's small object argument, this coproduct is replaced by a left Kan extension of the functor $J : \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ along itself. Write $L^0 f = \mathrm{Lan}_J J(f)$ for the step zero comonad, called the density comonad in the literature. When $\mathcal{J}$ is discrete, $L^0 f$ is the usual coproduct. In the general case, this arrow is a quotient of the usual coproduct.

The step-one factorization of both small object arguments is obtained the same way: by factoring the counit of the density comonad as a pushout followed by a square with an identity arrow on top.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "L^0 f"'] & \cdot \arrow[r, equal] \arrow[d, "L^1 f"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \cdot \arrow[r, "R^1 f"'] & \cdot
\end{tikzcd} \tag{I.4.1}$$

Concretely, $L^1 f$ is the pushout of $L^0 f$ along the canonical arrow from the domain of $L^0 f$ to the domain of f. The arrow f and the arrow from the codomain of $L^0 f$ to the codomain of f form a cone under this pushout diagram; the unique map given by the universal property is $R^1 f$. By the universal property of the pushout (I.4.1), specifying an $(R^1, \vec{\eta}^1)$-algebra structure for f is equivalent to specifying a lifting function ϕ such that $(f, \phi) \in \mathcal{J}^{\boxslash}$.

The most significant difference between Garner's and Quillen's small object argument appears in the inductive steps that follow. For Quillen's small object argument the above processes are repeated with the arrow $R^\alpha f$ in place of f. We take the left Kan extension (coproduct over squares) and then pushout to obtain an arrow that is composed with the preceding left factors to obtain $L^{\alpha+1} f$. The map induced by the universal property is $R^{\alpha+1} f$. Transfinite composition is used to obtain L^α and R^α for limit ordinals. We choose to halt this process at some predetermined "sufficiently large" ordinal, yielding the final functorial factorization.

For Garner's small object argument, this process is modified to include additional quotienting. At step two and all subsequent steps, the beginning is the same. We pushout L^0R^1f along the canonical arrow to obtain L^1R^1f. But then L^2f is defined to be L^1f composed with the coequalizer of two arrows from L^1f to L^1R^1f. As in previous steps, this is a quotient of Quillen's definition. In the language of cell complexes, the arrow L^1R^1f freely attaches new "cells" to the "spheres" in the domain of R^1f, while L^1f includes those "cells" attached to "spheres" in the domain of f into their image in the domain of R^1f. The coequalizer then avoids redundancy by identifying those "cells" attached to the same "spheres" in different stages.

Unlike Quillen's small object argument, this quotienting means that when the category $\mathcal{M}$ permits the small object argument this process *converges*; there is no need for an artificial termination point. The resulting object through which the arrow f factors is in some sense "smaller" than for the factorizations produced by Quillen's small object argument because cells are attached only once, not repeatedly. The monad $\mathbb{R}$ is algebraically-free on the pointed endofunctor R^1, which says that $\mathbb{R}$-**alg** $\cong (R^1, \vec{\eta}^1)$-alg $\cong \mathcal{J}^{\boxslash}$.

I.4.2 Pointwise algebraic weak factorization systems

We now turn our attention to pointwise awfs. Because **CAT** is cartesian closed, we have isomorphisms $(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}} \cong \mathcal{M}^{\mathcal{A}\times\mathbf{2}} \cong (\mathcal{M}^{\mathbf{2}})^{\mathcal{A}}$, which we use to regard a natural transformation α as a functor $\alpha\colon \mathcal{A} \to \mathcal{M}^{\mathbf{2}}$. On objects, this functor picks out the constituent morphisms of α; the image of a morphism in $\mathcal{A}$ is the corresponding naturality square. Morphisms $(\phi, \psi) : \alpha \Rightarrow \beta$ in the category of functors $\mathcal{A} \to \mathcal{M}^{\mathbf{2}}$ consist of a pair of morphisms ϕ, ψ in $\mathcal{M}^{\mathcal{A}}$ such that the vertical composites $\beta\phi$ and $\psi\alpha$ are equal.

Given an awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{M}$, we use these isomorphisms to define $L^{\mathcal{A}}$ to be the functor

$$(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}} \cong (\mathcal{M}^{\mathbf{2}})^{\mathcal{A}} \xrightarrow{(L)_*} (\mathcal{M}^{\mathbf{2}})^{\mathcal{A}} \cong (\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$$

induced by post-composition by L; similarly for $R^{\mathcal{A}}$. We define the natural transformations $\vec{\epsilon}^{\mathcal{A}}, \vec{\delta}^{\mathcal{A}}, \vec{\eta}^{\mathcal{A}}, \vec{\mu}^{\mathcal{A}}$ that make $L^{\mathcal{A}}$ and $R^{\mathcal{A}}$ into a comonad and monad as follows. Given an object α of $(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ regarded as a functor $\mathcal{A} \to \mathcal{M}^{\mathbf{2}}$, the arrow $\vec{\epsilon}^{\mathcal{A}}_{\alpha}$ is obtained by "whiskering" $\vec{\epsilon}$ with α, as depicted below.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\alpha"] & \mathcal{M}^{\mathbf{2}} \arrow[r, bend left=50, "L"{name=U}] \arrow[r, bend right=50, equal, ""{name=D}] & \mathcal{M}^{\mathbf{2}}
\arrow[from=U, to=D, Rightarrow, "\vec{\epsilon}"]
\end{tikzcd}$$

All of the other natural transformations are defined similarly. It is easy to see that $\mathbb{L}^{\mathcal{A}} = (L^{\mathcal{A}}, \vec{\epsilon}^{\mathcal{A}}, \vec{\delta}^{\mathcal{A}})$ and $\mathbb{R}^{\mathcal{A}} = (R^{\mathcal{A}}, \vec{\eta}^{\mathcal{A}}, \vec{\mu}^{\mathcal{A}})$ define an awfs because all the definitions are given by simply post-composing a natural transformation with the old comonad and monad.

Remark I.4.2. Note however that the underlying wfs of the pointwise awfs $(\mathbb{L}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}})$ is not itself given pointwise by the underlying wfs of $(\mathbb{L}, \mathbb{R})$. This is because, unlike the case for the left and right factors, generic pointwise maps will not have natural lifts. This is one area where awfs behave better than ordinary wfs.

I.4.3 Cofibrantly generated case

Given a cofibrantly generated awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{M}$, is the resulting pointwise awfs $(\mathbb{L}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}})$ on $\mathcal{M}^{\mathcal{A}}$ cofibrantly generated? There are many reasons to suspect that this is not the case. For example, there is an awfs on **Set** generated by $\mathcal{J} = \{\emptyset \to 1\}$ for which the right class is the epimorphisms. The right class of the pointwise awfs on $\mathbf{Set}^{\mathcal{A}}$ consists of epis with a natural section. If this awfs is cofibrantly generated, it means that this class can be characterized by a lifting property. While right lifting properties can be used to specify additional *structure* on a class of maps, they are not typically known to impose *coherence* conditions.

Despite this worry, the answer is yes, the pointwise awfs is always cofibrantly generated when the original one is. In retrospect, the solution to the above concern is obvious: the generating category $\mathcal{J}_{\mathcal{A}}$ for the pointwise awfs will not be discrete (unless $\mathcal{A}$ is)! This is the first example known to the author where the extra generality allowed in Garner's small object argument is useful.

Theorem I.4.3. *Let* $J\colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ *be a small category over* $\mathcal{M}^{\mathbf{2}}$*, where* $\mathcal{M}$ *permits the small object argument, and let* $(\mathbb{L}, \mathbb{R})$ *be the awfs generated by* $\mathcal{J}$*. Let* $\mathcal{J}_{\mathcal{A}}$ *be the category* $\mathcal{A}^{\mathrm{op}} \times \mathcal{J}$ *equipped with the functor*

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{J} \xrightarrow{y \times J} \mathbf{Set}^{\mathcal{A}} \times \mathcal{M}^{\mathbf{2}} \xrightarrow{- \cdot -} (\mathcal{M}^{\mathcal{A}})^{\mathbf{2}},$$

where y denotes the Yoneda embedding and $- \cdot -$ *denotes the copower*[8] *(tensor) of an arrow in* $\mathcal{M}$ *by a* **Set***-valued functor from* $\mathcal{A}$*. Then the pointwise awfs* $(\mathbb{L}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}})$ *is generated by* $\mathcal{J}_{\mathcal{A}}$*.*

In keeping with the previous notational conventions, we regard $\mathcal{J}_{\mathcal{A}}$ as the category with objects $\mathcal{A}(a,-) \cdot j$ for $a \in \mathcal{A}$ and $j \in \mathcal{J}$. Morphisms are generated by maps $\mathcal{A}(a,-) \cdot j \Rightarrow \mathcal{A}(a,-) \cdot j'$ for every $j \Rightarrow j'$ in $\mathcal{J}$ and by maps $f^* \colon \mathcal{A}(b,-) \cdot j \Rightarrow \mathcal{A}(a,-) \cdot j$ for every $f \colon a \to b$ in $\mathcal{A}$. We prefer to write $J_{\mathcal{A}} \colon \mathcal{J}_{\mathcal{A}} \to (\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ for the composite functor defined above.

Proof of Theorem I.4.3. We don't know a priori whether $\mathcal{M}^{\mathcal{A}}$ permits the small object argument, but we can begin to apply that construction to the category $\mathcal{J}_{\mathcal{A}}$ over $(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ nonetheless. We will show that the functors $(L^{\mathcal{A}})^0$, $(L^{\mathcal{A}})^1$, $(R^{\mathcal{A}})^1$, $(L^{\mathcal{A}})^2$, $(R^{\mathcal{A}})^2$, etc that arise at each step agree with the functors L^0, L^1, R^1, etc pointwise. It will follow that our construction on $(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ converges to the awfs $(\mathbb{L}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}})$, which is therefore generated by $\mathcal{J}_{\mathcal{A}}$.

The beginning stage of the small object argument computes the step-zero comonad $(L^{\mathcal{A}})^0$ as the left Kan extension of $J_{\mathcal{A}} \colon \mathcal{J}_{\mathcal{A}} \to (\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ along itself. Note that $(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ is cocomplete, since $\mathcal{M}$ is. The familiar formula for Kan extensions gives

$$(L^{\mathcal{A}})^0 \alpha = \int^{(a,j) \in \mathcal{J}_{\mathcal{A}} \cong \mathcal{A}^{\mathrm{op}} \times \mathcal{J}} \mathrm{Hom}_{(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}}(\mathcal{A}(a,-) \cdot j, \alpha) \cdot (\mathcal{A}(a,-) \cdot j).$$

The natural transformation $\mathcal{A}(a,-) \cdot j$ is the image of j under a functor $\mathcal{M}^{\mathbf{2}} \to (\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ that is left adjoint to evaluation at a. By this adjunction, the above coend equals

$$\begin{aligned} &= \int^{\mathcal{J}_{\mathcal{A}} \cong \mathcal{A}^{\mathrm{op}} \times \mathcal{J}} \mathrm{Hom}_{\mathcal{M}^{\mathbf{2}}}(j, \alpha_a) \cdot (\mathcal{A}(a,-) \cdot j) \\ &= \int^{\mathcal{J}_{\mathcal{A}} \cong \mathcal{A}^{\mathrm{op}} \times \mathcal{J}} \mathrm{Sq}(j, \alpha_a) \cdot (\mathcal{A}(a,-) \cdot j) \end{aligned}$$

---

[8] The copower $S \cdot j$ of a set S with an object j of $\mathcal{M}^{\mathbf{2}}$ is the coproduct of copies of j indexed by the set S. When S is instead a set-valued functor, the copower $S \cdot j$ is a natural transformation with each constituent arrow having the description just given.

where we've written "Sq" to indicate that morphisms from j to α_a in $\mathcal{M}^{\mathbf{2}}$ are commutative squares. By Fubini's theorem and cocontinuity of the copower, we can use the isomorphism $\mathcal{J}_{\mathcal{A}} \cong \mathcal{A}^{\mathrm{op}} \times \mathcal{J}$ to compute the coend over $\mathcal{J}$ first, yielding

$$\begin{aligned} &= \int^{\mathcal{A}^{\mathrm{op}}} \mathcal{A}(a,-) \cdot \left(\int^{\mathcal{J}} \mathrm{Sq}(j, \alpha_a) \cdot j \right) \\ &= \int^{\mathcal{A}^{\mathrm{op}}} \mathcal{A}(a,-) \cdot L^0 \alpha_a \end{aligned}$$

where L^0 is the step-zero comonad for $\mathcal{J}$. We now express this coend as a coequalizer

$$= \mathrm{coeq}\left(\coprod_{f \colon a \to b} \mathcal{A}(b,-) \cdot L^0 \alpha_a \rightrightarrows \coprod_{a} \mathcal{A}(a,-) \cdot L^0 \alpha_a \right)$$

where the top arrow is induced by $f^* \colon \mathcal{A}(b,-) \to \mathcal{A}(a,-)$ and the bottom arrow is induced by L^0 applied to the naturality square for f, which is a morphism from α_a to α_b in $\mathcal{M}^{\mathbf{2}}$. We compute this coequalizer pointwise; by inspection at an object $c \in \mathcal{A}$, the coequalizer in $\mathcal{M}^{\mathbf{2}}$ is $L^0 \alpha_c$ with $\mathcal{A}(a,c) \cdot L^0 \alpha_a \Rightarrow L^0 \alpha_c$ given by the evaluation map. This object and morphism satisfy the required universal property: the map out of $L^0 \alpha_c$ can be found by restricting to the identity component of the copower $\mathcal{A}(c,-) \cdot L^0 \alpha_c$.

The remaining steps in the small object argument are constructed from previous ones by applying the comonad $(L^{\mathcal{A}})^0$ and taking pushouts, coequalizers, and transfinite composites, which are all computed pointwise. As we've shown that $(L^{\mathcal{A}})^0$ is also computed pointwise, we are done. Because $\mathcal{M}$ permits the small object argument, this process will converge for each arrow α_a at some time (ordinal) β, which means that the naturally constructed arrows from step β to step $\beta + 1$ are isomorphisms. It follows that there is a natural isomorphism from step β on $(\mathcal{M}^{\mathcal{A}})^{\mathbf{2}}$ to step $\beta + 1$, which tells us that the construction converges. This completes the proof that Garner's small object argument applied to $\mathcal{J}_{\mathcal{A}}$ will give the pointwise monad and comonad of $(\mathbb{L}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}})$. Hence, $(\mathbb{L}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}})$ is the awfs generated by $\mathcal{J}_{\mathcal{A}}$. □

Example I.4.4. When $\mathcal{A}$ is a small category, the awfs of Lack's trivial model structure on the 2-category $\mathbf{Cat}^{\mathcal{A}}$ are pointwise awfs. In the case $\mathcal{A} = \mathbf{2}$, Lack proves [Lac07, Proposition 3.19] that his trivial model structure is not cofibrantly generated in Quillen's sense. By contrast, Theorem

I.4.3 can be used to show that this is an algebraic model structure, which is cofibrantly generated in Garner's sense.

I.4.4 Algebraic projective model structures

Using Theorem I.3.10 and the pointwise algebraic weak factorization system described above, we can prove that any cofibrantly generated algebraic model structure on a category $\mathcal{M}$ induces a cofibrantly generated *projective algebraic model structure* on the diagram category $\mathcal{M}^{\mathcal{A}}$. The awfs of this model structure are not the pointwise awfs on $\mathcal{M}^{\mathcal{A}}$; instead, the generating categories are discrete, at least when the original generators $\mathcal{I}$ and $\mathcal{J}$ are. The underlying model structure agrees with the usual projective model structure on a diagram category: weak equivalences are pointwise weak equivalences and fibrations are pointwise fibrations.

The generating categories $\mathcal{I}_{\text{proj}}$ and $\mathcal{J}_{\text{proj}}$ for the projective model structure look familiar; in the case where $\mathcal{I}$ and $\mathcal{J}$ are discrete these are the usual generating sets in the classical theory. Objects of $\mathcal{I}_{\text{proj}}$ are functors $\mathcal{A}(a,-)\cdot i$, for all $a \in \mathcal{A}$ and $i \in \mathcal{I}$. Each morphism $i \Rightarrow i'$ in $\mathcal{I}$ gives rise to a morphism $\mathcal{A}(a,-)\cdot i \Rightarrow \mathcal{A}(a,-)\cdot i'$ in $\mathcal{I}_{\text{proj}}$; there are no others. The category $\mathcal{J}_{\text{proj}}$ is defined similarly.

Theorem I.4.5. *Let $\mathcal{M}$ have an algebraic model structure, generated by $\mathcal{I}$ and $\mathcal{J}$, with weak equivalences $\mathcal{W}_{\mathcal{M}}$. Then the categories $\mathcal{I}_{\text{proj}}$ and $\mathcal{J}_{\text{proj}}$ give rise to a cofibrantly generated algebraic model structure on $\mathcal{M}^{\mathcal{A}}$, which we will call the projective algebraic model structure.*

Proof. Write $\mathcal{A}_0$ for the discrete subcategory of objects of $\mathcal{A}$. We first show that the algebraic model structure on $\mathcal{M}$ induces a model structure on the diagram category $\mathcal{M}^{\mathcal{A}_0}$. We then use an adjunction to pass this across to the projective model structure on $\mathcal{M}^{\mathcal{A}}$.

Arrows of $\mathcal{M}^{\mathcal{A}_0}$ are natural transformations with no naturality conditions, i.e., collections α of morphisms α_a in $\mathcal{M}$ for each $a \in \mathcal{A}_0$. The categories $\mathcal{I}$ and $\mathcal{J}$ induce a pair of pointwise awfs on $\mathcal{M}^{\mathcal{A}_0}$. By Theorem I.4.3, these awfs are generated by $\mathcal{I}_{\mathcal{A}_0}$ and $\mathcal{J}_{\mathcal{A}_0}$.[9] The comparison map of the algebraic model structure on $\mathcal{M}$ gives the elements of $\mathcal{J}$ coalgebra structures for the comonad generated by $\mathcal{I}$ that are natural with respect to morphisms in $\mathcal{J}$. Pointwise, this functor can be used to define a functor from $\mathcal{J}_{\mathcal{A}_0}$ to the category of coalgebras for the comonad induced by $\mathcal{I}_{\mathcal{A}_0}$. By Remark I.3.6, this induces a comparison map between the awfs generated by $\mathcal{J}_{\mathcal{A}_0}$ and $\mathcal{I}_{\mathcal{A}_0}$.

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[9] Of course, it is also possible to prove this directly as an easier special case of that theorem.

We quickly prove that this gives an algebraic model structure. As in the proof of Theorem I.3.10, the existence of this comparison map implies that trivial cofibrations are cofibrations and trivial fibrations are fibrations. Let $\mathcal{W}_0$ be the class of morphisms of $\mathcal{M}^{\mathcal{A}_0}$ that are pointwise weak equivalences. With this definition it is clear that trivial cofibrations and trivial fibrations are weak equivalences. So to show that $\mathcal{I}_{\mathcal{A}_0}$ and $\mathcal{J}_{\mathcal{A}_0}$ give rise to an algebraic model structure, it remains only to show that fibrations that are weak equivalences are trivial fibrations.

More precisely, we need to show is that algebras for the monad induced by $\mathcal{J}_{\mathcal{A}_0}$ that are pointwise weak equivalences have an algebra structure for the monad induced by $\mathcal{I}_{\mathcal{A}_0}$. Since the category $\mathcal{A}_0$ is discrete, a collection α of morphisms α_a has an algebra structure for the monad induced by $\mathcal{J}_{\mathcal{A}_0}$ just when each α_a is an algebra for the monad induced by $\mathcal{J}$. Here, each α_a is a trivial fibration for the algebraic model structure on $\mathcal{M}$; by Lemma I.2.30 this means that it has an algebra structure for the monad induced by $\mathcal{I}$. Again because $\mathcal{A}_0$ is discrete, this means that the collection α has an algebra structure for the monad induced by $\mathcal{I}_{\mathcal{A}_0}$, which is what we wanted to show. So the categories $\mathcal{I}_{\mathcal{A}_0}$ and $\mathcal{J}_{\mathcal{A}_0}$ generate an algebraic model structure on $\mathcal{M}^{\mathcal{A}_0}$.

Let $i\colon \mathcal{A}_0 \hookrightarrow \mathcal{A}$ be the canonical inclusion. Then left Kan extension along i gives rise to an adjunction

$$\begin{tikzcd}\mathrm{Lan}_i\colon \mathcal{M}^{\mathcal{A}_0} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}^{\mathcal{A}}\colon i^* \arrow[l, shift left]\end{tikzcd}$$

Here i^* might be thought of as an "evaluation" map; it takes a functor $G\colon \mathcal{A} \to \mathcal{M}$ to the collection of objects in its image and a natural transformation α to its collection of constituent arrows. Using the usual formula for left Kan extensions, the left adjoint takes an arrow $\alpha \in \mathcal{M}^{\mathcal{A}_0}$ to the disjoint union $\sqcup_{c\in\mathcal{A}_0}\mathcal{A}(c,-)\cdot\alpha_c$. Objects in $\mathcal{I}_{\mathcal{A}_0}$ are natural transformations $\mathcal{A}_0(a,-)\cdot i$ for some $a \in \mathcal{A}$ and $i \in \mathcal{I}$. As $\mathcal{A}_0$ is discrete, this natural transformation consists of the arrow i at the component for a and the identity arrow at the initial object of $\mathcal{M}$ at all other objects of $\mathcal{A}$. The image of this object under Lan_i is $\mathcal{A}(a,-)\cdot i$, by the above formula. It follows that

$$\mathrm{Lan}_i\,\mathcal{I}_{\mathcal{A}_0} = \mathcal{I}_{\mathrm{proj}} \quad \text{and} \quad \mathrm{Lan}_i\,\mathcal{J}_{\mathcal{A}_0} = \mathcal{J}_{\mathrm{proj}}.$$

In order to apply Theorem I.3.10 and conclude that $\mathcal{M}^{\mathcal{A}}$ has an algebraic model structure generated by $\mathcal{I}_{\mathrm{proj}}$ and $\mathcal{J}_{\mathrm{proj}}$, we must show that the right adjoint i^* takes the underlying maps of the coalgebras for the comonad generated by $\mathcal{J}_{\mathrm{proj}}$ to weak equivalences in $\mathcal{M}^{\mathcal{A}_0}$. In other words, we must show that the coalgebras for the comonad generated by $\mathcal{J}_{\mathrm{proj}}$ are pointwise weak equivalences.

Coalgebras for the comonad generated by $\mathcal{J}_{\text{proj}}$ are in the left class of the underlying wfs that this category generates, that is, they are arrows satisfying the LLP with respect to the underlying class of $\mathcal{J}_{\text{proj}}^{\boxslash}$. From the adjunction, we know that the underlying class of $\mathcal{J}_{\text{proj}}^{\boxslash} = (\text{Lan}_i\, \mathcal{J}_{\mathcal{A}_0})^{\boxslash} = (i^*)^{-1}(\mathcal{J}_{\mathcal{A}_0}^{\boxslash})$ is the class of pointwise algebras for the original monad generated by $\mathcal{J}$ on $\mathcal{M}^{\mathbf{2}}$.

Let α be an element of the left class generated by $\mathcal{J}_{\text{proj}}$ and factor α using the pointwise awfs $(\mathbb{C}_t^{\mathcal{A}}, \mathbb{F}^{\mathcal{A}})$ on $\mathcal{M}^{\mathcal{A}}$, not the awfs generated by $\mathcal{J}_{\text{proj}}$. The components of the right factor $F^{\mathcal{A}}\alpha$ are algebras for the monad $\mathbb{F}$ generated by $\mathcal{J}$ because $F^{\mathcal{A}}\alpha$ is an algebra for the monad $\mathbb{F}^{\mathcal{A}}$. So $F^{\mathcal{A}}\alpha \in \mathcal{J}_{\text{proj}}^{\boxslash}$ and hence α lifts against $F^{\mathcal{A}}\alpha$, which means that α is a retract of $C_t^{\mathcal{A}}\alpha$. The constituent maps $(C_t^{\mathcal{A}}\alpha)_a = C_t(\alpha_a)$ are coalgebras for the comonad on $\mathcal{M}^{\mathbf{2}}$ generated by $\mathcal{J}$; in particular they are weak equivalences, since $\mathcal{J}$ is the generating category of trivial cofibrations. So pointwise the arrows of α are retracts of weak equivalences; hence α consists of pointwise weak equivalences. Theorem I.3.10 may now be used to establish the projective algebraic model structure. □

I.5 Recognizing cofibrations

In previous sections, we have seen that cofibrantly generated model structures can be "algebraicized," so that the constituent wfs are in fact awfs. This gives all fibrations the structure of algebras for a monad and some cofibrations the structure of coalgebras for a comonad. This extra algebraic structure is unobtrusive, in the sense that it can be forgotten at any point to yield an ordinary notion of a model structure, with the added benefit that the factorizations constructed by Garner's small object argument are somehow "smaller."

However, we have not yet given a convincing argument that this extra algebra structure is useful, allowing us to prove theorems that were intractable otherwise. In this section, we will provide the first such examples, illustrating the following point: one pleasant feature of this algebraic data is it gives a technique for proving that certain maps are cofibrations.

I.5.1 Coalgebra structures for the comparison map

One such example is the following theorem, which is joint work with Richard Garner and Mike Shulman. In this theorem, we will require that the comparison map arise from a functor $\tau \colon \mathcal{J} \to \mathcal{I}$ over $\mathcal{M}^{\mathbf{2}}$ of a particularly nice form. Firstly, we require that it be a full inclusion (full, faithful, and injective on objects). Secondly, we require that $\mathcal{I}$ decompose as a coproduct $\tau(\mathcal{J}) \sqcup \mathcal{I}'$, i.e., that there are no morphisms from objects in the image of τ to objects not in the image.[10] Note that when $\mathcal{J}$ and $\mathcal{I}$ are sets, these requirements simply mean that the generating trivial cofibrations $\mathcal{J}$ are a subset of the generating cofibrations $\mathcal{I}$.

Theorem I.5.1. *Let $\mathcal{J}$ and $\mathcal{I}$ be categories that generate an algebraic model structure on $\mathcal{M}$ and such that we have an inclusion $\tau\colon \mathcal{J} \to \mathcal{I}$ over $\mathcal{M}^{\mathbf{2}}$ of the form described above. Suppose also that the cofibrations are monomorphisms in $\mathcal{M}$. Then the components of the comparison map $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ produced by Garner's small object argument are cofibrations and, furthermore, coalgebras for the comonad $\mathbb{C}$.*

Let us provide some intuition for this result. Given an arrow f, we construct Qf from Rf by attaching more "cells." Because the cofibrations are monomorphisms, the "cells" we had attached previously to form Rf are not killed by the quotienting involved in the construction of Qf. Hence the arrow $\xi_f\colon Rf \to Qf$ is itself a cofibration, and furthermore, because it was constructed cellularly, ξ_f is a $\mathbb{C}$-coalgebra.

It takes some effort to describe the comparison map explicitly and accordingly it will take some work to translate the above intuition into a rigorous argument. When the awfs are cofibrantly generated, the comparison map $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ is induced by the cone produced by the right-hand factorization over the colimits of the left-hand factorization. These are each constructed by various colimiting processes at a number of stages, and the proof will accordingly involve a transfinite induction corresponding to each stage.

Specifically, for each ordinal α, the small object argument produces functorial factorizations (C_t^α, F^α) and (C^α, F_t^α). Let $Q^\alpha, R^\alpha : \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ denote the functors accompanying each functorial factorization, i.e., so that $f\colon X \to Z$ factors as

$$\begin{tikzcd}
 & X \arrow[dl, "C_t^\alpha f"'] \arrow[dr, "C^\alpha f"] & \\
R^\alpha f \arrow[rr, "\xi_f^\alpha"] \arrow[dr, "F^\alpha f"'] & & Q^\alpha f \arrow[dl, "F_t^\alpha f"] \\
 & Z &
\end{tikzcd}$$

---

10. We suspect that this second condition is unnecessary but include it to simplify the arguments given below.

with ξ^{α}, the component at f of the step-α comparison map, as depicted.

Recall, the step-one factorization (C^1_t, F^1) is constructed by factoring the counit of the density comonad of $J\colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ as a pushout followed by a square with domain equal to the identity, as indicated below.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "C^0_t f"'] & X \arrow[r, equal] \arrow[d, "C^1_t f"] & X \arrow[d, "f"] \\
\cdot \arrow[r] & R^1 f \arrow[r, "F^1 f"'] \arrow[ul, phantom, "\ulcorner", very near start] & Z
\end{tikzcd} \tag{I.5.2}$$

In the familiar case when $\mathcal{J}$ is discrete, $C^0_t f = \mathrm{Lan}_J J(f)$ is the coproduct of elements $j \in \mathcal{J}$ over commutative squares from j to f, and the top and bottom horizontal composites are the canonical arrows induced from these coproducts.

A key step in the proof of Theorem I.5.1 is the following lemma, which will imply that the step-one comparison map $\xi^1_f\colon R^1 f \to Q^1 f$ is a $\mathbb{C}$-coalgebra. The general form of this lemma will enable multiple applications.

Lemma I.5.3. *Let $\mathcal{J}$ and $\mathcal{I}$ be small categories with an inclusion $\mathcal{J} \to \mathcal{I}$ over $\mathcal{M}^{\mathbf{2}}$ as described above and let (C^1_t, F^1) and (C^1, F^1_t) be the step-one factorizations they produce. Given any commutative triangle*

$$\begin{tikzcd}
X \arrow[rr, tail, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

such that h is a cofibration and a monomorphism, then the map $\xi^1\colon R^1 f \to Q^1 g$ induced by the colimit is a cofibration. If furthermore h is a $\mathbb{C}$-coalgebra, then so is ξ.

Remark I.5.4. The proof of Lemma I.5.3 will require some basic facts about coalgebras for a comonad. We say a morphism $(u, v)\colon f \Rightarrow g$ in $\mathcal{M}^{\mathbf{2}}$ is a map of $\mathbb{C}$-coalgebras if it lifts to the category $\mathbb{C}$-**coalg** (where we usually have particular coalgebra structures for f and g in mind). In particular, if f has a coalgebra structure and g is a pushout of f, the pushout square is a map of coalgebras, when g is given the canonical coalgebra structure of the pushout. (See the example in Footnote 5.) Similarly, if g is a colimit of any diagram in $\mathcal{M}^{\mathbf{2}}$ whose objects are coalgebras and whose arrows are maps of coalgebras, then g inherits a canonical coalgebra structure such that the legs of the colimit cone are maps of coalgebras. (This is a consequence of Theorem I.2.16.) Finally, when $\mathbb{C}$ is the comonad of an awfs, $\mathbb{C}$-coalgebras are closed under composition, as we saw in Lemma I.2.22, in such a way that $(1,g)\colon f \Rightarrow gf$ is a map of coalgebras if f and g are coalgebras.

We will use all of these facts in the proofs that follow.

Proof of Lemma I.5.3. The defining pushouts for the arrows $C^1_t f$ and $C^1 g$ are the top and bottom faces of the cube below.

$$\begin{tikzcd}
& \cdot \arrow[rr, "e"] \arrow[dl, tail, "C^0_t f"'] \arrow[dd, "i_S"] \arrow[dr, phantom, "\ulcorner", very near end] & & X \arrow[dl, tail, "C^1_t f"'] \arrow[dd, tail, "h"] \\
\cdot \arrow[rr, "p"] \arrow[dd, "i_D"'] & & R^1 f \arrow[dd, dashed, "\xi^1"] & \\
& \cdot \arrow[rr, "e'"] \arrow[dl, tail, "C^0 g"'] \arrow[dr, phantom, "\ulcorner", very near end] & & Y \arrow[dl, tail, "C^1 g"] \\
W \arrow[rr, "p'"'] & & Q^1 g &
\end{tikzcd} \tag{I.5.5}$$

The notation (i_S, i_D) is meant to evoke the inclusions of the indexing sets for the coproducts of spheres and disks, with the familiar generating set of cofibrations $\{S^{n-1} \to D^n\}$ in mind. The map ξ^1 is induced by the universal property of the top pushout.

We begin by defining the pushout in the right face of the cube (I.5.5).

$$\begin{tikzcd}
R^1 f \arrow[dr, tail, "l"] \arrow[dd, "\xi^1"'] & & X \arrow[ll, "C^1_t f"'] \arrow[dd, tail, "h"] \\
& P^1 g^{\urcorner} \arrow[dl, dashed, "w"] & \\
Q^1 g & & Y \arrow[ll, "C^1 g"] \arrow[ul, "k"']
\end{tikzcd}$$

Because h is a cofibration, l is as well. If h is a coalgebra, then l inherits a canonical coalgebra structure as a pushout of h. Because cofibrations and coalgebras for the comonad of an awfs are closed under composition, it suffices to show that w is a cofibration, and a coalgebra whenever h is a coalgebra. Actually w is always a coalgebra. We will use:

Lemma I.5.6. *Given a commutative cube in which the top and bottom faces are pushouts, form the pushouts in the left and right faces, as in the diagram below.*

$$\begin{tikzcd}
& & \bullet \arrow[rrr] \arrow[dll] \arrow[dd] & & & \bullet \arrow[dll] \arrow[dd] \\
\bullet \arrow[rrr] \arrow[ddd] \arrow[ddr] & & & \bullet \arrow[ddd] \arrow[ddr] & & \\
& & \bullet \arrow[rrr] \arrow[dl] \arrow[ddll] & & & \bullet \arrow[dl] \arrow[ddll] \\
& \bullet \arrow[rrr, dashed] \arrow[dl, dashed] & & & \bullet \arrow[dl, dashed] & \\
\bullet \arrow[rrr] & & & \bullet & &
\end{tikzcd}$$

Then the square created by these pushouts (with three edges dotted in the diagram) is itself a pushout square.

Proof. Easy diagram chase. □

By Lemma I.5.6, w is a pushout of the map c in the diagram below,

$$\begin{tikzcd}
\bullet \arrow[dd, "i_D"'] \arrow[dr] & & \bullet \arrow[ll, "C^0_t f"'] \arrow[dd, "i_S"] \\
& P \arrow[dl, dashed, "c"] & \\
\bullet & & \bullet \arrow[ll, tail, "C^0 g"] \arrow[ul]
\end{tikzcd}$$

so it remains to show that this is a cofibration and coalgebra.

In the case where $\mathfrak{J}$ and $\mathfrak{I}$ are discrete, $C^0_t f$ is the disjoint union of arrows j of $\mathfrak{J}$ over commutative squares from j to f, and $C^0 g$ is the disjoint union of arrows i of $\mathfrak{I}$ over squares from i to g. The square (i_S, i_D) maps the first coproduct into the second, sending j to its image under the functor $\tau\colon \mathfrak{J} \to \mathfrak{I}$ and a square from j to f to the composite of this square with $(h, 1)\colon f \Rightarrow g$. As h is monic, (i_S, i_D) is an inclusion, so we can separate the coproduct $C^0 g$ into the image of this inclusion and the rest. If we write $h_*\colon \mathrm{Sq}(\mathfrak{J}, f) \to \mathrm{Sq}(\mathfrak{I}, g)$ for the (injective) function that takes a square from j to f for some $j \in \mathcal{J}$ and composes with $(h, 1)$ to get a square from τj to g, then

$$C^0 g = \left(\coprod_{\mathrm{Sq}(\mathcal{J},f)} \tau j \right) \sqcup \left(\coprod_{\mathrm{Sq}(\mathcal{I},g) \backslash \mathrm{im}\, h_*} i \right)$$

With this notation, $C^0 g$ factors through P as the composite of first the arrow

$$\left(\coprod_{\mathrm{Sq}(\mathcal{J},f)} \tau j \right) \sqcup \left(\coprod_{\mathrm{Sq}(\mathcal{I},g) \backslash \mathrm{im}\, h_*} 1_{\mathrm{dom} i} \right) \text{then} \left(\coprod_{\mathrm{Sq}(\mathcal{J},f)} 1_{\mathrm{cod} \tau j} \right) \sqcup \left(\coprod_{\mathrm{Sq}(\mathcal{I},g) \backslash \mathrm{im}\, h_*} i \right).$$

This second arrow is c, which gets a canonical $\mathbb{C}$-coalgebra structure as a coproduct of arrows in $\mathcal{I}$ with identities, which are always coalgebras.

The proof of the general case where $\mathcal{J}$ and $\mathcal{I}$ are not discrete is similar. In this case, $C^0_t f$ is a quotient of the disjoint union described above, and similarly for $C^0 g$. But $C^0 g$ can still be separated into the disjoint union of the image of (i_S, i_D) and its complement by the hypotheses we made on the inclusion $\mathcal{J} \to \mathcal{I}$. So c has essentially the same description as above, except it is a quotient of a coproduct. This time, c is a colimit of a diagram whose objects are either identities or generating cofibrations, so to prove c is a coalgebra we must check that the maps of the diagram are maps of coalgebras. This is true because the morphisms in the formula for a coend are either arrows of $\mathcal{I}$, which are canonically coalgebra maps, or they are coproduct inclusions, which are always coalgebra maps. In any case, the above conclusion still stands: c is canonically a coalgebra, so w is as well. Hence ξ^1 is a cofibration, and a $\mathbb{C}$-coalgebra when h is. ☐

Remark I.5.7. It is possible to prove directly that ξ^1 is a cofibration by showing that it lifts against all trivial fibrations. But this proof can only show ξ^1 is a cofibration, not that it has a $\mathbb{C}$-coalgebra structure when h does, and it is this stronger fact that we will need in the proof of Theorem I.5.1.

Remark I.5.8. The argument of Lemma I.5.3 holds more generally than stated. In particular, it is not necessary that the arrows in the positions of $C^0_t f$ and $C^0 g$ be coends over *all* possible squares. As long as these arrows are constructed as coends over *some* squares such that (i_S, i_D) is an inclusion, the conclusion follows. In applications, we will often require this slightly more general result, for reasons that will become clear in a moment.

We will now use Lemma I.5.3 to prove Theorem I.5.1. Our proof used a modified version of the small object argument, suggested by Richard Garner in private communication, that can be used whenever the elements of the left class of the underlying wfs are monomorphisms. Steps zero and one, as depicted in (I.5.2) are the same as before. At this point, Quillen's small object argument has us freely attach "cells" to fill "spheres" in the object $R^1 f$ by repeating steps zero and one for the map $F^1 f$. Garner's small object argument does the same thing, but then takes a coequalizer to identify the "cells" in the "spheres" that were filled twice, once in step one and once in step two. In the modified version, we never attach these extraneous "cells" at all; instead, we only attach "cells" to fill "spheres" in $R^1 f$ that weren't filled already in step one. For this modification to work, it is essential that the cofibrations are monomorphisms; otherwise, "cells" that are needed to fill "spheres" at some intermediate stage may become redundant later. With this modification, solutions to lifting problems $\mathcal{J} \boxslash Ff$ factor uniquely through a minimal stage; colloquially every "sphere" that is filled in the object Rf is filled at some minimal step and is filled uniquely at this step. This gives a new form of the small object argument that produces the same factorizations as in Garner's version but with no need for taking coequalizers, which are the most difficult to manipulate. This modification of the small object argument was independently suggested by [RB99] in the special case of fibrant replacement.

Proof of Theorem I.5.1. We use the preceding lemma and transfinite induction. Applying Lemma I.5.3 in the case $h = 1_X$ and $f = g$ shows that $\xi_f^1 \colon R^1 f \to Q^1 f$ is a cofibration and a $\mathbb{C}$-coalgebra. Because the cofibrations are assumed to be monomorphisms, we may use the modified version of Garner's small object argument described above. In the modified version, step two applies factorizations that are similar to the step one factorizations to each of the vertical morphisms in the triangle

$$\begin{tikzcd}
R^\beta f \arrow[rr, tail, "\xi_f^\beta"] \arrow[dr, "F^\beta f"'] & & Q^\beta f \arrow[dl, "F_t^\beta f"] \\
& Y &
\end{tikzcd} \tag{I.5.9}$$

with ordinal $\beta = 1$ in this case. The difference between these factorizations and the step one factorizations is that some squares are left out of the step zero coproducts. By Remark I.5.8, we nonetheless deduce that ξ_f^2 is a $\mathbb{C}$-coalgebra. Likewise, applying Lemma I.5.3 to the triangles (I.5.9) produced at each stage, we conclude that each map $\xi_f^{\beta+1} \colon R^{\beta+1} f \to Q^{\beta+1} f$ is a

$\mathbb{C}$-coalgebra, assuming ξ_f^β is.

For limit ordinals α, the maps $R^\alpha f \to Q^\alpha f$ are created as colimits of the diagrams of the $\xi_f^\beta \colon R^\beta f \to Q^\beta f$ for ordinals $\beta < \alpha$, which by the inductive hypothesis are cofibrations and $\mathbb{C}$-coalgebras. As usual, we must check that the morphisms $\xi_f^\beta \Rightarrow \xi_f^{\beta+1}$ in this diagram are maps of coalgebras. When we apply Lemma I.5.3 to (I.5.9), the square $\xi_f^\beta \Rightarrow \xi_f^{\beta+1}$ is the right face of the cube (I.5.5), reproduced below

$$\begin{tikzcd}
\cdot \arrow[dd, tail, "\xi^{\beta+1}"'] \arrow[dr, tail, "l"] & & \cdot \arrow[ll, "c"'] \arrow[dd, tail, "\xi^{\beta}"] \\
& \cdot \arrow[dl, dashed, "w"] \arrow[ul, phantom, very near start, "\ulcorner"] & \\
\cdot & & \cdot \arrow[ll, "d"] \arrow[ul, "k"']
\end{tikzcd}$$

The arrow l inherits its coalgebra structure as a pushout of ξ^β, and this construction makes (c, k) a map of coalgebras. Similarly, $\xi^{\beta+1}$ inherits its coalgebra structure as a composite of the coalgebras l and w, and this construction makes $(1, w)$ a map of coalgebras. The morphism $(c, d) \colon \xi_f^\beta \Rightarrow \xi_f^{\beta+1}$ is a composite of maps of coalgebras, and hence a map of coalgebras. Hence the colimit ξ_f^α has a canonical coalgebra structure created by this diagram. By transfinite induction, the comparison map ξ is a pointwise cofibration with each component a $\mathbb{C}$-coalgebra. □

I.5.2 Algebraically fibrant objects revisited

One consequence of Lemma I.5.3 and the proof of Theorem I.5.1 is the following corollary, which says that fibrant replacement monads that are constructed algebraically preserve certain trivial cofibrations, assuming the trivial cofibrations are monomorphisms.

Corollary I.5.10. *Let $\mathcal{M}$ be a category that permits the small object argument equipped with a model structure such that the trivial cofibrations are monomorphisms. Suppose there exists a category $\mathfrak{J}$ of trivial cofibrations that detects algebraically fibrant objects, in the sense that an object X is fibrant if and only if $X \to *$ underlies some object of $\mathfrak{J}^{\boxslash}$, and let $\mathbb{R}$ be the fibrant replacement monad on $\mathcal{M}$ induced from the awfs $(\mathbb{C}_t, \mathbb{F})$ generated by $\mathfrak{J}$. Then if $f \colon X \to Z$ is a $\mathbb{C}_t$-coalgebra, $Rf \colon RX \to RZ$ is a $\mathbb{C}_t$-coalgebra.*

Proof. The arrow Rf is constructed by an inductive process, analogous to the construction of the comparison map, that begins by applying Lemma I.5.3 to the triangle

$$\begin{tikzcd}
X \arrow[rr, tail, "f"] \arrow[dr] & & Z \arrow[dl] \\
& * &
\end{tikzcd}$$

with both awfs taken to be $(\mathbb{C}_t, \mathbb{F})$. Hence this result. □

The algebras for the monad $\mathbb{R}$ are precisely the "algebraically fibrant objects" of $\mathcal{M}$, i.e., objects with chosen lifts against the generators, subject to any coherence conditions imposed by morphisms in the category $\mathcal{J}$. By Lemma I.2.30 every fibrant object has some algebra structure making it an algebraically fibrant object. It always suffices to take $\mathcal{J}$ to be the generating trivial cofibrations, assuming they exist, but in some examples it is preferable to use a smaller generating category.

As for any category of algebras for a monad, we have an adjunction

$$\begin{tikzcd}
\mathcal{M} \arrow[r, shift left=1.5, "T"] \arrow[r, phantom, "\perp"] & \mathbb{R}\text{-}\mathbf{alg} \arrow[l, shift left=1.5, "U"]
\end{tikzcd}$$

where T takes an object $X \in \mathcal{M}$ to the free $\mathbb{R}$-algebra (RX, μ_X). In practice, the category $\mathbb{R}$-**alg** may fail to be cocomplete, in which case it is not a suitable category for a model structure. But when $\mathbb{R}$-**alg** is cocomplete, as is the case when $\mathcal{M}$ is locally presentable and $\mathbb{R}$ arises from a cofibrantly generated awfs for example, Corollary I.5.10 provides some hope that one could build a model structure on $\mathbb{R}$-**alg** such that $T \dashv U$ is a Quillen adjunction. One feature of such a model structure is that its objects would all be fibrant. In fact, it follows easily that any such Quillen adjunction is in fact a Quillen equivalence.

One such example is the Quillen model structure on $\mathbf{Ch}_A$, the category of chain complexes of A-modules for some commutative ring A, though this example is rather unsatisfactory because the objects of $\mathbf{Ch}_A$ are already fibrant. More interestingly, Thomas Nikolaus has proven that the categories of algebraic Kan complexes and algebraic quasi-categories can be given a model structure, lifted in the first case from Quillen's and the second from Joyal's model structure on simplicial sets [Nik10]. For the latter case, we prefer to let the set $\mathcal{J}$ in Corollary I.5.10 be the inner horn inclusions, rather than the generating trivial cofibrations. For all of these examples, Theorems I.3.10 and I.3.13 imply that the resulting model structures are algebraic and the Quillen equivalences are algebraic Quillen adjunctions.

I.6 Adjunctions of awfs

We now return to the material previewed in Section I.3.4. In this section, we study adjunctions of awfs, which we define precisely below. To motivate this definition, we consider an important class of examples: suppose $\mathcal{J}$ generates an awfs $(\mathbb{C}, \mathbb{F})$ on $\mathcal{M}$ and $T\mathcal{J}$ generates an awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$, where $T \colon \mathcal{M} \to \mathcal{K}$ is the left adjoint of a specified adjunction. A main task of this section is to prove Theorem I.6.15, which says that there is a canonical adjunction of awfs in this situation.

A direct proof is possible but technically difficult. Instead, we present a more conceptual, though somewhat circuitous argument, that is nonetheless shorter. After preliminary explorations, we reintroduce the three notions of morphisms between awfs on different categories, each extending Definition I.2.14. In order to prove Theorem I.6.15, we use Theorem I.2.24, which says that an awfs $(\mathbb{L}, \mathbb{R})$ is equivalently characterized by a natural composition law on the category of algebras for a monad over cod. We prove a lemma that allows us to use this recognition principle to easily identify lax morphisms of awfs, which for categorical reasons, suffices to prove Theorem I.6.15.

However, Theorem I.6.15 is not quite strong enough to prove Theorem I.3.13, establishing the existence of an important class of algebraic Quillen adjunctions. In order to prove the naturality component of this result, we must show that the "unit" functors (I.2.26) constructed in Garner's small object argument satisfy a stronger universal property than was previously known. In [Gar09], Garner shows that these functors are universal among morphisms of awfs. In Section I.6.4, we show that they are universal among all adjunctions of awfs, a result that should be of independent categorical interest. In particular, it follows that two canonical methods of assigning coalgebra structures to generating cofibrations in the image of a left adjoint are the same.

I.6.1 Algebras and adjunctions

Consider an adjunction $T \colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ where $\mathcal{J}$ generates an awfs $(\mathbb{C}, \mathbb{F})$ on $\mathcal{M}$ and $T\mathcal{J}$ generates an awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$. If $(\overline{\mathcal{C}}, \mathcal{F})$ is the wfs underlying $(\mathbb{C}, \mathbb{F})$ and $(\overline{\mathcal{L}}, \mathcal{R})$ is the wfs underlying

$(\mathbb{L}, \mathbb{R})$, then

$$T(\overline{\mathcal{C}}) \subset \overline{\mathcal{L}} \quad \text{and} \quad S(\mathcal{R}) \subset \mathcal{F}$$

because the defining lifting properties are adjunct.

The next few sections work towards an algebraization of this result. Because the awfs are cofibrantly generated, it will be considerably easier to prove statements involving the categories of algebras.

Theorem I.6.1. *For any adjunction* $T \colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ *where a small category* $\mathcal{J}$ *generates an awfs* $(\mathbb{C}, \mathbb{F})$ *on* $\mathcal{M}$ *and* $T\mathcal{J}$ *generates an awfs* $(\mathbb{L}, \mathbb{R})$ *on* $\mathcal{K}$*, the right adjoint* S *lifts to a functor*

$$\begin{tikzcd}
\mathbb{R}\text{-}\mathbf{alg} \arrow[r, dashed, "\tilde{S}"] \arrow[d, "U"'] & \mathbb{F}\text{-}\mathbf{alg} \arrow[d, "U"] \\
\mathcal{K}^{\mathbf{2}} \arrow[r, "S"] & \mathcal{M}^{\mathbf{2}}
\end{tikzcd}$$

Proof. Because the awfs are cofibrantly generated, we have isomorphisms $\mathbb{R}\text{-}\mathbf{alg} \cong T\mathcal{J}^{\boxslash}$ and $\mathbb{F}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$ that commute with the forgetful functors to the underlying arrow categories. Using the notation of the proof of Theorem I.3.10, let $(f, \psi) \in T\mathcal{J}^{\boxslash}$ and define $\tilde{S}(f, \psi) := (Sf, \psi^{\sharp}) \in \mathcal{J}^{\boxslash}$. Given $(u, v) \colon (f, \psi) \to (g, \phi) \in T\mathcal{J}^{\boxslash}$,

$$\tilde{S}(u, v) := (Su, Sv) \colon (Sf, \psi^{\sharp}) \to (Sg, \phi^{\sharp})$$

is a morphism in $\mathcal{J}^{\boxslash}$ by naturality of the adjunction. This defines the desired lift. □

There is a well-known categorical result [Joh75, Lemma 1], which says that lifts of S to functors $\tilde{S} : \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg}$ are in bijective correspondence with natural transformations $\vec{\rho} \colon FS \Rightarrow SR$ satisfying

$$\begin{tikzcd}
 & S \arrow[dl, "\vec{\eta}_S"'] \arrow[dr, "S\vec{\eta}"] & \\
FS \arrow[rr, "\vec{\rho}"] & & SR
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & FSR \arrow[dr, "\vec{\rho}_R"] & \\
FFS \arrow[ur, "F\vec{\rho}"] \arrow[d, "\vec{\mu}_S"'] & & SRR \arrow[d, "S\vec{\mu}"] \\
FS \arrow[rr, "\vec{\rho}"] & & SR
\end{tikzcd}
\tag{I.6.2}$$

A pair $(S, \vec{\rho})$ satisfying these conditions is called a *lax morphism of monads*.

Let $Q \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ and $E \colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ be the functors accompanying the functorial factorizations of $(\mathbb{C}, \mathbb{F})$ and $(\mathbb{L}, \mathbb{R})$, respectively. Because F and R are monads over cod, $\vec{\rho} = (\rho, 1)$ where

$\rho\colon QS \Rightarrow SE$ is a natural transformation satisfying

$$\begin{tikzcd} \mathrm{dom}S \arrow[r, "SL"] \arrow[d, "CS"'] & SE \arrow[d, "SR"] \\ QS \arrow[ur, "\rho"] \arrow[r, "FS"'] & \mathrm{cod}S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & & QSR \arrow[drr, "\rho_R"] & & \\ QFS \arrow[urr, "Q(\rho{,}1)"] \arrow[dr, "\mu_S"'] & & & & SER \arrow[dl, "S\mu"] \\ & QS \arrow[rr, "\rho"] & & SE & \end{tikzcd} \tag{I.6.3}$$

The functor $\tilde{S}$ is defined by mapping the $\mathbb{R}$-algebra $(g, t\colon Eg \to \mathrm{dom}\, g)$ to the $\mathbb{F}$-algebra $(Sg, St \cdot \rho_g\colon QSg \to \mathrm{dom}, Sg)$.

If the direction of ρ is reversed, a pair $(S, (\rho, 1))$ satisfying diagrams analogous to (I.6.2) is called a *colax morphism of monads*. If the monads are replaced by their corresponding comonads and the direction of ρ is unchanged, a pair $(S, (1, \rho))$ satisfying diagrams analogous to (I.6.2) is called a *lax morphism of comonads*. If the direction of ρ is reversed as well, the pair $(S, (1, \rho))$ is called a *colax morphism of comonads*. This last type of morphism is in bijective correspondence with lifts of S to a functor between the categories of coalgebras for the comonads by the dual of the lemma mentioned above.

For the lift $\tilde{S}$ of Theorem I.6.1, it is not easy to describe ρ explicitly because we cannot easily write down the inverse to the isomorphism (I.2.27). Surprisingly, in light of the definitions of the next sections, this will be no great obstacle.

I.6.2 Lax morphisms and colax morphisms of awfs

The statement analogous to Theorem I.6.1 for the left adjoint and categories of coalgebras is considerably harder to prove. In fact, we will prove a stronger result and deduce this as a corollary. First, we establish the relevant terminology, which extends the lax and colax morphisms of monads and comonads, introduced in the last section.

For the following definitions let $(\mathbb{C}, \mathbb{F})$ be an awfs on a category $\mathcal{M}$ with $Q\colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$ the functor accompanying its functorial factorization, and let $(\mathbb{L}, \mathbb{R})$ be an awfs on $\mathcal{K}$ with $E\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ accompanying its functorial factorization.

Definition I.6.4. A *lax morphism of awfs* $(S, \rho)\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ consists of a functor $S : \mathcal{K} \to \mathcal{M}$ and a natural transformation $\rho\colon QS \Rightarrow SE$ such that $(1, \rho)\colon CS \Rightarrow SL$ is a lax morphism of comonads and $(\rho, 1)\colon FS \Rightarrow SR$ is a lax morphism of monads, i.e., such that the following commute:

$$\begin{tikzcd}
\text{dom}S \arrow[r, "SL"] \arrow[d, "CS"'] & SE \arrow[d, "SR"] \\
QS \arrow[r, "FS"'] \arrow[ur, "\rho"] & \text{cod}S
\end{tikzcd}
\begin{tikzcd}[column sep=small]
 & QS \arrow[rr, "\rho"] \arrow[dl, "\delta_S"'] & & SE \arrow[dr, "S\delta"] & \\
QCS \arrow[drr, "Q(1{,}\rho)"] & & & & SEL \\
 & & QSL \arrow[urr, "\rho_L"] & &
\end{tikzcd}
\begin{tikzcd}[column sep=small]
 & & QSR \arrow[drr, "\rho_R"] & & \\
QFS \arrow[urr, "Q(\rho{,}1)"] \arrow[dr, "\mu_S"'] & & & & SER \arrow[dl, "S\mu"] \\
 & QS \arrow[rr, "\rho"] & & SE &
\end{tikzcd}
\tag{I.6.5}$$

Lax morphisms of monads $(\rho, 1)$ are in bijection with lifts of S to functors $\tilde{S}\colon \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg}$. Lax morphisms of comonads $(1, \rho)$ are in bijection with extensions of S to functors $\hat{S}\colon \mathbf{coKl}(\mathbb{L}) \to \mathbf{coKl}(\mathbb{C})$.

Definition I.6.6. A *colax morphism of awfs* $(T, \lambda)\colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ consists of a functor $T\colon \mathcal{M} \to \mathcal{K}$ and a natural transformation $\lambda\colon TQ \Rightarrow ET$ such that $(1, \lambda)\colon TC \Rightarrow LT$ is a colax morphism of comonads and $(\lambda, 1)\colon TF \Rightarrow RT$ is a colax morphism of monads, i.e., such that the following commute:

$$\begin{tikzcd}
\text{dom}T \arrow[r, "LT"] \arrow[d, "TC"'] & ET \arrow[d, "RT"] \\
TQ \arrow[r, "TF"'] \arrow[ur, "\lambda"] & \text{cod}T
\end{tikzcd}
\begin{tikzcd}[column sep=small]
 & TQ \arrow[rr, "\lambda"] \arrow[dl, "T\delta"'] & & ET \arrow[dr, "\delta_T"] & \\
TQC \arrow[drr, "\lambda_C"] & & & & ELT \\
 & & ETC \arrow[urr, "E(1{,}\lambda)"] & &
\end{tikzcd}
\begin{tikzcd}[column sep=small]
 & & ETF \arrow[drr, "E(\lambda{,}1)"] & & \\
TQF \arrow[urr, "\lambda_F"] \arrow[dr, "T\mu"'] & & & & ERT \arrow[dl, "\mu_T"] \\
 & TQ \arrow[rr, "\lambda"] & & ET &
\end{tikzcd}
\tag{I.6.7}$$

Colax morphisms of comonads $(1, \lambda)$ are in bijection with lifts of T to functors $\tilde{T}\colon \mathbb{C}\text{-}\mathbf{coalg} \to \mathbb{L}\text{-}\mathbf{coalg}$. Colax morphisms of monads $(\lambda, 1)$ are in bijection with extensions of T to functors $\hat{T}\colon \mathbf{Kl}(\mathbb{F}) \to \mathbf{Kl}(\mathbb{R})$.

Example I.6.8. A morphism of awfs $\rho\colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$, defined in I.2.14, is simultaneously a lax morphism of awfs $(1, \rho)\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ and a colax morphism of awfs $(1, \rho)\colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$. Conversely, every lax or colax morphism of awfs over an identity functor is a morphism of awfs.

Lax and colax morphisms of awfs can be identified by the following recognition principle, which extends the material of Section I.2.5.

Lemma I.6.9. *Suppose* $(\mathbb{L},\mathbb{R})$ *and* $(\mathbb{C},\mathbb{F})$ *are awfs and* $(S,\vec{\rho})\colon \mathbb{R} \to \mathbb{F}$ *is a lax morphism of monads corresponding to a lift* $\tilde{S} : \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg}$ *of* S. *Then* $(S,\rho)\colon (\mathbb{L},\mathbb{R}) \to (\mathbb{C},\mathbb{F})$ *is a lax morphism of awfs if and only if* $\tilde{S}$ *preserves the canonical composition of algebras. Dually, a colax morphism between the comonads of awfs is a colax morphism of awfs if and only if the lifted functor preserves the canonical composition of coalgebras.*

Proof. Suppose (S,ρ) is a lax morphism of awfs and let (f,s) and (g,t) be composable $\mathbb{R}$-algebras. By definition

$$\begin{aligned}\tilde{S}(g,t) \bullet \tilde{S}(f,s) &= (Sg \cdot Sf, (St \cdot \rho_g) \bullet (Ss \cdot \rho_f)) \\ &= (S(gf), Ss \cdot \rho_f \cdot Q(1, St \cdot \rho_g \cdot Q(Sf,1)) \cdot \delta_{S(gf)}),\end{aligned}$$

while

$$\begin{aligned}\tilde{S}((g,t) \bullet (f,s)) &= \tilde{S}(gf, s \cdot E(1, t \cdot E(f,1)) \cdot \delta_{gf}) \\ &= (S(gf), Ss \cdot SE(1, t \cdot E(f,1)) \cdot S\delta_{gf} \cdot \rho_{gf}).\end{aligned}$$

The diagram

$$\begin{tikzcd}
& SEgf \arrow[rr, "S\delta_{gf}"] & & SELgf \arrow[dr, "SE(1{,}t\cdot E(f{,}1))"] & \\
QSgf \arrow[ur, "\rho_{gf}"] \arrow[dr, "\delta_{S(gf)}"'] & & QSLgf \arrow[ur, "\rho_{Lgf}"'] \arrow[dr, "Q(1{,}St\cdot SE(f{,}1))"] & & SEf \\
& QCSgf \arrow[ur, "Q(1{,}\rho_{gf})"] \arrow[rr, "Q(1{,}St\cdot SE(f{,}1)\cdot\rho_{gf})"'] & & QSf \arrow[ur, "\rho_f"'] &
\end{tikzcd}$$

which commutes by (I.6.5) and naturality of ρ shows that both algebra structures are the same.

Conversely, we must show that the center diagram of $(I.6.5)$ commutes if the functor $\tilde{S}$ defined via ρ preserves composition of algebras. The proof requires a straightforward diagram chase:

$$\begin{aligned}
S\delta \cdot \rho &= S(\mu \bullet \mu_L) \cdot SE(L^2, 1) \cdot \rho && \text{definition of } \delta \\
&= S(\mu \bullet \mu_L) \cdot \rho_{R \cdot RL} \cdot Q(SL^2, 1) && \text{naturality of } \rho \\
&= ((S\mu \cdot \rho_R) \bullet (S\mu_L \cdot \rho_{RL})) \cdot Q(SL^2, 1) && \tilde{S} \text{ preserves composition} \\
&= S\mu_L \cdot \rho_{RL} \cdot Q(1, S\mu) \cdot Q(1, \rho_R) \\
&\quad \cdot Q(1, Q(SRL, 1)) \cdot \delta_{SR \cdot SRL} \cdot Q(SL^2, 1) && \text{defn. of comp. in } \mathbb{F}\text{-}\mathbf{alg} \\
&= S\mu_L \cdot \rho_{RL} \cdot Q(1, S\mu) \cdot Q(1, \rho_R) \\
&\quad \cdot Q(1, Q(SL, 1)) \cdot Q(SL^2, 1) \cdot \delta_S && \text{nat. of } \delta\text{; functoriality of } Q \\
&= S\mu_L \cdot \rho_{RL} \cdot Q(1, S\mu) \cdot Q(1, SE(L, 1)) \\
&\quad \cdot Q(SL^2, 1) \cdot Q(1, \rho) \cdot \delta_S && \text{nat. of } \rho\text{; functoriality of } Q \\
&= S\mu_L \cdot \rho_{RL} \cdot Q(SL^2, 1) \cdot Q(1, \rho) \cdot \delta_S && \text{monad triangle identity} \\
&= S\mu_L \cdot SE(L^2, 1) \cdot \rho_L \cdot Q(1, \rho) \cdot \delta_S && \text{naturality of } \rho \\
&= \rho_L \cdot Q(1, \rho) \cdot \delta_S && \text{monad triangle identity}
\end{aligned}$$

□

I.6.3 Adjunctions of awfs

The notions of lax and colax morphisms of awfs are closely related. In fact, given a lax morphism of awfs $(S, \rho)\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ and an adjunction (T, S, ι, ν) where $T \dashv S$ and ι and ν are the unit and counit, there is a canonical natural transformation $\lambda\colon TQ \Rightarrow ET$ such that $(T, \lambda)\colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ is a colax morphism of awfs. The dual result holds as well. Combining the data of the corresponding lax and colax morphisms of awfs we obtain the concept of an *adjunction of awfs*, defined below. But first, we need the following categorical concept to explain the relationship between ρ and λ.

Given functors as in the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "I"] \arrow[d, shift right=1.5ex, "F"'] \arrow[d, phantom, "\dashv"] & \mathcal{C} \arrow[d, shift right=1.5ex, "H"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{B} \arrow[r, "J"'] \arrow[u, shift right=1.5ex, "G"'] & \mathcal{D} \arrow[u, shift right=1.5ex, "K"']
\end{tikzcd}$$

with η and ϵ the unit and counit for $F \dashv G$ and ι and ν the unit and counit for $H \dashv K$, there is a bijection between natural transformations

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "I"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "H"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{B} \arrow[r, "J"'] & \mathcal{D}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{A} \arrow[r, "I"] \arrow[dr, Rightarrow, "\beta"] & \mathcal{C} \\
\mathcal{B} \arrow[u, "G"] \arrow[r, "J"'] & \mathcal{D} \arrow[u, "K"']
\end{tikzcd}$$

given by the formulae

$$\beta = KJ\epsilon \cdot K\alpha_G \cdot \iota_{IG} \quad \text{and} \quad \alpha = \nu_{JF} \cdot H\beta_F \cdot HI\eta.$$

The corresponding natural transformations α and β are called *mates* [KS74].

Definition I.6.10. An *adjunction of awfs* $(T, S, \lambda, \rho) \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ consists of an adjoint pair of functors $T \dashv S$ together with mates λ and ρ such that $(T, \lambda) \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ is a colax morphism of awfs and $(S, \rho) \colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ is a lax morphism of awfs.

The natural transformations λ and ρ should be mates with respect to the functors

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "Q"] \arrow[d, shift right, "T"'] \arrow[d, phantom, "\dashv"] & \mathcal{M} \arrow[d, shift right, "T"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{K}^{\mathbf{2}} \arrow[r, "E"'] \arrow[u, shift right, "S"'] & \mathcal{K} \arrow[u, shift right, "S"']
\end{tikzcd} \tag{I.6.11}$$

As alluded to above, the criteria on λ and ρ in Definition I.6.10 are overdetermined:

Lemma I.6.12. *Suppose we have an adjunction* $(T, S, \iota, \nu) \colon \mathcal{M} \rightleftarrows \mathcal{K}$ *where* $\mathcal{M}$ *has an awfs* $(\mathbb{C}, \mathbb{F})$ *and* $\mathcal{K}$ *has an awfs* $(\mathbb{L}, \mathbb{R})$. *Let* λ *and* ρ *be mates with respect to the functors of* (I.6.11). *Then* $(S, \rho) \colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ *is a lax morphism of awfs if and only if* $(T, \lambda) \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ *is a colax morphism of awfs, in which case* (T, S, λ, ρ) *is an adjunction of awfs.*

Proof. Each diagram of Definition I.6.4 is satisfied by ρ if and only if its mate λ satisfies the corresponding diagram of Definition I.6.6, as can be verified by a diagram chase. Or see [Kel74]. □

Remark I.6.13. The proof of Lemma I.6.12 implies a conclusion slightly stronger than the statement. Given mates ρ and λ as above with respect to $T \dashv S$, to show that (T, S, λ, ρ) is an adjunction of awfs, it suffices to show that either ρ is a lax or λ is a colax morphism of monads and that either ρ is a lax or λ is a colax morphism of comonads.

Example I.6.14. The comparison map $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ in an algebraic model structure specifies an adjunction of awfs, where the adjunction is the trivial one with functors, unit, and counit all identities. In this case, ξ is its own mate and the requirements that $(1, \xi)\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ is a colax morphism of awfs and that $(1, \xi)\colon (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{C}_t, \mathbb{F})$ is a lax morphism of awfs are equivalent.

Less trivially, there exists a canonical adjunction of awfs $(\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ whenever $(\mathbb{C}, \mathbb{F})$ is generated by $\mathcal{J}$ and $(\mathbb{L}, \mathbb{R})$ is generated by $T\mathcal{J}$ for some adjunction $T \dashv S$. By Theorem I.6.1, there exists a natural transformation $\rho\colon QS \Rightarrow SE$ such that $(\rho, 1)$ is a lax morphism of monads. We will show that $(S, \rho)\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$ is a lax morphism of awfs. It follows from Lemma I.6.12 that this situation gives rise to an adjunction of awfs (T, S, λ, ρ), where λ is the mate of ρ, proving the following theorem.

Theorem I.6.15. *Consider an adjunction* $T\colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}\colon S$ *where* $\mathcal{J}$ *generates an awfs* $(\mathbb{C}, \mathbb{F})$ *on* $\mathcal{M}$ *and* $T\mathcal{J}$ *generates an awfs* $(\mathbb{L}, \mathbb{R})$ *on* $\mathcal{K}$*. Let* ρ *be the natural transformation determined by Theorem I.6.1 and let* λ *be its mate. Then* $(T, S, \lambda, \rho)\colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ *is an adjunction of awfs.*

First, we must show that (S, ρ) is a lax morphism of awfs. Doing so directly is possible but quite hard, because a concrete understanding of ρ is only obtained by laboriously computing λ, vis-à-vis running through the details of the small object argument. We use Lemma I.6.9 instead.

Proof of Theorem I.6.15. By Lemma I.6.9, it suffices to show that the functor $\tilde{S}$ defined in the proof of Theorem I.6.1 preserves the canonical composition of algebras. Then Lemma I.6.12 implies that (T, S, λ, ρ) is an adjunction of awfs, where λ is the mate of ρ.

Suppose (f, ϕ) and (g, ψ) are composable objects of $T\mathcal{J}^{\boxslash}$. By definition, the functor $\tilde{S}$ takes the composite, described in Example I.2.32, to the morphism $S(gf)$ with the lifting function:

$$(\psi \bullet \phi)^{\sharp}(j, a, b) = S\phi(Tj, \nu \cdot Ta, \psi(Tj, f \cdot \nu \cdot Ta, \nu \cdot Tb)) \cdot \iota$$

where ι and ν are the unit and counit of $T \dashv S$. By contrast

$$\begin{aligned}\psi^\sharp \bullet \phi^\sharp(j,a,b) &= \phi^\sharp(j,a,\psi^\sharp(j,Sf\cdot a,b))\\ &= S\phi(Tj,\nu\cdot Ta,\nu\cdot TS\psi(Tj,\nu\cdot TSf\cdot Ta,\nu\cdot Tb)\cdot T\iota)\cdot\iota\\ &= S\phi(Tj,\nu\cdot Ta,\psi(Tj,f\cdot\nu\cdot Ta,\nu\cdot Tb)\cdot\nu_T\cdot T\iota)\cdot\iota,\end{aligned}$$

which is the same as above, after application of a triangle identity for the adjunction $T \dashv S$. □

Theorem I.6.15 extends Theorem I.6.1 and the corresponding result for coalgebras, which we note, for completeness sake, as an immediate corollary.

Corollary I.6.16. *For any adjunction* $T\colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{K}\colon S$ *where a small category* $\mathcal{J}$ *generates an awfs* $(\mathbb{C},\mathbb{F})$ *on* $\mathcal{M}$ *and* $T\mathcal{J}$ *generates an awfs* $(\mathbb{L},\mathbb{R})$ *on* $\mathcal{K}$*, the left adjoint* T *lifts to a functor*

$$\begin{tikzcd}\mathbb{C}\text{-}\mathbf{coalg} \arrow[r, dashed, "\tilde{T}"] \arrow[d, "U"'] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[d, "U"] \\ \mathcal{M}^{\mathbf{2}} \arrow[r, "T"] & \mathcal{K}^{\mathbf{2}}\end{tikzcd}$$

It is an easy exercise to check that adjunctions of awfs are composable, i.e., given adjunctions of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L}',\mathbb{R}')$ and $(\mathbb{L}',\mathbb{R}') \to (\mathbb{L},\mathbb{R})$, the composite adjoint pair of functors and pasted natural transformations form an adjunction of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$. Hence, the following corollary combines Example I.6.14 and Theorem I.6.15 to find adjunctions of awfs in a weaker situation, opening up an array of potential examples.

Corollary I.6.17. *Suppose we have an adjunction* $T\colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{K}\colon S$ *where* $\mathcal{J}$ *generates an awfs* $(\mathbb{C},\mathbb{F})$ *on* $\mathcal{M}$ *and* $\mathcal{K}$ *has an awfs* $(\mathbb{L},\mathbb{R})$*, not necessarily cofibrantly generated. Suppose also that* $\mathcal{K}$ *permits the small object argument and that we have a functor* $\mathcal{J} \to \mathbb{L}$-**coalg** *lifting* T*. Then* T *and* S *give rise to an adjunction of awfs* $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$*.*

Proof. By Theorem I.2.28, there exists an awfs $(\mathbb{L}',\mathbb{R}')$ on $\mathcal{K}$ that is cofibrantly generated by $T\mathcal{J}$. By Theorem I.6.15, T and S give rise to an adjunction of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L}',\mathbb{R}')$. The functor $\mathcal{J} \to \mathbb{L}$-**coalg** lifting T is equivalently described as a functor $T\mathcal{J} \to \mathbb{L}$-**coalg** over $\mathcal{K}^{\mathbf{2}}$. By the universal property of $T\mathcal{J} \to \mathbb{L}'$-**coalg**, there exists a morphism of awfs $(\mathbb{L}',\mathbb{R}') \to (\mathbb{L},\mathbb{R})$, which is equivalently an adjunction of awfs $(\mathbb{L}',\mathbb{R}') \to (\mathbb{L},\mathbb{R})$ over the identity functors on $\mathcal{K}^{\mathbf{2}}$. We obtain the desired adjunction of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ by composing the above two. □

I.6.4 Change of base in Garner's small object argument

In the situation of Theorem I.6.15, there are two canonical methods for assigning $\mathbb{L}$-coalgebra structures to the objects Tj of the generating category $T\mathcal{J}$. One method applies the functor T to the canonical $\mathbb{C}$-coalgebra structure for j and then composes with the natural transformation λ accompanying the lift $\tilde{T}\colon \mathbb{C}\text{-}\mathbf{coalg} \to \mathbb{L}\text{-}\mathbf{coalg}$. The other simply assigns Tj the canonical coalgebra structure given by Garner's small object argument via the functor (I.2.26). We might hope that the two results are the same. This is the content of an immediate corollary to the main theorem of this section.

Corollary I.6.18. *Given an adjunction $T \dashv S$ between categories $\mathcal{M}$ and $\mathcal{K}$, consider a category $\mathcal{J}$ which generates an awfs $(\mathbb{C}, \mathbb{F})$ on $\mathcal{M}$ and such that $T\mathcal{J}$ generates an awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$. Then the functor $\tilde{T}$ arising from the canonical adjunction of awfs $(\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ commutes with the units exhibiting the "freeness" of cofibrantly generated awfs, i.e., the diagram*

$$\begin{tikzcd}
\mathcal{J} \arrow[dr, "\gamma^{\mathcal{M}}"] \arrow[drr, bend left, "\gamma^{\mathcal{K}}"] \arrow[dd] & & \\
& \mathbb{C}\text{-}\mathbf{coalg} \arrow[r, "\tilde{T}"] \arrow[dl] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[dl] \\
\mathcal{M}^{\mathbf{2}} \arrow[r, "T"'] & \mathcal{K}^{\mathbf{2}} &
\end{tikzcd} \tag{I.6.19}$$

commutes.

Given a category $\mathcal{M}$ that permits the small object argument, Garner's construction produces a reflection of any small category $\mathcal{J}$ over $\mathcal{M}^{\mathbf{2}}$ along the so-called "semantics" functor

$$\begin{tikzcd}
\mathcal{G} = \mathbf{AWFS}(\mathcal{M}) \arrow[r, "\mathcal{G}_1"] & \mathbf{LAWFS}(\mathcal{M}) \arrow[r, "\mathcal{G}_2"] & \mathbf{Cmd}(\mathcal{M}^{\mathbf{2}}) \arrow[r, "\mathcal{G}_3"] & \mathbf{CAT}/\mathcal{M}^{\mathbf{2}} \\
(\mathbb{C}, \mathbb{F}) \arrow[r, mapsto] & (\mathbb{C}, Q) \arrow[r, mapsto] & \mathbb{C} \arrow[r, mapsto] & \mathbb{C}\text{-}\mathbf{coalg}
\end{tikzcd} \tag{I.6.20}$$

from the category of awfs on $\mathcal{M}$ and morphisms of awfs to the slice category over $\mathcal{M}^{\mathbf{2}}$ [Gar09, §4]. Here, $\mathbf{Cmd}(\mathcal{M}^{\mathbf{2}})$ is the category of comonads on $\mathcal{M}^{\mathbf{2}}$ and comonad morphisms and $\mathbf{LAWFS}(\mathcal{M})$ is the full subcategory of comonads over dom, or equivalently the category of functorial factorizations, whose left functor is a comonad.

The component of the unit of this reflection at a small category $\mathcal{J}$ generating an awfs $(\mathbb{C}, \mathbb{F})$ is the functor $\mathcal{J} \to \mathbb{C}\text{-}\mathbf{coalg}$ over $\mathcal{M}^{\mathbf{2}}$ of (I.2.26), which is universal with respect to morphisms of awfs. We prove that these maps are universal with respect to all adjunctions of awfs. To find an appropriate categorical context for the statement and proof of this result, we must enlarge the categories of (I.6.20). The new domain is the category $\mathbf{AWFS}_{\mathrm{ladj}}$, whose objects are awfs and whose morphisms are adjunctions of awfs. In analogy with (I.6.20), there is a forgetful functor to $\mathbf{CAT}/(-)^{\mathbf{2}}_{\mathrm{ladj}}$, whose objects are categories sliced over arrow categories, whose morphisms are adjunctions between the base categories (before applying $(-)^{\mathbf{2}}$) together with a chosen lift of the left adjoint to the fibers. This "semantics" functor factors as:

$$\mathcal{G}^{\mathrm{ladj}} = \mathbf{AWFS}_{\mathrm{ladj}} \xrightarrow{\mathcal{G}_1^{\mathrm{ladj}}} \mathbf{LAWFS}_{\mathrm{ladj}} \xrightarrow{\mathcal{G}_2^{\mathrm{ladj}}} \mathbf{Cmd}(-)^{\mathbf{2}}_{\mathrm{ladj}} \xrightarrow{\mathcal{G}_3^{\mathrm{ladj}}} \mathbf{CAT}/(-)^{\mathbf{2}}_{\mathrm{ladj}} \tag{I.6.21}$$

Here, $\mathbf{Cmd}(-)^{\mathbf{2}}_{\mathrm{ladj}}$ is the category of comonads on arrow categories and colax morphisms of comonads whose functor is the left adjoint of a specified adjunction, and $\mathbf{LAWFS}_{\mathrm{ladj}}$ is the full subcategory of comonads over dom.

When restricted to objects whose base categories are cocomplete, each category in (I.6.21) is cofibered over $\mathbf{CAT}^{\mathbf{2}}_{\mathrm{ladj}}$, the category of arrow categories and adjunctions of underlying categories, regarded as morphisms in the direction of the left adjoint. The fibers over the identity arrows are exactly the categories of (I.6.20). For $\mathbf{AWFS}_{\mathrm{ladj}}$, we perhaps need to insist that the categories be locally finitely presentable, in which case this statement says that any awfs can be lifted along an adjunction, even if it is not cofibrantly generated. This decidedly non-trivial result is due to Richard Garner.

In [Gar09], Garner shows that when $\mathcal{M}$ permits the small object argument, any small category can be reflected along (I.6.20). We show that this construction gives a reflection of these objects along (I.6.21), which is precisely what is needed for the desired corollary.

Theorem I.6.22. *For any small category $\mathcal{J}$ over the arrow category of a category $\mathcal{M}$ that permits the small object argument, the unit functor constructed by Garner's small object argument is universal among adjunctions of awfs.*

Proof. It suffices to show that such $\mathcal{J}$ can be reflected along each of the $\mathcal{G}_i^{\mathrm{ladj}}$, i.e., the unit functor constructed at each step in [Gar09, §4] satisfies the appropriate universal property.

The reflection of $\mathcal{J}$ along $\mathcal{G}_3^{\mathrm{ladj}}$ is its density comonad $\mathbb{C}^0$, i.e., the left Kan extension of $J \colon \mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ along itself. If $T \colon \mathcal{M} \to \mathcal{K}$ is a left adjoint and $\mathbb{L}$ is an arbitrary comonad on $\mathcal{K}^{\mathbf{2}}$, functors $\mathcal{J} \to \mathbb{L}$-**coalg** lifting T are in bijection with natural transformations $TJ \Rightarrow LTJ$, which are in bijection with natural transformations $TC^0 \Rightarrow LT$ because left adjoints preserve left Kan extensions. By the universal property of the density comonad, or alternatively, by a straightforward diagram chase, such natural transformations are always comonad morphisms [Dub70, Chapter II]. This shows that the unit $\mathcal{J} \to \mathbb{C}^0$-**coalg** is universal with respect to comonad morphisms lifting left adjoints and hence gives a reflection of $\mathcal{J}$ along $\mathcal{G}_3^{\mathrm{ladj}}$.

The refection of $\mathbb{C}^0$ along $\mathcal{G}_2^{\mathrm{ladj}}$ is given by an ofs (see Example I.2.12) on arrow categories, which factors a given morphism (a square in the underlying category) as a pushout square followed by a square whose domain component is an identity. Explicitly, the reflection $\mathbb{C}^1$ is obtained by factoring the counit of $\mathbb{C}^0$ as depicted below:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "C^0 f"'] & \cdot \arrow[r, equal] \arrow[d, "C^1 f"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[from=1-1, to=2-2, phantom, "\ulcorner", very near end] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Let $\psi\colon C^0 \Rightarrow C^1$ be the natural transformation whose component at f is the left-hand square depicted above. Given a colax morphism of comonads $(T\colon \mathcal{M} \to \mathcal{K}, \lambda\colon TC^0 \Rightarrow LT)$ where T is a left adjoint and $\mathbb{L}$ is a comonad on $\mathcal{K}^{\mathbf{2}}$ over $\mathrm{dom}\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, we have a commutative square

$$\begin{tikzcd}
TC^0 \arrow[r, "\lambda"] \arrow[d, "T\psi"'] & LT \arrow[d, "\epsilon T"] \\
TC^1 \arrow[r, "T\epsilon"'] & T
\end{tikzcd}$$

because λ is a comonad morphism and the lower left composite is T applied to the counit of C^0. The left arrow $T\psi$ is in the left class of the ofs described above because T, as a left adjoint between the underlying categories, preserves pushouts, and so the components of $T\psi$ are pushout squares. The right arrow is in the right class because $\mathbb{L}$ was assumed to be a comonad over dom. The ofs described above, this time on $\mathcal{K}$, solves the lifting problem to obtain the components of a unique natural transformation $\lambda'\colon TC^1 \Rightarrow LT$. By setting up appropriate lifting problems and using the fact that any solutions that exist must be unique, we can easily check that λ' is a comonad morphism, as desired. This shows that the unit functor $\mathbb{C}^0$-**coalg**$\to$ $\mathbb{C}^1$-**coalg** is universal with respect to colax morphisms of comonads over dom that lift a left adjoint; hence, it exhibits $\mathbb{C}^1$ as the reflection of $\mathbb{C}^0$ along $\mathcal{G}_2^{\mathrm{ladj}}$.

It remains only to consider the reflection of $\mathbb{C}^1$ along $\mathcal{G}_1^{\mathrm{ladj}}$. For each category $\mathcal{M}$, there is a strict two-fold monoidal category $\mathbf{FF}(\mathcal{M})$ of functorial factorizations of $\mathcal{M}$ (see [Gar07, Gar09] and [BFSV03]), for which $\mathbf{LAWFS}(\mathcal{M})$ is the category of $\odot$-comonoids and $\mathbf{AWFS}(\mathcal{M})$ is the category of $\otimes, \odot$-bialgebras. The product $\otimes$ (resp. $\odot$) uses the second awfs to re-factor the right (resp. left) half of the factorization produced by the first awfs. To reflect from $\odot$-comonoids into bialgebras, Garner uses Max Kelly's construction of the free $\otimes$-monoid on a pointed object [Kel80], in this case the unique arrow $I \to \vec{Q}^1$ from the unit for $\otimes$, which is initial in $\mathbf{FF}(\mathcal{M})$, to the functorial factorization of $\mathbb{C}^1$.

Let $\mathbf{FF}_{\mathrm{ladj}}$ be the category of functorial factorizations over an arbitrary base whose morphisms are *colax morphisms of functorial factorizations* lifting left adjoints. If $\vec{X}$ is a functorial factorization on $\mathcal{M}$ and $\vec{Y}$ is a functorial factorization on $\mathcal{K}$, then a morphism $\phi\colon \vec{X} \to \vec{Y}$ lifting a left adjoint $T\colon \mathcal{M} \to \mathcal{K}$ is a natural transformation $\phi\colon TX \Rightarrow YT$ such that the two triangles analogous to the left-hand diagram of (I.6.7) commute. This category is not two-fold monoidal, as we have no way to combine objects in different fibers. However, given objects $\vec{X}$ and $\vec{Z}$ in the fiber over $\mathcal{M}$ and $\vec{Y}$ and $\vec{W}$ in the fiber over $\mathcal{K}$ together with morphisms $\phi\colon \vec{X} \to \vec{Y}$ and $\psi\colon \vec{Z} \to \vec{W}$ lifting the same left adjoint $T\colon \mathcal{M} \to \mathcal{K}$, we do obtain lifts $\phi \otimes \psi\colon \vec{X} \otimes \vec{Z} \to \vec{Y} \otimes \vec{W}$ and $\phi \odot \psi\colon \vec{X} \odot \vec{Z} \to \vec{Y} \odot \vec{W}$ of T.

Furthermore, if ϕ and ψ are $\odot$-comonoid morphisms, then so is $\phi \otimes \psi$. The proof uses the fact that $\odot$ distributes over $\otimes$ in each fiber [Gar07, §3.2], and the canonical arrows α exhibiting this distributivity are natural with respect to colax morphisms of functorial factorizations:

$$\begin{tikzcd}
(\vec{X} \odot \vec{X}') \otimes (\vec{Z} \odot \vec{Z}') \arrow[r, "\alpha"] \arrow[d, "(\phi\odot\phi')\otimes(\psi\odot\psi')" description] & (\vec{X} \otimes \vec{Z}) \odot (\vec{X}' \otimes \vec{Z}') \arrow[d, "(\phi\otimes\psi)\odot(\phi'\otimes\psi')" description] \\
(\vec{Y} \odot \vec{Y}') \otimes (\vec{W} \odot \vec{W}') \arrow[r, "\alpha"'] & (\vec{Y} \otimes \vec{W}) \odot (\vec{Y}' \otimes \vec{W}')
\end{tikzcd}$$

In other words, if ϕ and ψ are morphisms in $\mathbf{LAWFS}_{\mathrm{ladj}}$, so is $\phi \otimes \psi$. This is all the structure we need to prove the unit satisfies the desired universal property; we need not consider the inner workings of the category $\mathbf{LAWFS}_{\mathrm{ladj}}$ any further.

Given a morphism from a pointed object $I \to \vec{X}$ in the fiber over $\mathcal{M}$ to a $\otimes$-monoid $\vec{Y}$ in the fiber over $\mathcal{K}$, we inductively obtain morphisms from the colimits involved in Kelly's transfinite construction to $\vec{Y} \otimes \vec{Y}$ and thus to $\vec{Y}$ by applying the multiplication $\mu\colon \vec{Y} \otimes \vec{Y} \to \vec{Y}$. By the universal property, the resulting morphism from the free $\otimes$-monoid on $I \to \vec{X}$ to $\vec{Y}$ is a $\otimes$-monoid morphism in the category of $\odot$-comonoids and $\odot$-comonoid morphisms, and is unique. Applying this to the situation at hand, a colax morphism of comonads $(\mathbb{C}^1, Q^1) \to (\mathbb{L}, E)$ lifting the left adjoint of a specified adjunction and whose target underlies an awfs $(\mathbb{L}, \mathbb{R})$ factors through a unique colax morphism of awfs $(\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$. By Lemma I.6.12, this determines a unique adjunction of awfs. Hence, the unit of this reflection satisfies the desired universal property, completing the proof. □

The desired corollary now follows immediately from the universal property of $\gamma^{\mathcal{M}}$. We will need this result in the next section.

I.7 Algebraic Quillen adjunctions

We can now prove that the adjunction between the algebraic model structures of Theorem I.3.10 is canonically an algebraic Quillen adjunction.

Recall the following definition.

Definition I.3.11. Let $\mathcal{M}$ have an algebraic model structure $\xi^{\mathcal{M}} \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ and let $\mathcal{K}$ have an algebraic model structure $\xi^{\mathcal{K}} \colon (\mathbb{L}_t, \mathbb{R}) \to (\mathbb{L}, \mathbb{R}_t)$. An adjunction $T \colon \mathcal{M} \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathcal{K} \colon S$ is an *algebraic Quillen adjunction* if there exist natural transformations λ_t, λ, ρ_t, and ρ determining five adjunctions of awfs

$$\begin{tikzcd}
(\mathbb{C}_t, \mathbb{F}) \arrow[r, "{(T,S,\lambda_t,\rho)}"] \arrow[d, "{(1,1,\xi^{\mathcal{M}},\xi^{\mathcal{M}})}"'] \arrow[dr, "{(T,S,\lambda\cdot T\xi^{\mathcal{M}},S\xi^{\mathcal{K}}\cdot\rho)}" description] & (\mathbb{L}_t, \mathbb{R}) \arrow[d, "{(1,1,\xi^{\mathcal{K}},\xi^{\mathcal{K}})}"] \\
(\mathbb{C}, \mathbb{F}_t) \arrow[r, "{(T,S,\lambda,\rho_t)}"'] & (\mathbb{L}, \mathbb{R}_t)
\end{tikzcd} \tag{I.7.1}$$

such that both triangles commute.

Theorem I.3.13. *Let* $T \colon \mathcal{M} \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathcal{K} \colon S$ *be an adjunction. Suppose* $\mathcal{M}$ *has an algebraic model structure, generated by* $\mathcal{I}$ *and* $\mathcal{J}$*, with comparison map* $\xi^{\mathcal{M}}$*. Suppose* $\mathcal{K}$ *has the algebraic model structure, generated by* $T\mathcal{I}$ *and* $T\mathcal{J}$*, with canonical comparison map* $\xi^{\mathcal{K}}$*. Then* $T \dashv S$ *is canonically an algebraic Quillen adjunction.*

Proof. Write Q_t, Q, E_t, and E for the functors accompanying the functorial factorizations of the awfs $(\mathbb{C}_t, \mathbb{F})$, $(\mathbb{C}, \mathbb{F}_t)$, $(\mathbb{L}_t, \mathbb{R})$, and $(\mathbb{L}, \mathbb{R}_t)$, respectively. Then by Theorem I.6.15 the natural transformations

$$\lambda_t\colon TQ_t \Rightarrow E_tT,\ \lambda\colon TQ \Rightarrow ET,\ \rho\colon Q_tS \Rightarrow SE_t,\ \text{and}\, \rho_t\colon QS \Rightarrow SE$$

arising from the canonical lifts of S give rise to adjunctions of awfs

$$(T, S, \lambda_t, \rho)\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{L}_t, \mathbb{R}) \quad \text{and} \quad (T, S, \lambda, \rho_t)\colon (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{L}, \mathbb{R}_t).$$

Composing the left-hand adjunction with $\xi^{\mathcal{K}}$ and the right-hand adjunction with $\xi^{\mathcal{M}}$, which we saw in Example I.6.14 are themselves adjunctions of awfs, we obtain two canonical adjunctions of awfs

$$\begin{tikzcd}(\mathbb{C}_t, \mathbb{F}) \arrow[r, shift left, "{(T,S,\xi^{K}T\cdot\lambda_t,S\xi^{\mathcal{K}}\cdot\rho)}"] \arrow[r, shift right, "{(T,S,\lambda\cdot T\xi^{\mathcal{M}},\rho_t\cdot\xi^{\mathcal{M}}S)}"'] & (\mathbb{L}, \mathbb{R}_t).\end{tikzcd} \tag{I.7.2}$$

We'll show that the corresponding natural transformations are the same.

By the correspondence between colax morphisms of comonads and natural transformations [Joh75], to show that

$$\begin{tikzcd} TQ_t \arrow[r, "\lambda_t"] \arrow[d, "T\xi^{\mathcal{M}}"'] & E_tT \arrow[d, "\xi^{\mathcal{K}}T"] \\ TQ \arrow[r, "\lambda"'] & ET \end{tikzcd} \tag{I.7.3}$$

commutes, it suffices to show that both composites correspond to the same lift of T to a functor $\mathbb{C}_t$-**coalg** → $\mathbb{L}$-**coalg**.

Let $\tilde{T}_t\colon \mathbb{C}_t$-**coalg** → $\mathbb{L}_t$-**coalg** and $\tilde{T}\colon \mathbb{C}$-**coalg** → $\mathbb{L}$-**coalg** denote the lifts of T corresponding to λ_t and λ, respectively. We must show the the right-hand diagram of (I.3.12) commutes. By the definition of $\xi^{\mathcal{K}}$ in the proof of Theorem I.3.10, the outer rectangle of

$$\begin{tikzcd} \mathcal{J} \arrow[r, "\gamma^{\mathcal{M}}"] \arrow[d] & \mathbb{C}_t\text{-}\mathbf{coalg} \arrow[r, "(\xi^{\mathcal{M}})_*"] \arrow[d, "\tilde{T}_t"'] & \mathbb{C}\text{-}\mathbf{coalg} \arrow[d, "\tilde{T}"] \\ T\mathcal{J} \arrow[r, "\gamma^{\mathcal{K}}"'] & \mathbb{L}_t\text{-}\mathbf{coalg} \arrow[r, "(\xi^{\mathcal{K}})_*"'] & \mathbb{L}\text{-}\mathbf{coalg} \end{tikzcd}$$

commutes. By Corollary I.6.18, the left-hand square commutes. By Theorem I.6.22, the unit $\gamma^{\mathcal{M}}$ is universal among adjunctions of awfs, which implies that the right-hand square commutes.

The other half of the proof now follows formally, using the fact that the natural transformations ρ and ρ_t defining the lifts $\tilde{S}$ and $\tilde{S}_t$ of (I.3.12) are mates of the natural transformations λ_t and λ defining the lifts $\tilde{T}_t$ and $\tilde{T}$. If ι and ν are the unit and counit of $T \dashv S$, the commutative diagram

$$\begin{tikzcd}
Q_tS \arrow[r, "\iota Q_tS"] \arrow[d, "\xi^{\mathcal{M}}S"'] & STQ_tS \arrow[r, "S\lambda_tS"] \arrow[d, "ST\xi^{\mathcal{M}}S"'] & SE_tTS \arrow[r, "SE_t(\nu{,}\nu)"] \arrow[d, "S\xi^{\mathcal{K}}TS"] & SE_t \arrow[d, "S\xi^{\mathcal{K}}"] \\
QS \arrow[r, "\iota QS"'] & STQS \arrow[r, "S\lambda S"'] & SETS \arrow[r, "SE(\nu{,}\nu)"'] & SE
\end{tikzcd} \tag{I.7.4}$$

says that $S\xi^{\mathcal{K}} \cdot \rho = \rho_t \cdot \xi^{\mathcal{M}}S$. This tells us that the diagram of functors on the left-hand side of (I.3.12) commutes, which proves that the two adjunctions (I.7.2) are the same and that $T \dashv S$ is an algebraic Quillen adjunction. □

Note that a diagram like (I.7.4), which shows that the natural transformations $\lambda\cdot T\xi^{\mathcal{M}} \colon TQ_t \Rightarrow ET$ and $S\xi^{\mathcal{K}} \cdot \rho \colon Q_tS \Rightarrow SE$ are mates, appears in the proof that adjunctions of awfs can be composed. In light of Corollary I.6.17, we expect that many other naturally occurring examples of Quillen adjunctions can be algebraicized to give algebraic Quillen adjunctions.

Part II

Monoidal algebraic model structures

II.1 Introduction

Algebraic model structures, introduced in [Rie11], Part I of this thesis, are a structural extension of Quillen's model categories [Qui67] in which cofibrations and fibrations are "algebraic," i.e., equipped with specified retractions to their left or right factors which can be used to solve all lifting problems. The factorizations themselves are much more than functorial: the map from an arrow to its right factor is a monad and the map to its left factor is a comonad on the arrow category. In particular, the data of an algebraic model category determines a fibrant replacement monad and a cofibrant replacement comonad.

Despite the stringent structural requirements of this definition, algebraic model structures are quite abundant. A modified small object argument, due to Richard Garner, which can be run in many categories, produces an algebraic model structure in place of an ordinary cofibrantly generated one [Gar09]. The difference is that the components of a model structure—the *weak factorization systems* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$—are replaced with *algebraic weak factorization systems* $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$, which are categorically better behaved.

We find the weak factorization system perspective on model categories clarifying. The overdetermination of the model category axioms and the closure properties of the classes of cofibrations and fibrations are all more evident on the weak factorization system level. Quillen's small object argument is really a construction of functorial factorizations for a cofibrantly generated weak factorization system; the model structure context is beside the point. Also, the equivalence of various definitions of Quillen adjunction has to do with the interaction between the adjunction and each weak factorization system independently. See §II.3.1 for more details.

More precisely, an *algebraic model structure* on a category $\mathcal{M}$ with weak equivalences $\mathcal{W}$ consists of two algebraic weak factorization systems (henceforth, *awfs* for both the singular and the plural) together with a morphism $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ between them such that the underlying weak factorization systems form a model structure in the usual sense. Here $\mathbb{C}_t$ and $\mathbb{C}$ are comonads and $\mathbb{F}_t$ and $\mathbb{F}$ are monads on the arrow category $\mathcal{M}^{\mathbf{2}}$ that send an arrow to its appropriate factor with respect to the functorial factorizations of the model structure. We write $R, Q\colon \mathcal{M}^{\mathbf{2}} \rightrightarrows \mathcal{M}$ for the functors that assign to an arrow the object through which it factors. The notation is meant to evoke fibrant/cofibrant replacement: slicing over the terminal object or under the initial object gives the fibrant replacement monad and cofibrant replacement comonad, also denoted R and Q.

The natural transformation ξ, which we call the *comparison map*, plays a number of roles. Its components

$$\begin{tikzcd}
& \mathrm{dom} f \arrow[dl, "C_t f"'] \arrow[dr, "Cf"] & \\
Rf \arrow[rr, "\xi_f"] \arrow[dr, "Ff"'] & & Qf \arrow[dl, "F_t f"] \\
& \mathrm{cod} f &
\end{tikzcd} \tag{II.1.1}$$

are natural solutions to the lifting problem (II.1.1) that compares the two functorial factorizations of $f \in \mathcal{M}^{\mathbf{2}}$. Additionally, ξ must satisfy two pentagons: one involving the comultiplications of the comonads and one involving the multiplications of the monads. Under these hypothesis, ξ determines functors over $\mathcal{M}^{\mathbf{2}}$

$$\xi_* \colon \mathbb{C}_t\textbf{-coalg} \to \mathbb{C}\textbf{-coalg} \qquad \xi^* \colon \mathbb{F}_t\textbf{-alg} \to \mathbb{F}\textbf{-alg} \tag{II.1.2}$$

between the categories of coalgebras for the comonads and algebras for the monads.

Elements of, e.g., the category $\mathbb{F}$-**alg** are called *algebraic fibrations*; their images under the forgetful functor to $\mathcal{M}^{\mathbf{2}}$ are in particular fibrations in the model structure. The algebra structure associated to an algebraic fibration determines a canonical solution to any lifting problem of that arrow against an algebraic trivial cofibration. The functors (II.1.2) together with naturality of ξ imply that there is also a single canonical solution to any lifting problem of an algebraic trivial cofibration against an algebraic trivial fibration: the solution constructed using ξ_* and the awfs $(\mathbb{C}, \mathbb{F}_t)$ and the solution constructed using ξ^* and the awfs $(\mathbb{C}_t, \mathbb{F})$ agree.

For certain lifting problems, these canonical solutions themselves assemble into a natural transformation. For instance, the natural solution to the usual lifting problem that compares the two bifibrant replacements of an object defines a natural transformation $RQ \Rightarrow QR$ that turns out to be a distributive law of the fibrant replacement monad over the cofibrant replacement comonad. It follows that Q lifts to a comonad on the category $\mathbb{R}$-**alg** of algebraic fibrant objects, and dually R lifts to a monad on $\mathbb{Q}$-**coalg**. The coalgebras for the former and algebras for the later coincide, defining a category of algebraic bifibrant objects.

Any ordinary cofibrantly generated model structure gives rise to an algebraic model structure thanks to a modified form of Quillen's small object argument due to Richard Garner. As a result, this algebraic structure is much more common that might be supposed. Whenever the category permits the small object argument, he constructs an algebraic weak factorization system from any small category of generating cofibrations that satisfies two universal properties, both of which we frequently exploit [Gar07, Gar09].

Awfs were introduced to improve the categorical properties of ordinary weak factorization systems [GT06]. One feature of awfs is that the left and right classes are closed under colimits and limits, respectively, in the following precise sense. By standard monadicity results, the forgetful functors $\mathbb{C}$-**coalg** $\to \mathcal{M}^{\mathbf{2}}$, $\mathbb{F}$-**alg** $\to \mathcal{M}^{\mathbf{2}}$ create all colimits and limits, respectively, existing in $\mathcal{M}^{\mathbf{2}}$. In the context of algebraic model structures, this gives a new recognition principle for cofibrations constructed as colimits and fibrations constructed as limits. Familiarly, a colimit (in the arrow category) of cofibrations is not necessarily a cofibration. But if the cofibrations admit coalgebra structures that are preserved by the maps in the diagram, then the colimit is canonically a coalgebra and hence a cofibration.

When the model structure is cofibrantly generated, all fibrations and all trivial fibrations are *algebraic*, i.e., admit algebra structures; interestingly the dual statements do not hold. Transfinite composites of pushouts of coproducts of generating cofibrations $i \in \mathcal{J}$—the class of maps denoted $\mathcal{J}$-cell in the classical literature [Hov99, Hir03]—are necessarily algebraic cofibrations. Accordingly, we call the class of cofibrations that admit a $\mathbb{C}$-coalgebra structure the *cellular* cofibrations; a cofibration is cellular if and only if it can be made algebraic. In certain examples, the cellular cofibrations are precisely the class $\mathcal{J}$-cell. In all examples, the class of all cofibrations is the retract closure of the class of cellular ones. Cellularity will play an interesting and important role in the new results that follow.

The basic theory of algebraic model structures is developed in Part I. In particular, we define an *algebraic Quillen adjunction*, which is an ordinary Quillen adjunction such that the right adjoint lifts to commuting functors between the algebraic (trivial) fibrations and the left adjoint lifts to commuting functors between the categories of algebraic (trivial) cofibrations. This should be thought of an algebraization of the usual condition that the right adjoint preserves fibrations and trivial fibrations and left adjoint preserves cofibrations and trivial cofibrations. We also ask that the lifts of one adjoint determine the lifts of the other in a sense made precise below, a condition that mirrors the classical fact that a Quillen adjunction can be detected by examining the left or right adjoint alone. Algebraic Quillen adjunctions exist in an important class of examples: when a cofibrantly generated algebraic model structure is lifted along an adjunction, the resulting Quillen adjunction is canonically algebraic. For instance, this situation describes the usual adjunction between spaces and simplicial sets.

A classical categorical results characterizes lifted functors of algebraic fibrations, i.e., functors between the categories of algebras for the monads, as certain natural transformations sometimes called *lax monad morphisms*, but this condition alone fails to capture the symmetry of the classical situation where a right adjoint preserves fibrations if and only if its left adjoint preserves trivial cofibrations. There are two ways to describe the desired additional hypothesis. One, the approach emphasized in Part I, is to ask that the *mate* of the natural transformation characterizing the lifted functor of algebraic fibrations defines the lifted functor of algebraic trivial cofibrations. An equivalent condition is to ask that the lifted functor of algebraic fibrations is in fact a lifted *double functor* between double categories of algebraic fibrations, suitably defined. Both approaches are described in §II.3 below, which reviews the development of the theory of algebraic Quillen adjunctions.

In this paper, we extend these results in order to define monoidal algebraic model structures. Much of the structure of a closed monoidal category or a tensored and cotensored enriched category is encoded in a *two-variable adjunction*. For enriched categories, the constituent bifunctors are commonly denoted

$$\mathcal{V} \times \mathcal{M} \xrightarrow{-\odot-} \mathcal{M} \qquad \mathcal{V}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\{-,-\}} \mathcal{M} \qquad \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\hom(-,-)} \mathcal{V}$$

and come equipped with hom-set isomorphisms

$$\mathcal{M}(V \odot M, N) \cong \mathcal{M}(M, \{V, N\}) \cong \mathcal{V}(V, \hom(M, N)) \tag{II.1.3}$$

natural in all three variables. Fixing any one variable, two-variable adjunctions give rise to ordinary adjunctions, e.g., $- \odot M \dashv \hom(M, -)$.

The monoidal case necessarily precedes the enriched one but also inherits nearly all of its complexity. A closed monoidal category with an algebraic model structure is a *monoidal algebraic model category* if tensoring with the cofibrant replacement of the monoidal unit sends cofibrant objects to weak equivalences and if the closed monoidal structure is an *algebraic Quillen two-variable adjunction*. Such an adjunction consists of three lifts of the so-called “pushout-product”

$$\begin{aligned} \mathbb{C}_t\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} &\longrightarrow \mathbb{C}_t\textbf{-coalg} \\ \mathbb{C}\textbf{-coalg} \times \mathbb{C}_t\textbf{-coalg} &\longrightarrow \mathbb{C}_t\textbf{-coalg} \\ \mathbb{C}\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} &\longrightarrow \mathbb{C}\textbf{-coalg} \end{aligned}$$

such that the mates of the characterizing natural transformations determine similar lifts of the left and right closures. In the best cases, these functors satisfy three evident coherence conditions which say that various canonical coalgebra structures agree, but we shall see that such coherence is too much to ask for in general. One could also give weaker definitions of an algebraic Quillen bifunctor applying to monoidal and enriched model categories in which some of the adjoint bifunctors don't exist. This is much less categorically challenging than the theory presented here, so the details may be safely left to the reader.

There are three main technical theorems that allow us to identify algebraic Quillen two-variable adjunctions in practice. The first describes a composition criterion that identifies when a lifted bifunctor is part of a two-variable adjunction of awfs, the version of algebraic Quillen two-variable adjunctions for categories equipped with a single awfs in place of a full algebraic model structure. The other two results, which we call the cellularity and uniqueness theorems, combine to characterize two-variable adjunctions of awfs in the case when the awfs are cofibrantly generated. The cellularity theorem says that a two-variable adjunction of awfs arises from any assignment of coalgebra structures to the pushout-product of the generators; hence, such structures exist if and only if the pushout-product of the generators is cellular. The uniqueness theorem says that such an assignment completely determines the lifted functors, so at most one two-variable adjunction of awfs can be obtained in this way.

Several new categorical results were necessary to make all of this precise. Of most general categorical interest is the theory of *parameterized mates*, introduced in §II.2 below. This theory describes the relationship between the natural transformations characterizing the lifts of the three functors constituting a two-variable adjunction and their interactions with ordinary adjunctions of awfs.

Other results appearing below are designed to deal with complications arising in the proofs of the cellularity and uniqueness theorems. The main technical difficulty is quite simply accounted for: the only adjunctions considered in Part I between arrow categories were those, now denoted $T^{\mathbf{2}} \dashv S^{\mathbf{2}} \colon \mathcal{M}^{\mathbf{2}} \rightleftarrows \mathcal{K}^{\mathbf{2}}$, defined pointwise by an ordinary adjunction $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ between the base categories. However, the adjunctions on arrow categories arising from two-variable adjunctions on the bases no longer have this form and in particular don't preserve composability of arrows. Thus, double categorical composition criterion we use to great effect in the previous paper to characterize lifted left adjoints that determine lifts of right adjoints must take on a new form.

In §II.2, we introduce double categories, mates, and parameterized mates and prove some elementary lemmas which will be used frequently in what follows. In §II.3, we give a streamlined review of the theory of algebraic Quillen adjunctions. The same general ideas will be used to prove analogous theorems for the two-variable case, though the categorical structures that appear are somewhat more complicated. In §II.4, we define algebraic Quillen two-variable adjunctions and state the cellularity and uniqueness theorems. In §II.5 and §II.6, the most technical sections, we describe the new composition criterion, which together with an extension of the universal property of Garner's small object argument allows us to prove these theorems. In §II.5, we consider first the adjunctions of a single variable arising from two-variable adjunctions and then in §II.6 consider the full bifunctors. A reader who is willing to take these facts on faith could skip these sections and jump straight to §II.7, where we define monoidal algebraic model structures and give examples.

II.2 Double categories, mates, parameterized mates

The calculus of *mates* will play an important conceptual and calculational role in what follows. To streamline later proofs, we take a few moments in §II.2.1 to outline the important features without getting mired in technical details. The canonical reference is [KS74]; we also like [Shu11].

Bifunctors, meaning functors whose domain is the product of two categories, are determined by the collection of ordinary functors obtained when one of the variables is fixed together with the natural transformations between such functors arising from morphisms in that category. This fact is often expressed by saying that category **CAT** of categories is cartesian closed. For this simple reason, the classical theory of mates extends to a new theory of *parameterized mates*, outlined in §II.2.2 below.

II.2.1 Double categories and mates

A *double category* $\mathbb{D}$ is a category internal to **CAT**:

$$\begin{tikzcd}
\mathbb{D}_1 \underset{\mathbb{D}_0}{\times} \mathbb{D}_1 \arrow[r, "\circ"] & \mathbb{D}_1 \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathbb{D}_0 \arrow[l, "\mathrm{id}"' description]
\end{tikzcd}$$

The objects and arrows of $\mathbb{D}_0$ are called objects and horizontal arrows of $\mathbb{D}$ while the objects and arrows of $\mathbb{D}_1$ are called vertical arrows and squares. Via the functors $\mathrm{dom}, \mathrm{cod} \colon \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$, the sources and targets of vertical arrows are objects of $\mathbb{D}$, and likewise the squares can be depicted in the way their name suggests. Squares can be composed horizontally using composition in $\mathbb{D}_1$ and vertically using the functor $\circ$. In this paper, both horizontal and vertical composition of squares is strict and is preserved strictly. We refer to $\mathbb{D}_1$ as the *category of vertical arrows*; this category forgets the composition of vertical arrows and remembers only the horizontal composition of squares.

Example II.2.1. A category $\mathcal{M}$ gives rise to a double category $\mathbb{S}\mathbf{q}(\mathcal{M})$

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{3}} \cong \mathcal{M}^{\mathbf{2}} \underset{\mathcal{M}}{\times} \mathcal{M}^{\mathbf{2}} \arrow[r, "\circ"] & \mathcal{M}^{\mathbf{2}} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{M} \arrow[l, "\mathrm{id}"' description]
\end{tikzcd}$$

whose objects are objects of $\mathcal{M}$, horizontal and vertical arrows are morphisms of $\mathcal{M}$, and squares are commutative squares. The category of vertical arrows is usually called the arrow category—the notation will be explained in §II.3—and plays an essential role in what follows.

Given categories, functors, and adjunctions, as displayed below, there is a bijection between natural transformations in the square involving the left adjoints and natural transformations in the square involving the right adjoints

$$\begin{tikzcd}
\cdot \arrow[r, "H"] \arrow[d, shift right=2, "T"'] \arrow[d, phantom, "\dashv"] & \cdot \arrow[d, shift right=2, "T'"'] \arrow[d, phantom, "\dashv"] \\
\cdot \arrow[r, "K"'] \arrow[u, shift right=2, "S"'] & \cdot \arrow[u, shift right=2, "S'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "H"] \arrow[d, "T"'] & \cdot \arrow[d, "T'"] \arrow[dl, phantom, "\lambda\Swarrow"] \\
\cdot \arrow[r, "K"'] & \cdot
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\cdot \arrow[r, "H"] & \cdot \arrow[dl, phantom, "\Searrow\rho"] \\
\cdot \arrow[r, "K"'] \arrow[u, "S"] & \cdot \arrow[u, "S'"']
\end{tikzcd}
\tag{II.2.2}$$

given by the formulas

$$\rho = S'K\epsilon \cdot S'\lambda_S \cdot \iota_{HS} \quad \text{and} \quad \lambda = \nu_{KT} \cdot T'\rho_T \cdot T'H\eta, \tag{II.2.3}$$

where η and ϵ are the unit and counit for $T \dashv S$ and ι and ν are the unit and counit for $T' \dashv S'$. Corresponding λ and ρ are called *mates*.

Significant examples abound in this paper, but we also will make use of trivial ones.

Example II.2.4. A natural transformation $H \Rightarrow K$ is its own mate with respect to the identity adjunctions.

Example II.2.5. *Adjunct* arrows $f^\sharp \colon Tm \to k \in \mathcal{K}$, $f \colon m \to Sk \in \mathcal{M}$ corresponding under the adjunction $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ are mates in the following squares

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "m"] \arrow[d, "1"'] \arrow[dr, phantom, "f^\sharp \swarrow"] & \mathcal{M} \arrow[d, "T"] & \mathbf{1} \arrow[r, "m"] \arrow[dr, phantom, "\searrow f"] & \mathcal{M} \\
\mathbf{1} \arrow[r, "k"'] & \mathcal{K} & \mathbf{1} \arrow[r, "k"'] \arrow[u, "1"] & \mathcal{K} \arrow[u, "S"']
\end{tikzcd}$$

where $\mathbf{1}$ denotes the terminal category.

Example II.2.6. If $\mathcal{M}$ has a left-closed monoidal structure and $f \colon m' \to m \in \mathcal{M}$, then the induced natural transformations

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "1"] \arrow[d, "m\otimes-"'] \arrow[dr, phantom, "f\otimes- \swarrow"] & \mathcal{M} \arrow[d, "m'\otimes-"] & \mathcal{M} \arrow[r, "1"] \arrow[dr, phantom, "\searrow \hom_\ell(f,-)"] & \mathcal{M} \\
\mathcal{M} \arrow[r, "1"'] & \mathcal{M} & \mathcal{M} \arrow[r, "1"'] \arrow[u, "\hom_\ell(m,-)"] & \mathcal{M} \arrow[u, "\hom_\ell(m',-)"']
\end{tikzcd}$$

are mates. Analogous correspondences hold for any *parameterized adjunction* [ML98, IV.7.3].

There are double categories $\mathbb{L}$**adj** and $\mathbb{R}$**adj** whose objects are categories, horizontal arrows are functors, vertical arrows are adjunctions in the direction of the left adjoint, and whose squares are natural transformations as displayed in the middle and right-hand squares of (II.2.2), respectively. The mates correspondence is natural, or, more accurately, functorial, in the following precise sense.

Theorem II.2.7 (Kelly-Street [KS74, §2]). *The mates correspondence gives an isomorphism of double categories* $\mathbb{L}$**adj**$\cong$ $\mathbb{R}$**adj**.

This says that a natural transformation obtained by pasting squares in $\mathbb{L}$**adj** either vertically or horizontally is the mate of the natural transformation obtained by pasting the mates of these squares in $\mathbb{R}\mathbf{adj}$. The “calculus of mates” refers to this fact, which, when used in conjunction with Examples II.2.4–II.2.6, implies that mates satisfy dual diagrams.

For instance, suppose the functors H and K of (II.2.2) are monads $(H,\eta,\mu), (K,\eta,\mu)$ and suppose $T = T'$ and $S = S'$. A pair (S,ρ) as in the right square of (II.2.2) is a *lax morphism of monads* if

$$\begin{tikzcd} & S \arrow[dl, "\eta S"'] \arrow[dr, "S\eta"] & \\ HS \arrow[rr, "\rho"] & & SK \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & HSK \arrow[dr, "\rho K"] & \\ HHS \arrow[ur, "H\rho"] \arrow[d, "\mu S"'] & & SKK \arrow[d, "S\mu"] \\ HS \arrow[rr, "\rho"] & & SK \end{tikzcd} \tag{II.2.8}$$

commute. We will frequently use

Lemma II.2.9 (Appelgate [Joh75])**.** *A lax morphism of monads (S,ρ) determines and is determined by a lift of S to a functor from the category of K-algebras to the category of H-algebras.*

Proof. The $\mathbb{H}$-algebra structure assigned the image under S of a $\mathbb{K}$-algebra $t\colon Kx \to x$ is

$$HSx \xrightarrow{\rho x} SKx \xrightarrow{St} Sx$$

□

The dual notion, a *colax morphism of monads*, is a pair (S,ρ) satisfying diagrams analogous to (II.2.8) but with the direction of ρ reversed. Theorem II.2.7 can be used to prove

Lemma II.2.10. *Suppose (S,ρ) is a lax morphism of monads, $T \dashv S$, and λ is the mate of ρ with respect to this adjunction. Then (T,λ) is a colax morphism of monads.*

Proof. We show (T,λ) satisfies the pentagon and leave the triangle as an exercise. The pentagon for (S,ρ) says that the left pasted squares

$$\begin{tikzcd} \cdot \arrow[r, "HH"] & \cdot \\ \cdot \arrow[r, "H"'] \arrow[u, "1"] \arrow[ur, phantom, "\mu\Searrow"] & \cdot \arrow[u, "1"'] \\ \cdot \arrow[r, "K"'] \arrow[u, "S"] \arrow[ur, phantom, "\rho\Searrow"] & \cdot \arrow[u, "S"'] \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "H"] & \cdot \arrow[r, "H"] & \cdot \\ \cdot \arrow[r, "K"'] \arrow[u, "S"] \arrow[ur, phantom, "\rho\Searrow"] & \cdot \arrow[r, "K"'] \arrow[u, "S"] \arrow[ur, phantom, "\rho\Searrow"] & \cdot \arrow[u, "S"'] \\ \cdot \arrow[rr, "K"'] \arrow[u, "1"] \arrow[urr, phantom, "\mu\Searrow"] & & \cdot \arrow[u, "1"'] \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "HH"] \arrow[d, "1"'] \arrow[dr, phantom, "\mu\Swarrow"] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "H"'] \arrow[d, "T"'] \arrow[dr, phantom, "\lambda\Swarrow"] & \cdot \arrow[d, "T"] \\ \cdot \arrow[r, "K"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "H"] \arrow[d, "T"'] \arrow[dr, phantom, "\lambda\Swarrow"] & \cdot \arrow[r, "H"] \arrow[d, "T"'] \arrow[dr, phantom, "\lambda\Swarrow"] & \cdot \arrow[d, "T"] \\ \cdot \arrow[r, "K"'] \arrow[d, "1"'] \arrow[drr, phantom, "\mu\Swarrow"] & \cdot \arrow[r, "K"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[rr, "K"'] & & \cdot \end{tikzcd}$$

are equal in $\mathbb{R}\mathbf{adj}$. By Theorem II.2.7 the pasted composites of their mates in $\mathbb{L}\mathbf{adj}$, displayed on the right above, also agree. □

Of course, analogous results hold with any 2-category in place of **CAT**. At this level of generality, Theorem II.2.7 asserts that the functors $\mathbb{L}\mathbf{Adj}, \mathbb{R}\mathbf{Adj} \colon 2\text{-}\mathbf{CAT} \rightrightarrows \mathbf{DblCAT}$ are isomorphic.

II.2.2 Parameterized mates

By a lemma below, in the context of a two-variable adjunction, or more generally a parameterized adjunction, the mates correspondences for the adjunctions obtained by fixing the parameter are natural in the parameter. This means that the two sets of mates assemble into natural transformations of two variables. We say that natural transformations corresponding in this way are *parameterized mates*. We do not know if this correspondence has been studied before, but it is essential to describe the interactions between awfs and two-variable adjunctions. The following lemmas establish the bare bones of this theory.

First, we prove that if we fix one of the variables in a natural transformation between bifunctors which are pointwise adjoints and then take mates, the resulting *pointwise mates* assemble to give a natural transformation between the appropriate bifunctors.

Lemma II.2.11. *Suppose given a pair of left-closed bifunctors* $\otimes, \otimes'$*; ordinary functors* K, M, N*; and a natural transformation* $\lambda_{k,m} \colon Kk \otimes' Mm \to N(k \otimes m)$ *as displayed*

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{M} \arrow[r, "K \times M"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\lambda \Swarrow"] & \mathcal{K}' \times \mathcal{M}' \arrow[d, "\otimes'"] \\
\mathcal{N} \arrow[r, "N"'] & \mathcal{N}
\end{tikzcd}$$

Let $\rho_{k,-}$ *denote the mate of the natural transformation* $\lambda_{k,-}$ *with respect to the adjunctions* $k \otimes - \dashv \hom(k,-)$ *and* $Kk \otimes' - \dashv \hom'(Kk,-)$*. Then the* $\rho_{k,-}$ *are also natural in* $\mathcal{K}$ *and assemble into a natural transformation* $\rho_{k,n} \colon M \hom(k,n) \to \hom'(Kk, Nn)$

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "M"] \arrow[dr, phantom, "\Searrow \rho"] & \mathcal{M}' \\
\mathcal{K}^{\mathrm{op}} \times \mathcal{N} \arrow[u, "\hom"] \arrow[r, "K \times N"'] & \mathcal{K}'^{\mathrm{op}} \times \mathcal{N}' \arrow[u, "\hom'"']
\end{tikzcd}$$

Proof. Naturality of λ in $\mathcal{K}$ says that for any $f\colon k' \to k$ in $\mathcal{K}$, the pasted composites

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "1"] \arrow[d, "k\otimes-"'] \arrow[dr, phantom, "f\otimes- \Swarrow"] & \mathcal{M} \arrow[r, "M"] \arrow[d, "k'\otimes-" description] \arrow[dr, phantom, "\lambda_{k',-} \Swarrow"] & \mathcal{M}' \arrow[d, "Kk'\otimes'-" description] & = & \mathcal{M} \arrow[r, "M"] \arrow[d, "k'\otimes-" description] \arrow[dr, phantom, "\lambda_{k,-} \Swarrow"] & \mathcal{M}' \arrow[r, "1"] \arrow[d, "Kk\otimes'-" description] \arrow[dr, phantom, "Kf\otimes'- \Swarrow"] & \mathcal{M}' \arrow[d, "Kk'\otimes'-" description] \\
\mathcal{N} \arrow[r, "1"'] & \mathcal{N} \arrow[r, "N"'] & \mathcal{N}' & & \mathcal{N} \arrow[r, "N"'] & \mathcal{N}' \arrow[r, "1"'] & \mathcal{N}'
\end{tikzcd}$$

are equal. By Theorem II.2.7, the pasted composites

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "1"] & \mathcal{M} \arrow[r, "M"] & \mathcal{M}' & = & \mathcal{M} \arrow[r, "M"] & \mathcal{M}' \arrow[r, "1"] & \mathcal{M}' \\
\mathcal{N} \arrow[r, "1"'] \arrow[u, "\hom(k,-)" description] \arrow[ur, phantom, "\Searrow^{\hom(f,-)}"] & \mathcal{N} \arrow[r, "N"'] \arrow[u, "\hom(k',-)" description] \arrow[ur, phantom, "\Searrow^{\rho_{k',-}}"] & \mathcal{N}' \arrow[u, "\hom'(Kk',-)" description] & & \mathcal{N} \arrow[r, "N"'] \arrow[u, "\hom(k,-)" description] \arrow[ur, phantom, "\Searrow^{\rho_{k,-}}"] & \mathcal{N}' \arrow[r, "1"'] \arrow[u, "\hom'(Kk,-)" description] \arrow[ur, phantom, "\Searrow^{\hom'(Kf,-)}"] & \mathcal{N}' \arrow[u, "\hom'(Kk',-)" description]
\end{tikzcd}$$

are also equal, which says that the ρ_k are natural in $\mathcal{K}$. □

The following lemma establishes the *parameterized mates correspondence.*

Lemma II.2.12. *Suppose given two-variable adjunctions* $(\otimes, \hom_\ell, \hom_r)$, $(\otimes', \hom'_\ell, \hom'_r)$ *and functors* K, M, N *as below. There is a natural bijective correspondence between natural transformations*

$$\begin{tikzcd}
\mathcal{K}\times\mathcal{M} \arrow[r, "K\times M"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\lambda \Swarrow"] & \mathcal{K}'\times\mathcal{M}' \arrow[d, "\otimes'"] & \mathcal{M} \arrow[r, "M"] & \mathcal{M}' & \mathcal{K} \arrow[r, "K"] & \mathcal{K}' \\
\mathcal{N} \arrow[r, "N"'] & \mathcal{N} & \mathcal{K}^{\mathrm{op}}\times\mathcal{N} \arrow[r, "K^{\mathrm{op}}\times N"'] \arrow[u, "\hom_\ell"] \arrow[ur, phantom, "\Searrow \rho^\ell"] & \mathcal{K}'^{\mathrm{op}}\times\mathcal{N}' \arrow[u, "\hom'_\ell"'] & \mathcal{M}^{\mathrm{op}}\times\mathcal{N} \arrow[r, "M^{\mathrm{op}}\times N"'] \arrow[u, "\hom_r"] \arrow[ur, phantom, "\Searrow \rho^r"] & \mathcal{M}'^{\mathrm{op}}\times\mathcal{N}' \arrow[u, "\hom'_r"']
\end{tikzcd}$$

obtained by applying the pointwise mates correspondence to either variables.

Proof. By symmetry, it suffices to show that if we fix $\mathcal{K}$ and takes pointwise mates to define ρ^ℓ from λ and then fix $\mathcal{N}$ and take pointwise mates to define ρ^r from ρ^ℓ, the result is the same as fixing $\mathcal{M}$ and taking pointwise mates to define ρ^r from λ. This follows from the formulas (II.2.3), the compatible hom-set isomorphisms (II.1.3) and a diagram chase. We leave this as an exercise to the reader with the following hint: when in a sequence of composable arrows, one sees the unit followed by arrows in the image of the right adjoint, this asserts that the composite is adjunct to whatever remains when the unit and the right adjoint are erased. Dually, the composite of an arrow in the image of the left adjoint with the counit is adjunct to what remains when the counit and left adjoint are erased. We made frequent use of this observation. □

The lemma says that taking pointwise mates is a “Klein four groupoid,” by which we mean the chaotic groupoid on three objects. The point is that any distinct isomorphisms compose to the other.

Lemma II.2.13. *Composition of parameterized mates in any of the three variables with ordinary mates pointing in compatible directions is well-defined.*

Proof. Suppose α and β are mates with respect to the top squares and $\lambda, \rho^\ell, \rho^r$ are parameterized mates with respect to the bottom squares of the following diagram in $\mathbb{L}\mathbf{adj}\cong \mathbb{R}\mathbf{adj}$.

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "J"] \arrow[d, shift right=2, "T"'] \arrow[d, phantom, "\dashv"] & \mathcal{J}' \arrow[d, shift right=2, "T'"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{K} \arrow[r, "K"'] \arrow[u, shift right=2, "S"'] \arrow[d, shift right=2, "-\otimes m"'] \arrow[d, phantom, "\dashv"] & \mathcal{K}' \arrow[u, shift right=2, "S'"'] \arrow[d, shift right=2, "-\otimes' Mm"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N} \arrow[r, "N"'] \arrow[u, shift right=2, "\hom_r(m,-)"'] & \mathcal{N}' \arrow[u, shift right=2, "\hom'_r(Mm,-)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{J} \arrow[r, "J"] \arrow[d, shift right=2, "T"'] \arrow[d, phantom, "\dashv"] & \mathcal{J}' \arrow[d, shift right=2, "T'"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{K} \arrow[r, "K"'] \arrow[u, shift right=2, "S"'] \arrow[d, shift right=2, "\hom_\ell(-{,}n)"'] \arrow[d, phantom, "\dashv"] & \mathcal{K}' \arrow[u, shift right=2, "S'"'] \arrow[d, shift right=2, "\hom'_\ell(-{,}Nn)"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{M}^{\mathrm{op}} \arrow[r, "M"'] \arrow[u, shift right=2, "\hom_r(-{,}n)"'] & \mathcal{M}'^{\mathrm{op}} \arrow[u, shift right=2, "\hom'_r(-{,}Nn)"']
\end{tikzcd}$$

Applying Theorem II.2.7 and Lemma II.2.11 to the left-hand rectangle, we conclude that

$$T'J \otimes' M \xrightarrow{\alpha\otimes' 1} KT \otimes' M \xrightarrow{\lambda_{T,1}} N(T \otimes -) \qquad \text{and}$$

$$JS\hom_r(-,-) \xrightarrow{\beta_{\hom_r}} S'K\hom_r(-,-) \xrightarrow{S'\rho^r} S'\hom'_r(M,N)$$

are mates; from right-hand rectangle, we conclude that this second natural transformation and

$$M\hom_\ell(T,-) \xrightarrow{\rho^\ell_{T,1}} \hom'_\ell(KT,N) \xrightarrow{\hom'_\ell(\alpha,N)} \hom'_\ell(T'J,N)$$

are mates. By Lemma II.2.12, the three composite natural transformations are parameterized mates. □

As a consequence, algebraic Quillen two-variable adjunctions pointing in the direction of the left adjoints can be composed in any of their variables with algebraic Quillen adjunctions pointing also in the direction of the left adjoints; see Lemma II.6.11.

II.3 Algebraic Quillen adjunctions via double categories and cellularity

In this section, we give a streamlined presentation of one of the main topics of Part I: the definitions, characterizations, and examples of algebraic Quillen adjunctions. All the results are contained in Part I, but they are not presented in quite this way. This narrative will inspire the extension to algebraic Quillen two-variable adjunctions, the primary technical component of monoidal algebraic model structures. In §II.3.1, we discuss ordinary weak factorization systems and functorial factorizations, and introduce a category of solutions to lifting problems. In §II.3.2, we define awfs, characterize awfs as double categories, and state the universal properties associated to Garner's small object argument. In §II.3.3, we define adjunctions of awfs, the main structural components of algebraic Quillen adjunctions, and characterize those whose domain is cofibrantly generated. Such adjunctions of awfs arise precisely when the image of the generators under the left adjoint is cellular, in the sense described in the introduction. In §II.3.4, we define algebraic Quillen adjunctions and outline the proof that there are interesting examples.

II.3.1 Preliminaries

We write $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$, etc for the categories assigned to these ordinals; e.g., $\mathbf{2}$ is the "walking arrow" category, $\mathbf{3}$ is the free category containing a composable pair of arrows, and so on. The functor category $\mathcal{M}^{\mathbf{2}}$ is the category whose objects are arrows in $\mathcal{M}$, depicted vertically, and whose morphisms, denoted $(u, v)\colon f \Rightarrow g$, are commutative squares

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \tag{II.3.1}$$

Any such square presents a *lifting problem* of f against g; a solution would be an arrow from the bottom left to the upper right such that both resulting triangles commute. If every lifting problem presented by a morphism $f \Rightarrow g$ has a solution, we say that f has the *left lifting property* with respect to g and, equivalently, that g has the *right lifting property* with respect to f.

Definition (I.2.3, I.2.4)**.** A *weak factorization system* on $\mathcal{M}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

(factorization) every arrow of $\mathcal{M}$ can be factored as an arrow of $\mathcal{L}$ followed by an arrow of $\mathcal{R}$

(lifting) every lifting problem (II.3.1) with $f \in \mathcal{L}$ and $g \in \mathcal{R}$ has a solution

(closure) every arrow with the left lifting property with respect to every arrow in $\mathcal{R}$ is in $\mathcal{L}$ and every arrow with the right lifting property every arrow of $\mathcal{L}$ is in $\mathcal{R}$.

In the presence of the first two axioms, the third can be replaced by

(closure′) the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts

as a consequence of the so-called "retract argument," familiar from the model category literature.

A *homotopical category* $(\mathcal{M}, \mathcal{W})$ is a complete and cocomplete category $\mathcal{M}$ together with a class of morphisms $\mathcal{W}$ called *weak equivalences* that satisfy the 2-of-3 property. A model structure on a homotopical category is given by a pair of interacting weak factorization systems.

Definition II.3.2. A *model structure* on a homotopical category $(\mathcal{M}, \mathcal{W})$ consists of two classes of morphisms $\mathcal{C}, \mathcal{F}$ such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.

Adopting standard notation

$$\mathcal{L}^{\boxslash} = \{g \in \mathcal{M}^{\mathbf{2}} \mid g \text{ has the right lifting property with respect to all } f \in \mathcal{L}\}$$
$$^{\boxslash}\mathcal{R} = \{f \in \mathcal{M}^{\mathbf{2}} \mid f \text{ has the left lifting property with respect to all } g \in \mathcal{R}\}$$

the lifting and closure axioms combine to assert that $\mathcal{R} = \mathcal{L}^{\boxslash}$ and $\mathcal{L} = {}^{\boxslash}\mathcal{R}$. In particular, it is clear that either class determines the other. For any class of morphisms $\mathcal{R}$, the class ${}^{\boxslash}\mathcal{R}$ is closed under coproducts, pushouts, (transfinite) composition, retracts, and contains the isomorphisms: precisely the familiar closure properties for the cofibrations in a model category.

We will now "categorify" the notation just introduced.

Definition (I.2.25)**.** If $\mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ is some subcategory of arrows, not necessarily full, define $\mathcal{J}^{\boxslash}$ be the category whose objects are pairs (f, ϕ_f), where $f \in \mathcal{M}^{\mathbf{2}}$ and ϕ_f is a *lifting function* that specifies a solution

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "j"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "b"'] \arrow[ur, dashed, "\phi_f(j,a,b)" description] & \cdot \end{tikzcd}$$

to any lifting problem against some $j \in \mathcal{J}$ in such a way that the specified lifts commute with morphisms in $\mathcal{J}$. A morphism $(f, \phi_f) \to (g, \phi_g)$ is a morphism $f \Rightarrow g$ in $\mathcal{M}^{\mathbf{2}}$ that preserves the chosen lifts.

When $\mathcal{J}$ is discrete, the set of objects in the image of the forgetful functor $\mathcal{J}^{\boxslash} \to \mathcal{M}^{\mathbf{2}}$ is precisely the set $\mathcal{J}^{\boxslash}$, as defined above. In fact, $\mathcal{J}^{\boxslash}$ is the category of vertical arrows for a double category, which we also denote $\mathcal{J}^{\boxslash}$, with vertical composition defined as follows. If $(f, \phi_f), (g, \phi_g) \in \mathcal{J}^{\boxslash}$ with $\mathrm{cod} f = \mathrm{dom} g$, their composite is denoted $(gf, \phi_g \bullet \phi_f)$ where

$$\phi_g \bullet \phi_f(j, a, b) := \phi_f(j, a, \phi_g(j, fa, b)) \qquad \begin{tikzcd} \cdot \arrow[r, "a"] \arrow[dd, "j"'] & \cdot \arrow[d, "f"] \\ & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "b"'] \arrow[uur, dashed, "\phi_f(j,a,\phi_g)"] \arrow[ur, dashed, "\phi_g(j,fa,b)"'] & \cdot \end{tikzcd} \tag{II.3.3}$$

There is a forgetful double functor $\mathcal{J}^{\boxslash} \to \mathbb{S}\mathbf{q}(\mathcal{M})$ which restricts to the above forgetful functor on the categories of vertical arrows. The category ${}^{\boxslash}\mathcal{J}$ is defined dually, and also forms a double category.

A *functorial factorization* on $\mathcal{M}$ is a section $\vec{E} \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{3}}$ of the "composition" functor $\mathcal{M}^{\mathbf{3}} \to \mathcal{M}^{\mathbf{2}}$; $\vec{E}$ is often described by a pair of functors $L, R \colon \mathcal{M}^{\mathbf{2}} \rightrightarrows \mathcal{M}^{\mathbf{2}}$ whose respective codomain and domain define a common functor $E \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$, as depicted below

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \overset{\vec{E}}{\mapsto} \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ Ef \arrow[r, "E(u,v)"] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \overset{\circ}{\mapsto} \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \tag{II.3.4}$$

Throughout, the vector notation is used to decorate functors and natural transformations on arrow categories whose primary data is described by one component; e.g., E contains all the data of $\vec{E}$ on morphisms.

II.3.2 Algebraic weak factorization systems

The endofunctors L, R arising from a functorial factorization $\vec{E}$ are equipped with canonical natural transformations $\vec{\epsilon} \colon L \Rightarrow 1$, $\vec{\eta} \colon 1 \Rightarrow R$, described in §I.2.3. A functorial factorization gives rise to an algebraic weak factorization system when this data can be extended to a compatible comonad and a monad.

Definition (I.2.9)**.** An *algebraic weak factorization system* $(\mathbb{L}, \mathbb{R})$ on a category $\mathcal{M}$ consists of a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ and a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ arising from a functorial factorization and such that $(\delta, \mu)\colon LR \Rightarrow RL$ is a distributive law.

The underlying weak factorization system is $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$, the retract closures of the classes of maps admitting coalgebra and algebra structures, respectively. Unraveling the definition, an $\mathbb{R}$-algebra is an arrow f equipped with a specified lift against its left factor

$$\begin{tikzcd}
\mathrm{dom} f \arrow[r, equal] \arrow[d, "Lf"'] & \mathrm{dom} f \arrow[d, "f"] \\
Ef \arrow[r, "Rf"'] \arrow[ur, dashed, "s"] & \mathrm{cod} f
\end{tikzcd} \tag{II.3.5}$$

that is compatible with the multiplication μ_f. The arrow s can be used to define a canonical solution to any lifting problem against an $\mathbb{L}$-coalgebra in such a way that the canonical solution to the lifting problem posed in (II.3.5) is s; see I.2.10. Morphisms of $\mathbb{R}$-algebras preserve the chosen solutions to lifting problems. Dually, $\mathbb{L}$-coalgebra structures determine canonical solutions to lifting problems against $\mathbb{R}$-algebras, defining an embedding

$$\begin{tikzcd}
\mathbb{L}\text{-}\mathbf{coalg} \arrow[rr, "\mathrm{lift}"] \arrow[dr, "U"'] & & {}^{\boxslash}\mathbb{R}\text{-}\mathbf{alg} \arrow[dl, "U"] \\
& \mathcal{M}^{\mathbf{2}} &
\end{tikzcd} \tag{II.3.6}$$

Unusually for comonads on $\mathcal{M}^{\mathbf{2}}$, the category $\mathbb{L}$-**coalg** embeds as the vertical arrows and squares of a double category $\mathbb{C}$**oalg**$(\mathbb{L})$:

$$\begin{tikzcd}
\mathbb{L}\text{-}\mathbf{coalg} \underset{\mathcal{M}}{\times} \mathbb{L}\text{-}\mathbf{coalg} \arrow[r, "\circ"] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{M} \arrow[l, "\mathrm{id}" description]
\end{tikzcd}$$

Objects are objects of $\mathcal{M}$, horizontal arrows are morphisms of $\mathcal{M}$, vertical arrows are $\mathbb{L}$-coalgebras, and squares are maps of $\mathbb{L}$-coalgebras. The essential point is that $\mathbb{L}$-coalgebras have a canonical composition law—the functor $\circ$ above—that is functorial with respect to $\mathbb{L}$-coalgebra morphisms. This vertical composition, given by the formula (II.3.7) below, is derived from the embedding (II.3.6) and (II.3.3): if $(i, s), (j, t) \in \mathbb{L}$-**coalg** with $\mathrm{cod} i = \mathrm{dom} j$, then the arrow ji is canonically an $\mathbb{L}$-coalgebra with coalgebra structure $t \bullet s$ defined by

$$t \bullet s := \ \mathrm{cod}j \xrightarrow{t} Ej \xrightarrow{E(E(1,j)\cdot s,1)} ER(ji) \xrightarrow{\mu_{ji}} E(ji). \tag{II.3.7}$$

As in (II.3.3), $t \bullet s$ is defined to be the canonical solution to the lifting problem displayed on the left

$$\begin{tikzcd}
\mathrm{dom}j \arrow[r, "E(1,j)\cdot s"] \arrow[dd, "j"'] & E(ji) \arrow[dd, "R(ji)"] \\
{} \arrow[r, dashed, "E(E(1,j)s,1)"'] & {} \arrow[u, dashed, "\mu_{ji}"] \\
\mathrm{cod}j \arrow[u, dashed, "t"'] \arrow[r, equal] & \mathrm{cod}j
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{dom}i \arrow[r, equal] \arrow[dd, "i"'] & \mathrm{dom}i \arrow[r, "L(ji)"] \arrow[dd, "ji"'] & E(ji) \arrow[dd, "R(ji)"] \\
{} \arrow[r, dashed, "E(1,j)"] & {} \arrow[ur, dashed, "1"] \arrow[r, dashed, "E(L(ji),1)"'] & {} \arrow[u, dashed, "\mu_{ji}"] \\
\mathrm{cod}i = \mathrm{dom}j \arrow[u, dashed, "s"'] \arrow[r, "j"'] & \mathrm{cod}j \arrow[r, equal] & {}
\end{tikzcd}$$

whose top component, by a monad triangle identity, is the canonical solution to the lifting problem displayed on the right.

There is an obvious forgetful double functor $\mathbb{C}\mathbf{oalg}(\mathbb{L}) \to \mathbb{S}\mathbf{q}(\mathcal{M})$ which factors through the left class of the underlying wfs of $(\mathbb{L}, \mathbb{R})$. A double category $\mathbb{A}\mathbf{lg}(\mathbb{R})$ is defined similarly with composition law, dual to (II.3.7), arising from the vertical composition in $\mathbb{L}$-$\mathbf{coalg}^{\boxslash}$.

Lemma II.3.8. *For any awfs* $(\mathbb{L}, \mathbb{R})$*, the functor* $\mathbb{R}\text{-}\mathbf{alg} \xrightarrow{\text{lift}} \mathbb{L}\text{-}\mathbf{coalg}^{\boxslash}$ *over* $\mathcal{M}^{\mathbf{2}}$ *preserves composition of algebras.*

Proof. The functor "lift" assigns an $\mathbb{R}$-algebra (II.3.5) the lifting function $\phi_{(f,s)}$ defined in (I.2.10) using the awfs $(\mathbb{L}, \mathbb{R})$ and the algebra structure s. Given composable $(f, s), (g, t) \in \mathbb{R}$-$\mathbf{alg}$, we must show that $\phi_{(g,t)} \bullet \phi_{(f,s)}$, defined by the formula (II.3.3), equals $\phi_{(gf,t\bullet s)}$. Using the dual to (II.3.7), the chosen solution $\phi_{(gf,t\bullet s)}((j,z), a, b)$ to a lifting problem (II.3.3) against an $\mathbb{L}$-coalgebra (j,z) is

$$\begin{tikzcd}
\mathrm{cod}j \arrow[r, "z"] \arrow[dr, "z"'] & Ej \arrow[r, "E(a,b)"] \arrow[dr, "\delta_j"] & E(gf) \arrow[r, "\delta_{gf}"] & EL(gf) \arrow[r, "E(1,E(f,1))"] & E(Lg\cdot f) \arrow[r, "E(1,t)"] & Ef \arrow[r, "s"] & \mathrm{dom}f \\
& Ez \arrow[r, "E(1,z)"'] & ELj \arrow[ur, "E(a,E(a,b))"'] & & & &
\end{tikzcd}$$

by naturality of δ and the comultiplication compatibility condition for the $\mathbb{L}$-coalgebra z. By definition $\phi_{(g,t)} \bullet \phi_{(f,s)}((j,z),a,b)$ is

$$\operatorname{cod} j \xrightarrow{z} Ej \xrightarrow{E(a,\phi_g)} Ef \xrightarrow{s} \operatorname{dom} f$$

where ϕ_g is shorthand for $\phi_{(g,t)}((j,z),fa,b) := t \cdot E(fa,b) \cdot z$. The lifting problem $(a,\phi_g) : j \Rightarrow f$ factors as

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "j"'] & \cdot \arrow[r, "a"] \arrow[d, "Lj"] & \cdot \arrow[r, equal] \arrow[d, "L(gf)"] & \cdot \arrow[r, equal] \arrow[d, "Lg\cdot f"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "z"'] & \cdot \arrow[r, "{E(a,b)}"'] & \cdot \arrow[r, "{E(f,1)}"'] & \cdot \arrow[r, "t"'] & \cdot
\end{tikzcd}$$

$E(a,\phi_g)$ is the image of this factorization under $E \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$; hence $\phi_{(gf,t\bullet s)} = \phi_{(g,t)} \bullet \phi_{(f,s)}$. □

These double categories capture the entire structure of the awfs $(\mathbb{L},\mathbb{R})$.

Lemma II.3.9 (Garner, I.2.24)**.** *Either of the double categories* $\mathbb{C}\mathbf{oalg}(\mathbb{L})$ *or* $\mathbb{A}\mathbf{lg}(\mathbb{R})$ *completely determines the awfs* $(\mathbb{L},\mathbb{R})$.

Proof. Given $\mathbb{A}\mathbf{lg}(\mathbb{R})$, the functorial factorization $\vec{E}$, and in particular the functor L and counit $\vec{\epsilon}$, can be read off from the unit $\vec{\eta}$ of the monad $\mathbb{R}$. The comultiplication δ can be defined in terms of the algebra structure assigned to the composite of the free algebras $(Rf,\mu_f) \circ (RLf,\mu_{Lf})$ as follows:

$$\delta_f := Ef \xrightarrow{E(L^2f,1)} E(Rf \cdot RLf) \xrightarrow{\mu_f \bullet \mu_{Lf}} ELf \tag{II.3.10}$$

See §I.2.5 for more details. □

The following theorem enables the theory of algebraic model categories.

Theorem II.3.11 (Garner [Gar09])**.** *Suppose* $\mathcal{M}$ *permits the small object argument* (*see I.2.28*) *and* $\mathcal{J}$ *is any small category of arrows of* $\mathcal{M}$. *Then* $\mathcal{M}$ *has an awfs* $(\mathbb{L},\mathbb{R})$ *such that there is*

(I.2.26) a functor $\mathcal{J} \to \mathbb{L}$-$\mathbf{coalg}$ *over* $\mathcal{M}^{\mathbf{2}}$ *universal among morphisms of awfs*

(I.2.27) an isomorphism of categories $\mathbb{R}$-$\mathbf{alg} \cong \mathcal{J}^{\boxslash}$ *over* $\mathcal{M}^{\mathbf{2}}$

Here $\mathcal{J}^{\boxslash}$ is the category defined above whose objects are arrows in $\mathcal{M}$ which lift coherently against elements of $\mathcal{J}$ together with specified solutions to all lifting problems, which morphisms of $\mathcal{J}^{\boxslash}$ preserve. We make frequent use of both universal properties. Indeed, the universal property

(I.2.26) of the *unit functor* $\mathcal{J} \to \mathbb{L}$-**coalg** is even stronger than originally stated. We first extended it in §I.6.4, Theorem II.3.24 below, and will do so again in Theorem II.5.16.

The isomorphism (I.2.27) factors as

$$\mathbb{R}\text{-}\mathbf{alg} \xrightarrow{\text{lift}} \mathbb{L}\text{-}\mathbf{coalg}^{\boxslash} \xrightarrow{\text{res}} \mathcal{J}^{\boxslash},$$

the restriction along the unit functor. Many applications of the second universal property stem from the following consequence of Lemma II.3.8: (I.2.27) preserves vertical composition, defining an isomorphism of double categories $\mathbb{A}\mathbf{lg}(\mathbb{R}) \cong \mathcal{J}^{\boxslash}$.

The essential application of these universal properties is

Corollary II.3.12 (I.3.6)**.** *An ordinary cofibrantly generated model structure, with generating trivial cofibrations $\mathcal{J}$ and generating cofibrations $\mathcal{I}$, has an algebraic model structure with the same generators if and only if the elements of $\mathcal{J}$ are $\mathcal{I}$-cellular, i.e., if and only if there is a functor $\mathcal{J} \to \mathbb{C}$-**coalg** over $\mathcal{M}^{\mathbf{2}}$.*

Proof. Given such an algebraic model structure $(\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$, the functor $\mathbb{C}_t$-**coalg** $\to$ $\mathbb{C}$-**coalg** arising from the comparison map defines $\mathbb{C}$-coalgebra structures for the generating trivial cofibrations. Conversely, given $\mathcal{J} \to \mathbb{C}$-**coalg**, where $(\mathbb{C}, \mathbb{F}_t)$ is the awfs generated by $\mathcal{I}$, the universal property of Garner's small object argument tell us that this functor factors through the unit $\mathcal{J} \to \mathbb{C}_t$-**coalg** along a functor induced by a morphism of awfs $(\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$, where $(\mathbb{C}_t, \mathbb{F})$ is the awfs generated by $\mathcal{J}$. On account of the isomorphisms $\mathbb{F}$-**alg** $\cong \mathcal{J}^{\boxslash}, \mathbb{F}_t$-**alg** $\cong \mathcal{I}^{\boxslash}$ the underlying wfs of the awfs $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ coincide with the wfs in the ordinary model structure generated by $\mathcal{J}$ and $\mathcal{I}$. So this defines an algebraic model structure compatible with the original model structure, as desired. □

Remark II.3.13. In fact, any cofibrantly generated ordinary model structure gives rise to an algebraic model structure even if the elements of $\mathcal{J}$ aren't $\mathcal{I}$-cellular, though at the cost of changing one of the generating sets. See I.3.7 and I.3.8.

II.3.3 Adjunctions of algebraic weak factorization systems

The appropriate sorts of morphisms between categories equipped with a weak factorization system preserve one class or the other; both is too much to expect. By convention, *lax* morphisms preserve the right class and *colax* morphisms preserve the left class. An easy argument shows that, for any adjunction $T \dashv S$, the left adjoint T is colax if and only if the right adjoint S is lax.

In the algebraic setting, we ask that the right and left adjoints preserve algebraic right and left maps, in the sense that they lift to functors between the categories of algebras and coalgebras, respectively. But simply asking for lifted functors is not strong enough. Because a single lifted functor doesn't capture the full data of an awfs, in the way that it does in the non-algebraic setting, a lift of the left adjoint does not guarantee the existence of, much less determine, a lift of the right adjoint and conversely.

Suppose $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ is an adjunction, T the left adjoint, and let $\mathcal{M}$ and $\mathcal{K}$ have awfs $(\mathbb{C}, \mathbb{F})$ and $(\mathbb{L}, \mathbb{R})$, respectively. The most succinct statement of the correct definition uses Lemma II.3.9.

Definition II.3.14. An *adjunction of awfs* $(\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ is determined by either

- a double functor $\mathbb{C}\mathbf{oalg}(\mathbb{C}) \to \mathbb{C}\mathbf{oalg}(\mathbb{L})$ lifting the left adjoint
- a double functor $\mathbb{A}\mathbf{lg}(\mathbb{R}) \to \mathbb{A}\mathbf{lg}(\mathbb{F})$ lifting the right adjoint

On its own, the lifted double functor $\mathbb{C}\mathbf{oalg}(\mathbb{C}) \to \mathbb{C}\mathbf{oalg}(\mathbb{L})$ is called a *colax morphism of awfs* $(\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ and the lifted double functor $\mathbb{A}\mathbf{lg}(\mathbb{R}) \to \mathbb{A}\mathbf{lg}(\mathbb{F})$ is called a *lax morphism of awfs* $(\mathbb{L}, \mathbb{R}) \to (\mathbb{C}, \mathbb{F})$. What is far from obvious with this definition is that in the presence of an adjunction $T \dashv S$, lax morphisms of awfs lifting S determine colax morphisms of awfs lifting T and conversely. To prove this, we will work towards an alternate, equivalent definition.

Passing to the category of vertical arrows, the double functors in particular determine functors $\mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg**, $\mathbb{R}$-**alg** $\to$ $\mathbb{F}$-**alg** lifting T and S. The following extension of Lemma II.3.9 is essentially a tautology.

Lemma II.3.15. *A lifted double functor* $\mathbb{A}\mathbf{lg}(\mathbb{R}) \to \mathbb{A}\mathbf{lg}(\mathbb{F})$ *is precisely a lifted functor* $\mathbb{R}$-**alg** $\to$ $\mathbb{F}$-**alg** *that preserves the canonical composition of algebras. Dually, a lifted double functor* $\mathbb{C}\mathbf{oalg}(\mathbb{C}) \to \mathbb{C}\mathbf{oalg}(\mathbb{L})$ *is precisely a composition-preserving lifted functor* $\mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg** *of coalgebras.*

Proof. A double functor $\mathbb{A}\mathbf{lg}(\mathbb{R}) \to \mathbb{A}\mathbf{lg}(\mathbb{F})$ lifting S is determined by a commuting diagram of functors

$$\begin{tikzcd}
\mathbb{R}\text{-}\mathbf{alg} \times_{\mathcal{K}} \mathbb{R}\text{-}\mathbf{alg} \arrow[r, "\circ"] \arrow[d, "\tilde{S}\times_S\tilde{S}"'] & \mathbb{R}\text{-}\mathbf{alg} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] \arrow[d, "\tilde{S}"'] & \mathcal{K} \arrow[l, "\mathrm{id}" description] \arrow[d, "S"] \\
\mathbb{F}\text{-}\mathbf{alg} \times_{\mathcal{M}} \mathbb{F}\text{-}\mathbf{alg} \arrow[r, "\circ"] & \mathbb{F}\text{-}\mathbf{alg} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{M} \arrow[l, "\mathrm{id}" description]
\end{tikzcd}$$

□

A lifted functor, $\mathbb{R}$-**alg** → $\mathbb{F}$-**alg** preserves composition of certain free algebras if and only if the characterizing natural transformation satisfies a pentagon involving the comultiplication. This leads to an equivalent definition of lax and colax morphisms of awfs.

Lemma (I.6.9)**.** *Lax morphisms of awfs* $S\colon (\mathbb{L},\mathbb{R}) \to (\mathbb{C},\mathbb{F})$ *correspond bijectively to natural transformations* $\rho\colon QS \Rightarrow SE$ *satisfying diagrams* (I.6.5)*. Dually, colax morphisms of awfs* $T\colon (\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ *correspond to natural transformations* $\lambda\colon TQ \Rightarrow ET$ *satisfying diagrams* (I.6.7).

Proof. Use Lemma II.3.15 and a diagram chase; see I.6.9 for more details. □

Given $\rho\colon QS \Rightarrow SE$, its mate with respect to

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "Q"] \arrow[d, shift right=2, "T^{\mathbf{2}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{M} \arrow[d, shift right=2, "T"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{K}^{\mathbf{2}} \arrow[u, shift right=2, "S^{\mathbf{2}}"'] \arrow[r, "E"'] & \mathcal{K} \arrow[u, shift right=2, "S"']
\end{tikzcd} \tag{II.3.16}$$

is a natural transformation $\lambda\colon TQ \Rightarrow ET$. Using the previous two lemmas, we can restate Definition II.3.14 in such a way that it is apparent that the two defining conditions are equivalent.

Definition (I.6.10)**.** An *adjunction of awfs* $(T,S,\lambda,\rho)\colon (\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ consists of an adjoint pair of functors together with mates λ and ρ, as above, such that (S,ρ) is a lax morphism of awfs and (T,λ) is a colax morphism of awfs.

Corollary II.3.17. *The conditions defining an adjunction of awfs are equivalent.*

Proof. The mates correspondence associates each diagram of (I.6.5) to a corresponding diagram of (I.6.7). Hence, by Lemma II.2.10 and its dual, a natural arrow ρ gives rise to a lax morphism of awfs lifting S if and only if its mate λ gives rise to a colax morphism of awfs lifting T. □

Example (I.6.14). The comparison map in an algebraic model structure is an adjunction of awfs, the adjunction in question being the identity. Indeed, the morphisms of awfs defined in [Gar09] are exactly adjunctions of awfs lifting identity adjunctions: in such cases, the diagrams (I.6.5) and (I.6.7) agree and coincide with those diagrams defining morphisms of awfs.

Lemma II.3.15 can also be used to characterize the adjunctions of awfs whose domain is cofibrantly generated. The non-trivial direction of the following theorem was first suggested by Mike Shulman; his proof appears as I.6.17. Below, we give a streamlined proof, whose essential details are the same but whose argument is more conceptual.

Theorem II.3.18. *Suppose* $\mathcal{M}$ *has an awfs* $(\mathbb{C},\mathbb{F})$ *generated by* $\mathcal{J}$ *and* $\mathcal{K}$ *has an awfs* $(\mathbb{L},\mathbb{R})$*, not necessarily cofibrantly generated. An adjunction* $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ *is an adjunction of awfs if and only if there is a lift*

$$\begin{tikzcd}
\mathcal{J} \arrow[r, dashed] \arrow[d] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[d] \\
\mathcal{M}^{\mathbf{2}} \arrow[r, "T^{\mathbf{2}}"] & \mathcal{K}^{\mathbf{2}}
\end{tikzcd} \qquad \text{(II.3.19)}$$

in which case the adjunction of awfs $(T,S,\lambda,\rho) : (\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ *is canonically determined.*

In other words, there is an adjunction of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ if an only if the image of the generators under $T^{\mathbf{2}}$ is *cellular*. Write $T^{\mathbf{2}}\mathcal{J}$ for the category $\mathcal{J}$ over $\mathcal{K}^{\mathbf{2}}$; with this notation, the lifted functor of (II.3.19) is precisely a functor $T^{\mathbf{2}}\mathcal{J} \to \mathbb{L}$-**coalg** over $\mathcal{K}^{\mathbf{2}}$.

Proof of Theorem II.3.18. We employ Lemma II.3.15. A categorical expression for the familiar fact that adjunctions interact nicely with lifting problems is that

$$\begin{tikzcd}
(T^{\mathbf{2}}\mathcal{J})^{\boxslash} \arrow[r, "\text{adj}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{J}^{\boxslash} \arrow[d] \\
\mathcal{K}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"'] & \mathcal{M}^{\mathbf{2}}
\end{tikzcd} \qquad \text{(II.3.20)}$$

is a pullback in **CAT**. The functor adj: $(T^{\mathbf{2}}\mathcal{J})^{\boxslash} \to \mathcal{J}^{\boxslash}$ sends an arrow f with lifting function ϕ_f to the arrow Sf with lifting function $\phi^{\sharp}_f$, whose chosen solutions are adjunct to the solutions chosen by ϕ_f to the transposed lifting problem. Define $\mathbb{R}$-**alg** $\to \mathbb{F}$-**alg** $\cong \mathcal{J}^{\boxslash}$ to be the composite

$$\mathbb{R}\text{-}\mathbf{alg} \xrightarrow{\text{lift}} (\mathbb{L}\text{-}\mathbf{coalg})^{\boxslash} \xrightarrow{\text{res}} (T^{\mathbf{2}}\mathcal{J})^{\boxslash} \xrightarrow{\text{adj}} \mathcal{J}^{\boxslash} \qquad \text{(II.3.21)}$$

where the restriction is along the functor $T^{\mathbf{2}}\mathcal{J} \to \mathbb{L}$-**coalg**. Each functor preserves composition: the first by Lemma II.3.8, the second trivially, and the third by naturality of adjunctions—this last diagram chase is given in the proof of Theorem I.6.15. □

Remark II.3.22. In fact, all of the functors of (II.3.21) are double functors and the pullback (II.3.20) defines a pullback in **DblCAT**.

Remark II.3.23. The definition of (II.3.21) works for any adjunction $T \dashv S : \mathcal{M}^{\mathbf{2}} \rightleftarrows \mathcal{K}^{\mathbf{2}}$, not only for those defined pointwise by adjunctions $\mathcal{M} \rightleftarrows \mathcal{K}$. However in the general case, the functor $\mathrm{adj}\colon (T\mathcal{J})^{\boxslash} \to \mathcal{J}^{\boxslash}$ won't preserve vertical composition; indeed S and hence adj won't necessarily preserve composability of vertical arrows. This accounts for nearly all the technical difficulties in §II.5 and §II.6.

Conversely, there is a unique adjunction of awfs arising from a specified cellular structure for the generators $T^{\mathbf{2}}\mathcal{J}$. This is an immediate consequence of the following extension of the universal property of Theorem II.3.11.

Theorem II.3.24 (I.6.22)**.** *Garner's small object argument reflects small categories of arrows over categories permitting the small object argument along the forgetful functor*

$$\begin{tikzcd} \mathbf{AWFS}_{\mathrm{ladj}} \arrow[r] & \mathbf{CAT}/(-)^{\mathbf{2}}_{\mathrm{ladj}} \arrow[l, dashed, bend right] \end{tikzcd}. \tag{II.3.25}$$

In particular, if $(\mathbb{C}, \mathbb{F})$ *is generated by* $\mathcal{J}$*, the canonical functor* $\mathcal{J} \to \mathbb{C}$-**coalg** *is universal among adjunctions of awfs.*

$\mathbf{AWFS}_{\mathrm{ladj}}$ is the category of awfs and adjunctions of awfs, and $\mathbf{CAT}/(-)^{\mathbf{2}}_{\mathrm{ladj}}$ is the category of categories over an arrow category, where morphisms are left adjoints between the bases together with a specified lift to the fibers.

II.3.4 Algebraic Quillen adjunctions

An *algebraic Quillen adjunction* $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ between categories equipped with algebraic model structures is an adjunction of awfs with respect to the (trivial cofibration, fibration) and (cofibration, trivial fibration) awfs that satisfies an additional compatibility condition.

Definition (I.3.11)**.** Suppose $\mathcal{M}$ has an algebraic model structure $\xi^{\mathcal{M}}\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ and $\mathcal{K}$ has an algebraic model structure $\xi^{\mathcal{K}}\colon (\mathbb{L}_t, \mathbb{R}) \to (\mathbb{L}, \mathbb{R}_t)$. An *algebraic Quillen adjunction* is an adjunction $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ together with adjunctions of awfs

$$\begin{tikzcd}
(\mathbb{C}_t, \mathbb{F}) \arrow[r, "{(T,S)}"] \arrow[d, "\xi^{\mathcal{M}}"'] \arrow[dr, "{(T,S)}" description] & (\mathbb{L}_t, \mathbb{R}) \arrow[d, "\xi^{\mathcal{K}}"] \\
(\mathbb{C}, \mathbb{F}_t) \arrow[r, "{(T,S)}"'] & (\mathbb{L}, \mathbb{R}_t)
\end{tikzcd}$$

such that both triangles commute.

In particular, an algebraic Quillen adjunction consists of commuting lifted double functors

$$\begin{tikzcd}
\mathbb{A}\mathbf{lg}(\mathbb{R}_t) \arrow[r, "S^{\mathbf{2}}"] \arrow[d, "\xi^{\mathcal{K}}"'] & \mathbb{A}\mathbf{lg}(\mathbb{F}_t) \arrow[d, "\xi^{\mathcal{M}}"] & \text{and} & \mathbb{C}\mathbf{oalg}(\mathbb{C}_t) \arrow[r, "T^{\mathbf{2}}"] \arrow[d, "\xi^{\mathcal{M}}"'] & \mathbb{C}\mathbf{oalg}(\mathbb{L}_t) \arrow[d, "\xi^{\mathcal{K}}"] \\
\mathbb{A}\mathbf{lg}(\mathbb{R}) \arrow[r, "S^{\mathbf{2}}"'] & \mathbb{A}\mathbf{lg}(\mathbb{F}) & & \mathbb{C}\mathbf{oalg}(\mathbb{C}) \arrow[r, "T^{\mathbf{2}}"'] & \mathbb{C}\mathbf{oalg}(\mathbb{L})
\end{tikzcd} \tag{II.3.26}$$

Because all of these double functors are lifts, this compatibility condition is equivalent to (I.3.12), which asks that the ordinary lifted functors on algebraic (trivial) cofibrations and fibrations commute. Taking either perspective, functors on the left-hand or right-hand sides determine those on the other. In particular, it suffices to check commutativity of one of these two diagrams. For example

Theorem II.3.27. *Suppose $\mathcal{M}$ has an algebraic model structure $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$. Then the category $\mathcal{M}_*$ of pointed objects in $\mathcal{M}$ has an algebraic model structure such that the disjoint basepoint–forgetful adjunction $(-)_+ \dashv U\colon \mathcal{M} \rightleftarrows \mathcal{M}_*$ is an algebraic Quillen adjunction.*

Proof. The category $\mathcal{M}_*$ is isomorphic to the slice category $*/\mathcal{M}$, where $*$ denotes the terminal object. An arrow or a commutative square in $\mathcal{M}_*$ is determined by the arrow or square in the image of the forgetful functor together with the basepoint of its initial object; the other basepoints are defined by composition. This says that

$$\begin{tikzcd}
(\mathcal{M}_*)^{\mathbf{2}} \arrow[r, "U^{\mathbf{2}}"] \arrow[d, "\mathrm{dom}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}^{\mathbf{2}} \arrow[d, "\mathrm{dom}"] \\
\mathcal{M}_* \arrow[r, "U"'] & \mathcal{M}
\end{tikzcd}$$

is a pullback. We will see that this implies that the entire structure of the algebraic model structure on $\mathcal{M}$ can be lifted along U to define an algebraic model structure on $\mathcal{M}_*$.

The comonad $\mathbb{C}$ is domain-preserving, so its constituent functor and natural transformations can be pulled back to $(\mathcal{M}_*)^{\mathbf{2}}$; this works for the 2-cells because limits in **CAT** are also 2-limits [Kel89].

$$\begin{tikzcd}
(\mathcal{M}_*)^{\mathbf{2}} \arrow[rr, "U^{\mathbf{2}}"] \arrow[dr, dashed, "C_*"] \arrow[ddr, bend right, "\mathrm{dom}"'] & & \mathcal{M}^{\mathbf{2}} \arrow[dr, "C"] \arrow[ddr, "\mathrm{dom}"'] & \\
& (\mathcal{M}_*)^{\mathbf{2}} \arrow[rr, "U^{\mathbf{2}}"] \arrow[d, "\mathrm{dom}"] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathcal{M}^{\mathbf{2}} \arrow[d, "\mathrm{dom}"] \\
& \mathcal{M}_* \arrow[rr, "U"'] & & \mathcal{M}
\end{tikzcd}$$

The multiplication for the monads also lifts to $\mathcal{M}_*$: e.g., the basepoint of FRf is the image of the basepoint of $\mathrm{dom} f$, which maps to the basepoint of Rf, which proves that μ_f preserves basepoints. For similar reasons, the comparison map lifts along U. This defines an algebraic model structure we denote $\xi_* \colon ((\mathbb{C}_t)_*, \mathbb{F}_*) \to (\mathbb{C}_*, (\mathbb{F}_t)_*)$ on $\mathcal{M}_*$.

Algebra structures for fibrations in $\mathcal{M}_*$ are precisely algebra structures for the underlying fibrations in $\mathcal{M}$: the basepoint of Rf is in the image of the basepoint of $\mathrm{dom} f$ and hence maps via the algebra structure map back to the basepoint of $\mathrm{dom} f$. It follows that the left-hand diagram

$$\begin{tikzcd}
\mathbb{A}\mathbf{lg}(\mathbb{F}_*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}\mathbf{lg}(\mathbb{F}) \arrow[d] & & \mathbb{A}\mathbf{lg}((\mathbb{F}_t)_*) \arrow[r] \arrow[d, "\xi_*"'] & \mathbb{A}\mathbf{lg}(\mathbb{F}_t) \arrow[d, "\xi"] \\
\mathbb{S}\mathbf{q}(\mathcal{M}_*) \arrow[r, "U"'] & \mathbb{S}\mathbf{q}(\mathcal{M}) & & \mathbb{A}\mathbf{lg}(\mathbb{F}_*) \arrow[r] & \mathbb{A}\mathbf{lg}(\mathbb{F})
\end{tikzcd}$$

is a pullback in **DblCAT**. By this fact and the definition of ξ_*, the right-hand square commutes, establishing the algebraic Quillen adjunction. □

We use the fact that it suffices to verify the compatibility condition (II.3.26) on the level of the categories of algebraic (trivial) cofibrations together with Theorems II.3.18 and II.3.24 to prove

Theorem II.3.28. *Suppose that $\mathcal{M}$ and $\mathcal{K}$ have algebraic model structures, as above, such that the algebraic model structure on $\mathcal{M}$ is generated by categories $\mathcal{J}$ and $\mathcal{I}$. Then $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ is* *an algebraic Quillen adjunction if and only if there exist lifts*

$$\begin{tikzcd}
\mathcal{J} \arrow[rr, dashed] \arrow[dd] & & \mathbb{L}_t\textbf{-coalg} \arrow[dd] & \\
& \mathcal{I} \arrow[rr, dashed] \arrow[dl] & & \mathbb{L}\textbf{-coalg} \arrow[dl] \\
\mathcal{M}^2 \arrow[rr, "T^2"'] & & \mathcal{K}^2 &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
\mathcal{J} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{L}_t\textbf{-coalg} \arrow[dd] \arrow[dr, "\xi^{\mathcal{K}}"] & \\
& \mathbb{C}\textbf{-coalg} \arrow[rr] \arrow[dl] & & \mathbb{L}\textbf{-coalg} \arrow[dl] \\
\mathcal{M}^2 \arrow[rr, "T^2"'] & & \mathcal{K}^2 &
\end{tikzcd}$$

commutes. In this case, the algebraic Quillen adjunction is canonically determined.

The first condition says that the images of $\mathcal{J}$ and $\mathcal{I}$ must be cellular for $\mathbb{L}_t$ and $\mathbb{L}$ respectively. The second condition says that the two canonical ways of assigning $\mathbb{L}$-coalgebra structures to $\mathcal{J}$—one using $\xi^{\mathcal{M}}$ and one lifted functor and the other using $\xi^{\mathcal{K}}$ and the other lifted functor—must agree.

Proof of Theorem II.3.28. By Theorem II.3.18, the lifts of T^2 give rise to adjunctions of awfs

$$(T,S)\colon (\mathbb{C}_t,\mathbb{F}) \to (\mathbb{L}_t,\mathbb{R}) \quad (T,S)\colon (\mathbb{C},\mathbb{F}_t) \to (\mathbb{L},\mathbb{R}_t).$$

These combine to specify an algebraic Quillen adjunction if and only if the lifted functors

$$\begin{tikzcd}
\mathbb{C}_t\textbf{-coalg} \arrow[rr] \arrow[dr, "\xi^{\mathcal{M}}"] \arrow[dd] & & \mathbb{L}_t\textbf{-coalg} \arrow[dd] \arrow[dr, "\xi^{\mathcal{K}}"] & \\
& \mathbb{C}\textbf{-coalg} \arrow[rr] \arrow[dl] & & \mathbb{L}\textbf{-coalg} \arrow[dl] \\
\mathcal{M}^2 \arrow[rr, "T^2"'] & & \mathcal{K}^2 &
\end{tikzcd} \tag{II.3.29}$$

commute. The reflection (II.3.25) defines the functor $\mathbb{C}_t$-**coalg** $\to$ $\mathbb{L}_t$-**coalg** by factoring $\mathcal{J} \to \mathbb{L}_t$-**coalg** through $\mathbb{C}_t$-**coalg**. By the universal property of $\mathcal{J} \to \mathbb{C}_t$-**coalg** in Theorem II.3.24, (II.3.29) commutes if and only if the restriction to $\mathcal{J}$ does, which was a hypothesis. □

In particular, the conditions of Theorem II.3.28 are satisfied if the algebraic model structure on $\mathcal{K}$ is constructed by lifting the algebraic model structure on $\mathcal{M}$ along an adjunction

Theorem (I.3.10, I.3.13)**.** *Suppose $\mathcal{M}$ has an algebraic model structure generated by $\mathcal{J}$ and $\mathcal{I}$, $T \dashv S : \mathcal{M} \rightleftarrows \mathcal{K}$ is an adjunction, and $\mathcal{K}$ permits the small object argument. If*

(††) S maps the $T^2\mathcal{J}$-cellular arrows into weak equivalences

then $T^{\mathbf{2}}\mathcal{J}$ and $T^{\mathbf{2}}\mathcal{I}$ generate an algebraic model structure on $\mathcal{K}$ such that $T \dashv S$ is canonically an algebraic Quillen adjunction.

This theorem gives an important class of algebraic Quillen adjunctions, including the geometric realization–total singular complex adjunction between simplicial sets and spaces, the adjunction between G-spaces and space-valued presheaves on the orbit category for a group G, the adjunctions establishing the projective model structure, as well as many other classical examples.

II.4 Algebraic Quillen two-variable adjunctions

A two-variable adjunction such as $(\otimes, \mathrm{hom}_\ell, \mathrm{hom}_r)\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ for a closed monoidal category, or $(\odot, \{-,-\}, \mathrm{hom})\colon \mathcal{V} \times \mathcal{M} \to \mathcal{M}$ for a tensored and cotensored $\mathcal{V}$-enriched category, or $(\otimes, \mathrm{hom}_\ell, \mathrm{hom}_r)\colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$ in general consists of three bifunctors

$$\mathcal{K} \times \mathcal{M} \xrightarrow{-\otimes-} \mathcal{N} \qquad \mathcal{K}^{\mathrm{op}} \times \mathcal{N} \xrightarrow{\mathrm{hom}_\ell(-,-)} \mathcal{M} \qquad \mathcal{M}^{\mathrm{op}} \times \mathcal{N} \xrightarrow{\mathrm{hom}_r(-,-)} \mathcal{K} \tag{II.4.1}$$

together with hom-set isomorphisms

$$\mathcal{N}(k \otimes m, n) \cong \mathcal{M}(m, \mathrm{hom}_\ell(k, n)) \cong \mathcal{K}(k, \mathrm{hom}_r(m, n)) \tag{II.4.2}$$

natural in all three variables. In particular, these form *parameterized adjunctions*: fixing any one variable gives rise to families of adjunctions in the ordinary sense. When $\mathcal{K}$ and $\mathcal{M}$ have pullbacks and $\mathcal{N}$ has pushouts, there is an induced two-variable adjunction

$$\mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \xrightarrow{-\hat{\otimes}-} \mathcal{N}^{\mathbf{2}} \qquad (\mathcal{K}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \xrightarrow{\hat{\mathrm{hom}}_\ell(-,-)} \mathcal{M}^{\mathbf{2}} \qquad (\mathcal{M}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \xrightarrow{\hat{\mathrm{hom}}_r(-,-)} \mathcal{K}^{\mathbf{2}} \tag{II.4.3}$$

defined in (II.5.2) below. The bifunctor $-\hat{\otimes}-$ is sometimes called the *pushout-product*; we call $\hat{\mathrm{hom}}_\ell$ and $\hat{\mathrm{hom}}_r$ *pullback-homs*.

If $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$ are model categories, the two-variable adjunction $(\otimes, \mathrm{hom}_\ell, \mathrm{hom}_r)$ is *Quillen* if the following equivalent conditions are satisfied [Hov99]:

(a) if $i \in \mathcal{K}^{\mathbf{2}}$ and $j \in \mathcal{M}^{\mathbf{2}}$ are cofibrations then $i \hat{\otimes} j \in \mathcal{N}^{\mathbf{2}}$ is a cofibration that is trivial if either i or j is, in which case we say that $\otimes$ is a *left Quillen bifunctor*

(b) if $i \in \mathcal{K}^{\mathbf{2}}$ is a cofibration and $f \in \mathcal{N}^{\mathbf{2}}$ is a fibration then $\widehat{\hom}_\ell(i, f) \in \mathcal{M}^{\mathbf{2}}$ is a fibration that is trivial if either i or f is, in which case we say that $\hom_\ell$ is a *right Quillen bifunctor*

(c) if $j \in \mathcal{M}^{\mathbf{2}}$ is a cofibration and $f \in \mathcal{N}^{\mathbf{2}}$ is a fibration then $\widehat{\hom}_r(j, f) \in \mathcal{K}^{\mathbf{2}}$ is a fibration that is trivial if either j or f is, in which case we say that $\hom_r$ is a *right Quillen bifunctor*

The equivalence of the three conditions rests on the interplay between adjunctions and lifting problems. This should be thought of as a strengthening of the usual lifting axiom. For instance, the corresponding axiom (c) for simplicial model categories implies that any two solutions to a lifting problem under a cofibrant object are homotopic relative to that object [GJ99, §II.3].

In analogy with §II.3, we say a two-variable Quillen adjunction $(\otimes, \hom_\ell, \hom_r)$ is *algebraic* if the two-variable adjunction $(\hat{\otimes}, \widehat{\hom}_\ell, \widehat{\hom}_r)$ lifts to functors of algebraic (trivial) cofibrations and fibrations as appropriate. Importantly, we can also capture the symmetry of the classical setting—the equivalence of conditions (a), (b), and (c)—by requiring that the parameterized mates of the natural transformation characterizing the lift of one of the functors $(\hat{\otimes}, \widehat{\hom}_\ell, \widehat{\hom}_r)$ characterizes the others. A priori, this would be rather difficult to check in practice because each natural transformation must satisfy a number of diagrams, but a theorem below gives a particularly simple necessary and sufficient condition for all of this structure to exist in the case that the model structures on $\mathcal{K}$ and $\mathcal{M}$ are cofibrantly generated.

The components of an algebraic Quillen two-variable adjunction are three *two-variable adjunctions of awfs*. The precise definition is somewhat technical and will be given in §II.6 below, but the main idea is simple enough to state. Suppose $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$ are equipped with awfs $(\mathbb{C}', \mathbb{F}')$, $(\mathbb{C}, \mathbb{F})$, and $(\mathbb{L}, \mathbb{R})$ respectively.

Definition II.4.4. A *two-variable adjunction of awfs* $\otimes \colon (\mathbb{C}', \mathbb{F}') \times (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ is a two-variable adjunction $(\otimes, \hom_\ell, \hom_r) \colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$ equipped with lifted functors

$$\begin{aligned} \hat{\otimes} \colon & \mathbb{C}'\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg} \\ \widehat{\hom}_\ell \colon & \mathbb{C}'\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg} \\ \widehat{\hom}_r \colon & \mathbb{C}\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}\textbf{-alg} \to \mathbb{F}'\textbf{-alg} \end{aligned}$$

such that their characterizing natural transformations, described in §II.6 below, are parameterized mates.

Now suppose $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$ have algebraic model structures

$$\xi^{\mathcal{K}} \colon (\mathbb{C}'_t, \mathbb{F}') \to (\mathbb{C}', \mathbb{F}'_t), \quad \xi^{\mathcal{M}} \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t), \quad \text{and} \quad \xi^{\mathcal{N}} \colon (\mathbb{L}_t, \mathbb{R}) \to (\mathbb{L}, \mathbb{R}_t).$$

Definition II.4.5. An *algebraic Quillen two-variable adjunction* $(\otimes, \hom_\ell, \hom_r) \colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$ consists of specified two-variable adjunctions of awfs

$$\begin{aligned} \otimes &\colon (\mathbb{C}', \mathbb{F}'_t) \times (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{L}, \mathbb{R}_t) \\ \otimes &\colon (\mathbb{C}'_t, \mathbb{F}') \times (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{L}_t, \mathbb{R}) \\ \otimes &\colon (\mathbb{C}', \mathbb{F}'_t) \times (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{L}_t, \mathbb{R}) \end{aligned}$$

The algebraic Quillen two-variable adjunction is *maximally coherent* if the lifted functors

$$\begin{tikzcd}
& \mathbb{C}'_t\textbf{-coalg} \times \mathbb{C}_t\textbf{-coalg} \arrow[dl, "1\times\xi^{\mathcal{M}}"'] \arrow[ddr, "\xi^{\mathcal{K}}\times 1"] \arrow[drr, dashed] & & & \\
\mathbb{C}'_t\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \arrow[rrr] \arrow[ddr, "\xi^{\mathcal{K}}\times 1"'] & & & \mathbb{L}_t\textbf{-coalg} \arrow[ddr, "\xi^{\mathcal{N}}"] & \\
& & \mathbb{C}'\textbf{-coalg} \times \mathbb{C}_t\textbf{-coalg} \arrow[ur] \arrow[dl, "1\times\xi^{\mathcal{M}}"] & & \\
& \mathbb{C}'\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \arrow[rrr] & & & \mathbb{L}\textbf{-coalg}
\end{tikzcd}$$

(II.4.6)

commute.

The condition (II.4.6) asks that three squares relating each pair of two-variable adjunctions of awfs commute. The square comparing the last two, together with Lemma II.6.11 below, defines a fourth two-variable adjunction of awfs $\otimes \colon (\mathbb{C}'_t, \mathbb{F}') \times (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{L}_t, \mathbb{R})$; compare with I.3.11. By the calculus of parameterized mates, the coherence conditions (II.4.6) are equivalent to coherence conditions for the lifts of $\hat{\hom}_\ell$ or $\hat{\hom}_r$ displayed in (II.6.13) below.

Evaluating a maximally coherent algebraic Quillen two-variable adjunction at an algebraic cofibrant object or an algebraic fibrant object gives rise to an ordinary algebraic Quillen adjunction.

Lemma II.4.7. *If* $(\otimes, \hom_\ell, \hom_r) \colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$ *is a maximally coherent algebraic Quillen two-variable adjunction and A is an algebraic cofibrant object of* $\mathcal{K}$*, then* $A \otimes - \dashv \hom_\ell(A, -) \colon \mathcal{M} \rightleftarrows \mathcal{N}$ *is canonically an algebraic Quillen adjunction. Dually, if X is an algebraic fibrant object of* $\mathcal{N}$*, then* $\hom_\ell(-, X) \dashv \hom_r(-, X) \colon \mathcal{K} \rightleftarrows \mathcal{M}^{\mathrm{op}}$ *is canonically an algebraic Quillen adjunction.*

Proof. Using the notation of Definition II.4.5, an algebraic cofibrant object A is a $\mathbb{C}'$-coalgebra $i\colon \emptyset \to A$. The adjunction $i\hat{\otimes}- \dashv \widehat{\hom}_\ell(i,-)$ coincides with the pointwise-defined adjunction

$$A \otimes - \dashv \hom_\ell(A,-)\colon \mathcal{M}^{\mathbf{2}} \rightleftarrows \mathcal{N}^{\mathbf{2}}.$$

Hence, upon evaluating at $i \in \mathbb{C}'$-**coalg**, the front rectangle of (II.4.6) exhibits the desired algebraic Quillen adjunction. □

An immediate corollary to two hard theorems, whose proofs are deferred to §II.5 and §II.6, gives a simple criterion characterizing algebraic Quillen two-variable adjunctions in the case where the algebraic model structures on $\mathcal{K}$ and $\mathcal{M}$ are cofibrantly generated. The first theorem constructs a two-variable adjunction of awfs assuming the pushout-product of the generators is cellular.

Theorem II.4.8 (Cellularity Theorem)**.** *Suppose* $\mathcal{I}$ *generates* $(\mathbb{C}',\mathbb{F}')$ *on* $\mathcal{K}$ *and* $\mathcal{J}$ *generates* $(\mathbb{C},\mathbb{F})$ *on* $\mathcal{M}$ *and* $\mathcal{N}$ *has an awfs* $(\mathbb{L},\mathbb{R})$*. Then* $(\otimes, \hom_\ell, \hom_r)$ *gives rise to a two-variable adjunction of awfs if and only if* $\mathcal{I}\hat{\otimes}\mathcal{J}$ *is cellular, that is, if and only if there is a lift*

$$\begin{tikzcd}
\mathcal{I} \times \mathcal{J} \arrow[r, dashed] \arrow[d] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[d] \\
\mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \arrow[r, "-\hat{\otimes}-"] & \mathcal{N}^{\mathbf{2}}
\end{tikzcd}$$

Conversely, a cellular structure for $\mathcal{I}\hat{\otimes}\mathcal{J}$ determines a unique adjunction of awfs.

Theorem II.4.9 (Uniqueness Theorem)**.** *There can be at most one two-variable adjunction of awfs* $(\mathbb{C}',\mathbb{F}')\times(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ *whose lifted left adjoint restricts along the unit functors to a given lifted functor* $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg***.*

Corollary II.4.10. *Suppose the algebraic model structures on* $\mathcal{K}$ *and* $\mathcal{M}$ *are cofibrantly generated, with generating cofibrations* $\mathcal{J}'$*,* $\mathcal{I}'$*,* $\mathcal{J}$*, and* $\mathcal{I}$*. Then* $(\otimes, \hom_\ell, \hom_r)$ *is an algebraic Quillen two-variable adjunction if and only if the category* $\mathcal{I}' \times \mathcal{I}$ *is* $\mathbb{L}$*-cellular and the categories* $\mathcal{J}' \times \mathcal{I}$

and $\mathcal{I}' \times \mathcal{J}$ are $\mathbb{L}_t$-cellular, and is maximally coherent if and only if

$$\begin{tikzcd}
 & \mathcal{I}' \times \mathcal{J} \arrow[rd] \arrow[ldd] & \\
\mathcal{J}' \times \mathcal{I} \arrow[rr, crossing over] \arrow[d] & & \mathbb{L}_t\textbf{-coalg} \arrow[d] \\
\mathbb{C}'\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \arrow[rr] & & \mathbb{L}\textbf{-coalg}
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
\mathcal{J}' \times \mathcal{J} \arrow[r] \arrow[d] & \mathbb{C}'_t\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \arrow[d] \\
\mathbb{C}'\textbf{-coalg} \times \mathbb{C}_t\textbf{-coalg} \arrow[r] & \mathbb{L}_t\textbf{-coalg}
\end{tikzcd}$$

commute.

At this point the definition of a monoidal algebraic model structure is rather obvious. A reader who is willing to take these results on faith and uninterested in the categorical work necessary to make these definitions precise might wish to skip directly to §II.7, perhaps detouring to absorb the composition criterion of Theorems II.5.12 and II.6.4.

In §II.5, we study single variable adjunctions on arrow categories arising from two-variable adjunctions, extending the definitions and theorems of §II.3.3 to include adjunctions of this particular sort. To a great extent, the two-variable case reduces to the single variable one. In particular, we prove Theorem II.4.9 at the end of §II.5. In §II.6, we focus on difficulties particular to bifunctors, giving an explicit description of the parameterized mates characterizing the lifted functors in a two-variable adjunction of awfs, and proving Theorem II.4.8.

Notation II.4.11. In §II.5 and §II.6, we only concern ourselves with the interactions between a two-variable adjunction and categories equipped with a single awfs each, for which we adopt the following notation. We write $(\mathbb{C}', \mathbb{F}')$, $(\mathbb{C}, \mathbb{F})$, and $(\mathbb{L}, \mathbb{R})$ for the awfs on $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$, respectively, with functors $Q' \colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, $Q \colon \mathcal{M}^{\mathbf{2}} \to \mathcal{M}$, and $E \colon \mathcal{N}^{\mathbf{2}} \to \mathcal{N}$ accompanying the functorial factorizations. We write $i \colon A \to B$, $j \colon K \to L$, and $f \colon X \to Y$ for generic elements of $\mathcal{K}^{\mathbf{2}}$, $\mathcal{M}^{\mathbf{2}}$, and $\mathcal{N}^{\mathbf{2}}$ respectively. Whenever we assume further that i has the structure of a $\mathbb{C}'$-coalgebra, j has the structure of a $\mathbb{C}$-coalgebra, or f has the structure of an $\mathbb{R}$-algebra, we always make this explicit.

II.5 Adjunctions of algebraic weak factorization systems, revisited

We now begin to get our hands dirty. In §II.5.1, we extend the definition of adjunction of awfs to include adjunctions $\mathcal{M}^{\mathbf{2}} \rightleftarrows \mathcal{N}^{\mathbf{2}}$ obtained by evaluating the two-variable adjunction (II.4.3) at fixed $i \in \mathcal{K}^{\mathbf{2}}$. In §II.5.2, we give a cellularity criterion analogous to Theorem II.3.18 that characterizes these sorts of adjunctions of awfs whenever the domain is cofibrantly generated. Of particular interest is new composition criterion, conceived to prove this result, that is analogous to, though considerably harder to state than, Lemma II.3.15. In later sections, we will see that this is essentially our only trick for recognizing two-variable adjunctions of awfs.

In §II.5.3 we further extend the universal property of the unit functor constructed via Garner's small object argument, proving that Theorem II.3.24 still holds with the extended terminology. The general structure of the proof parallels our original argument, though the technical details are somewhat more complicated.

II.5.1 Adjunctions arising from two-variable adjunctions

We consider adjunctions

$$i\hat{\otimes}-\colon \mathcal{M}^{\mathbf{2}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{N}^{\mathbf{2}} \colon \widehat{\hom}_\ell(i,-) \tag{II.5.1}$$

obtained by fixing $i\colon A \to B \in \mathcal{K}$ and evaluating the induced two-variable adjunction (II.4.3). Because the right closure $\hom_r$ won't appear in this section, we abbreviate $\widehat{\hom}_\ell$ to $\widehat{\hom}$ and use exponential notation for $\hom_\ell$. For $j\colon K \to L$ in $\mathcal{M}^{\mathbf{2}}$ and $f\colon X \to Y$ in $\mathcal{N}^{\mathbf{2}}$, the arrows $i\hat{\otimes}j$ and $\widehat{\hom}(i,f)$ are defined to be the "pushout-product" and the "pullback-hom" displayed below

$$\begin{tikzcd}
A\otimes K \arrow[r, "A\otimes j"] \arrow[d, "i\otimes K"'] & A\otimes L \arrow[d] \arrow[ddr, bend left, "i\otimes L"] & \\
B\otimes K \arrow[r] \arrow[drr, bend right, "B\otimes j"'] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "i\hat{\otimes}j"] & \\
 & & B\otimes L
\end{tikzcd}
\qquad
\begin{tikzcd}
X^B \arrow[dr, dashed, "\widehat{\hom}(i,f)"] \arrow[drr, bend left, "\widehat{\hom}(B,f)"] \arrow[ddr, bend right, "\widehat{\hom}(i,X)"'] & & \\
 & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^B \arrow[d, "\widehat{\hom}(i,Y)"] \\
 & X^A \arrow[r, "\widehat{\hom}(A,f)"'] & Y^A
\end{tikzcd}
\tag{II.5.2}$$

In addition to (II.5.1), these functors induce an adjunction on the categories of composable triples of arrows.

Lemma II.5.3. *For fixed $i : A \to B$, there is an adjunction*

$$(i\hat{\otimes}-, i\hat{\otimes}-)\colon \mathcal{M}^{\mathbf{4}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{N}^{\mathbf{4}} \colon (\widehat{\hom}(i,-), \widehat{\hom}(i,-))$$

Proof. The left and right adjoints are

$$\begin{tikzcd}
I \arrow[d, "j"'] & & A\otimes J \underset{A\otimes I}{\sqcup} B\otimes I \arrow[d, "i\hat{\otimes}j"] & & X \arrow[d, "f"'] & & X^B \arrow[d, "\mathrm{h\hat{o}m}(i,f)"] \\
J \arrow[d, "k"'] & \mapsto & B\otimes J \arrow[d, "\iota\circ(B\otimes k)"] & \text{and} & Y \arrow[d, "g"'] & \mapsto & Y^B \times_{Y^A} X^A \arrow[d, "g^B\circ\pi"] \\
K \arrow[d, "l"'] & & A\otimes L \underset{A\otimes K}{\sqcup} B\otimes K \arrow[d, "i\hat{\otimes}l"] & & Z \arrow[d, "h"'] & & Z^B \arrow[d, "\mathrm{h\hat{o}m}(i,h)"] \\
L & & B\otimes L & & W & & W^B \times_{W^A} Z^A
\end{tikzcd}$$

where ι and π are the obvious legs of the pushout and pullback cones. The diagram

$$\begin{tikzcd}
A\otimes J \underset{A\otimes I}{\sqcup} B\otimes I \arrow[r, "a\sqcup b"] \arrow[d, "i\hat{\otimes}j"'] & X \arrow[d, "f"] & & I \arrow[r, "b^\sharp"] \arrow[d, "j"'] & X^B \arrow[d, "\mathrm{h\hat{o}m}(i,f)"] \\
B\otimes J \arrow[r, "c"] \arrow[d, "B\otimes k"'] & Y \arrow[dd, "g"] & & J \arrow[r, "c^\sharp\times a^\sharp"] \arrow[dr, dashed, "c^\sharp"'] \arrow[dd, "k"'] & Y^B \times_{Y^A} X^A \arrow[d, "\pi"] \\
B\otimes K \arrow[d, "\iota"'] \arrow[dr, dashed, "e"] & & \leftrightsquigarrow & & Y^B \arrow[d, "g^B"] \\
A\otimes L \underset{A\otimes K}{\sqcup} B\otimes K \arrow[r, "d\sqcup e"'] \arrow[d, "i\hat{\otimes}l"'] & Z \arrow[d, "h"] & & K \arrow[r, "e^\sharp"] \arrow[d, "l"'] & Z^B \arrow[d, "\mathrm{h\hat{o}m}(i,h)"] \\
B\otimes L \arrow[r, "z"] & W & & L \arrow[r, "z^\sharp\times d^\sharp"] & W^B \times_{W^A} Z^A
\end{tikzcd}$$

exhibits the adjoint correspondence: the top and bottom squares of each diagram are adjunct under $i\hat{\otimes}- \dashv \mathrm{h\hat{o}m}(i,-)$ and the middle quadrangles are adjunct under $B\otimes - \dashv (-)^B$. □

Remark II.5.4. Despite the horrendous notation, we introduce these functors as a convenience, allowing us to quote Theorem II.2.7 rather than chase diagrams. But while there exist similar bifunctors

$$\mathcal{K}^{\mathbf{4}}\times\mathcal{M}^{\mathbf{4}}\to\mathcal{N}^{\mathbf{4}} \quad (\mathcal{K}^{\mathbf{4}})^{\mathrm{op}}\times\mathcal{N}^{\mathbf{4}}\to\mathcal{M}^{\mathbf{4}} \quad (\mathcal{M}^{\mathbf{4}})^{\mathrm{op}}\times\mathcal{N}^{\mathbf{4}}\to\mathcal{K}^{\mathbf{4}}$$

these no longer form any sort of adjunction. Hence, the categories of composable triples of arrows will not appear in §II.6.

Now adopt the notation of II.4.11. By Lemma II.2.9 and its dual, lifts of $i\hat{\otimes}-$ and $\hat{\text{hom}}(i,-)$ to functors on coalgebras and algebras correspond to natural transformations

$$i\hat{\otimes}C \xRightarrow{\lambda\vec{(i)}} L(i\hat{\otimes}-) \quad \text{and} \quad F\hat{\text{hom}}(i,-) \xRightarrow{\rho\vec{(i)}} \hat{\text{hom}}(i,R),$$

that satisfy (co)unit and (co)multiplication conditions. The (co)unit condition defines the domain of $\lambda\vec{(i)}$ and the codomain of $\rho\vec{(i)}$ in such a way that that component of the pentagon is for free. Write $\lambda(i) = \text{cod}\lambda\vec{(i)}$ and $\rho(i) = \text{dom}\rho\vec{(i)}$ for the non-trivial components. Their (co)unit conditions are usually expressed by saying that $\lambda(i)$ and $\rho(i)$ define colax and lax morphisms of functorial factorizations in a sense we will exhibit in (II.5.7) below. The (co)multiplication conditions for $\lambda(i)$ and $\rho(i)$ are non-trivial; these pentagons appear in the statement of Lemma II.5.9.

In analogy with Definition I.6.10 and (II.3.16), we define

Definition II.5.5. The functors $i\hat{\otimes}-$ and $\hat{\text{hom}}(i,-)$ form an *adjunction of awfs* $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ if there exist mates $\lambda(i)$ and $\rho(i)$ with respect to the adjunctions

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "Q"] \arrow[d, shift right=2, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] & \mathcal{M} \arrow[d, shift right=2, "B\otimes-"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "E"'] \arrow[u, shift right=2, "\hat{\text{hom}}(i,-)"'] & \mathcal{N} \arrow[u, shift right=2, "(-)^B"']
\end{tikzcd}$$

such that $\lambda\vec{(i)}$ and $\rho\vec{(i)}$ determine lifts of $i\hat{\otimes}-$ and $\hat{\text{hom}}(i,-)$.

We give another presentation of the mates correspondence of $\lambda(i)$ and $\rho(i)$ that captures the full data of $\lambda\vec{(i)}$ and $\rho\vec{(i)}$. Adopting simplicial notation, write s_1 for precomposition with the functor $\mathbf{4} \to \mathbf{3}$ that collapses the middle two objects of $\mathbf{4}$ to the middle object of $\mathbf{3}$. Mates with respect to the adjunctions

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "\vec{Q}"] \arrow[d, shift right=2, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] & \mathcal{M}^{\mathbf{3}} \arrow[r, "s_1"] & \mathcal{M}^{\mathbf{4}} \arrow[d, shift right=2, "(i\hat{\otimes}-{,}i\hat{\otimes}-)"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "\vec{E}"'] \arrow[u, shift right=2, "\hat{\text{hom}}(i,-)"'] & \mathcal{N}^{\mathbf{3}} \arrow[r, "s_1"'] & \mathcal{N}^{\mathbf{4}} \arrow[u, shift right=2, "(\hat{\text{hom}}(i,-){,}\hat{\text{hom}}(i,-))"']
\end{tikzcd} \tag{II.5.6}$$

are natural transformations

$$(\lambda(\vec{i}), \lambda'(\vec{i}))\colon (i\hat{\otimes}C-, i\hat{\otimes}C-) \Rightarrow L(i\hat{\otimes}-, i\hat{\otimes}-)$$

$$(\rho'(\vec{i}), \rho(\vec{i}))\colon F(\hat{\mathrm{hom}}(i,-), \hat{\mathrm{hom}}(i,-)) \Rightarrow (\hat{\mathrm{hom}}(i,R-), \hat{\mathrm{hom}}(i,R-))$$

whose components at $j\colon K \to L \in \mathcal{M}$ and $f\colon X \to Y \in \mathcal{N}$ are

$$\begin{tikzcd}
A\otimes Qj \underset{A\otimes K}{\sqcup} B\otimes K \arrow[r, "A\otimes Fj\sqcup 1"] \arrow[d, "i\hat{\otimes}Cj"'] & A\otimes L \underset{A\otimes K}{\sqcup} B\otimes K \arrow[d, "L(i\hat{\otimes}j)"] \\
B\otimes Qj \arrow[r, "\lambda(i)_j"] \arrow[d, "\iota"'] & E(i\hat{\otimes}j) \arrow[d, equal] \\
A\otimes L \underset{A\otimes Qj}{\sqcup} B\otimes Qj \arrow[r, "\lambda'(i)_j"] \arrow[d, "i\hat{\otimes}Fj"'] & E(i\hat{\otimes}j) \arrow[d, "R(i\hat{\otimes}f)"] \\
B\otimes L \arrow[r, equal] & B\otimes L
\end{tikzcd}$$

and

$$\begin{tikzcd}
X^B \arrow[r, equal] \arrow[d, "C\hat{\mathrm{hom}}(i,f)"'] & X^B \arrow[d, "\hat{\mathrm{hom}}(i,Lf)"] \\
Q\hat{\mathrm{hom}}(i,f) \arrow[r, "\rho'(i)_f"] \arrow[d, equal] & Ef^B \times_{Ef^A} X^A \arrow[d, "\pi"] \\
Q\hat{\mathrm{hom}}(i,f) \arrow[r, "\rho(i)_f"] \arrow[d, "F\hat{\mathrm{hom}}(i,f)"'] & Ef^B \arrow[d, "\hat{\mathrm{hom}}(i,Rf)"] \\
Y^B \underset{Y^A}{\times} X^A \arrow[r, "1\times Lf^A"'] & Y^B \times_{Y^A} Ef^A
\end{tikzcd}$$

(II.5.7)

Obviously $\lambda'(i)$ determines $\lambda(i)$ but, under the hypothesis that the left-hand side is the mate of a lax morphism of functorial factorizations, the converse also holds. One component of $\lambda'(i)$ is $\lambda(i)$ and the other is necessarily a component of $L(i\hat{\otimes}-)$ on account of the appearance of the functor L in the bottom component of the right-hand natural transformation. Similarly, when the mate of the right-hand side is a colax morphism of functorial factorizations, $\rho(i)$ determines $\rho'(i)$, whose other component is defined from $F\hat{\mathrm{hom}}(i,-)$.

Extending the notation introduced above, write $\lambda(\vec{i})$, $\lambda'(\vec{i})$, $\rho'(\vec{i})$, and $\rho(\vec{i})$ for the natural transformations of the upper left, lower left, upper right, and lower right squares, respectively. Note $\lambda(\vec{i})$ and $\rho'(\vec{i})$ are mates and $\lambda'(\vec{i})$ and $\rho(\vec{i})$ are mates with respect to

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "C"] \arrow[d, shift right=2, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] & \mathcal{M}^{\mathbf{2}} \arrow[d, shift right=2, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "L"'] \arrow[u, shift right=2, "\hat{\mathrm{hom}}(i,-)"'] & \mathcal{N}^{\mathbf{2}} \arrow[u, shift right=2, "\hat{\mathrm{hom}}(i,-)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "F"] \arrow[d, shift right=2, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] & \mathcal{M}^{\mathbf{2}} \arrow[d, shift right=2, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "R"'] \arrow[u, shift right=2, "\hat{\mathrm{hom}}(i,-)"'] & \mathcal{N}^{\mathbf{2}} \arrow[u, shift right=2, "\hat{\mathrm{hom}}(i,-)"']
\end{tikzcd}$$

respectively. Indeed:

Lemma II.5.8. *$\lambda(i)$ and $\rho(i)$ are mates if and only if $(\lambda(\vec{i}), \lambda'(\vec{i}))$ and $(\rho'(\vec{i}), \rho(\vec{i}))$ are mates.*

Proof. Pasting

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{4}} \arrow[r, "\mathrm{ev}_{12}"] \arrow[d, shift right=1.5ex, "(i\hat{\otimes}-{,}i\hat{\otimes}-)"'] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\Searrow\pi"] & \mathcal{M}^{\mathbf{2}} \arrow[r, "\mathrm{ev}_0"] \arrow[d, shift right=1.5ex, "(B\otimes-)^{\mathbf{2}}"'] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\Searrow\hom(B{,}-)"] & \mathcal{M} \arrow[d, shift right=1.5ex, "B\otimes-"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N}^{\mathbf{4}} \arrow[u, shift right=1.5ex, "(\hat{\hom}(i{,}-){,}\hat{\hom}(i{,}-))"'] \arrow[r, "\mathrm{ev}_{12}"'] \arrow[ur, phantom, "\iota\Swarrow"] & \mathcal{N}^{\mathbf{2}} \arrow[u, shift right=1.5ex, "\hom(B{,}-)^{\mathbf{2}}"'] \arrow[r, "\mathrm{ev}_1"'] \arrow[ur, phantom, "B\otimes-\Swarrow"] & \mathcal{N} \arrow[u, shift right=1.5ex, "\hom(B{,}-)"']
\end{tikzcd}$$

on the right of (II.5.6) defines $\lambda(i), \rho(i)$ in terms of $(\lambda(\vec{i}), \lambda'(\vec{i}))$ and $(\rho'(\vec{i}), \rho(\vec{i}))$. It follows from Theorem II.2.7 that if the latter are mates, so are the former. Conversely, using the definitions of $\lambda'(\vec{i})$ and $\rho'(\vec{i})$ above, a diagram chase shows that if $\lambda(i)$ and $\rho(i)$ are mates, then so are $(\lambda(\vec{i}), \lambda'(\vec{i}))$ and $(\rho'(\vec{i}), \rho(\vec{i}))$. ☐

With (II.5.7), we can now be more explicit about the conditions on the natural transformations that satisfy Definition II.5.5.

Lemma II.5.9. *The adjunction $(i\hat{\otimes}-, \hom(i,-))$ is an adjunction of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$ if either*

- *There exists $\lambda(i)$ as in* (II.5.7) *such that the pentagons*

$$\begin{tikzcd}
 & i\hat{\otimes}C \arrow[rr, "\lambda(\vec{i})"] \arrow[dl, "i\hat{\otimes}\vec{\delta}"'] & & L(i\hat{\otimes}-) \arrow[dr, "\vec{\delta}_{i\hat{\otimes}-}"] & \\
i\hat{\otimes}C^2 \arrow[dr, "\lambda(\vec{i})_C"'] & & & & L^2(i\hat{\otimes}-) \\
 & & L(i\hat{\otimes}C) \arrow[urr, "L\lambda(\vec{i})"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & R(i\hat{\otimes}F) \arrow[drr, "R\lambda'(\vec{i})"] & & \\
i\hat{\otimes}F^2 \arrow[urr, "\lambda'(\vec{i})_F"] \arrow[dr, "i\hat{\otimes}\vec{\mu}"'] & & & & R^2(i\hat{\otimes}-) \arrow[dl, "\vec{\mu}_{i\hat{\otimes}-}"] \\
 & i\hat{\otimes}F \arrow[rr, "\lambda'(\vec{i})"'] & & R(i\hat{\otimes}-) &
\end{tikzcd}$$

  commute, in which case we say that $(i\hat{\otimes}-, \lambda(i))$ is a colax morphism of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$.

- *There exists $\rho(i)$ as in* (II.5.7) *such that the pentagons*

$$\begin{tikzcd}
 & C\hat{\hom}(i,-) \arrow[rr, "\rho'(\vec{i})"] \arrow[dl, "\vec{\delta}_{\hat{\hom}(i,-)}"'] & & \hat{\hom}(i,L) \arrow[dr, "\hat{\hom}(i{,}\vec{\delta})"] & \\
C^2\hat{\hom}(i,-) \arrow[dr, "C\rho'(\vec{i})"'] & & & & \hat{\hom}(i,L^2) \\
 & & C\hat{\hom}(i,L) \arrow[urr, "\rho'(\vec{i})_L"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & F\hat{\hom}(i,R) \arrow[drr, "\rho(\vec{i})_R"] & & \\
F^2\hat{\hom}(i,-) \arrow[urr, "F\rho(\vec{i})"] \arrow[dr, "\vec{\mu}_{\hat{\hom}(i,-)}"'] & & & & \hat{\hom}(i,R^2) \arrow[dl, "\hat{\hom}(i{,}\vec{\mu})"] \\
 & F\hat{\hom}(i,-) \arrow[rr, "\rho(\vec{i})"'] & & \hat{\hom}(i,R) &
\end{tikzcd}$$

commute, in which case we say that $(\widehat{\hom}(i,-),\rho(i))$ *is a* lax morphism of awfs $(\mathbb{L},\mathbb{R}) \to (\mathbb{C},\mathbb{F})$.

Proof. When $\lambda(i)$ and $\rho(i)$ are mates, so are $\lambda(\vec{i})$ and $\rho'(\vec{i})$ and $\lambda'(\vec{i})$ and $\rho(\vec{i})$; hence, in each column, the top pentagon commutes if and only if the bottom one does by Theorem II.2.7. □

II.5.2 Composition and cellularity criteria

In practice, it is often easier to define a lifted functor than to describe its natural transformation, so it is useful to have a criterion to detect which lifted functors are (co)lax morphisms of awfs.

Recall that for cofibrantly generated awfs, vertical composition in $\mathbb{R}$-**alg** $\cong \mathcal{J}^{\boxslash}$ is particularly easy to describe. If $(f,\phi_f),(g,\phi_g) \in \mathcal{J}^{\boxslash}$ with $\mathrm{cod} f = \mathrm{dom} g$, their composite is $(gf, \phi_g \bullet \phi_f)$ where

$$\phi_g \bullet \phi_f(j,a,b) := \phi_f(j,a,\phi_g(j,fa,b)) \qquad \begin{tikzcd} \cdot \arrow[rr, "a"] \arrow[dd, "j"'] & & \cdot \arrow[d, "f"] \\ & & \cdot \arrow[d, "g"] \\ \cdot \arrow[rr, "b"'] \arrow[uurr, dashed, "\phi_f(j{,}a{,}\phi_g)"] \arrow[urr, dashed, "\phi_g(j{,}fa{,}b)"'] & & \cdot \end{tikzcd} \tag{II.5.10}$$

Note that $(f,1) : (gf,\phi_g \bullet \phi_f) \Rightarrow (g,\phi_g)$ is a morphism in $\mathcal{J}^{\boxslash}$, but $(1,g)\colon f \Rightarrow gf$ is not. However this latter arrow, appearing as the middle square below, does preserve solutions to some lifting problems: namely those whose bottom arrow is the solution ϕ_g to a lifting problem of the form

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "j"'] & \cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[r, "f"] \arrow[d, "gf"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, dashed] \arrow[urrr, dashed, "\phi_g(j{,}fa{,}b)"] \arrow[rrr, bend right, "b"'] & \cdot \arrow[r, "g"'] & \cdot \arrow[r, equal] & \cdot \end{tikzcd} \tag{II.5.11}$$

More generally, for any awfs $(\mathbb{L},\mathbb{R})$ and composable $\mathbb{R}$-algebras f and g, $(f,1)\colon gf \Rightarrow g$ is a morphism of $\mathbb{R}$-algebras, while $(1,g)\colon f \Rightarrow gf$ only preserves the canonical solutions to lifting problems of the form (II.5.11). This point will return in the proof of Theorem II.5.15.

Theorem II.5.12 (Composition criterion)**.** *Composing with the embedding (II.3.6), a lifted functor* $\hat{\hom}(i,-)\colon \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg}$ *gives rise to*

$$\begin{tikzcd}
\mathbb{R}\text{-}\mathbf{alg} \arrow[r] \arrow[d] & \mathbb{F}\text{-}\mathbf{alg} \arrow[r, "\text{lift}"] \arrow[d] & \mathbb{C}\text{-}\mathbf{coalg}^{\boxslash} \arrow[dl] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "\hat{\hom}(i,-)"] & \mathcal{M}^{\mathbf{2}} &
\end{tikzcd}$$

The functor $\mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg}$ *defines a lax morphism of awfs if and only if the lifting functions assigned to a composable pair* $f, g \in \mathbb{R}\text{-}\mathbf{alg}$ *have the following property: given a lifting problem* $(a, b \times c)$ *between* $j \in \mathbb{C}\text{-}\mathbf{coalg}$ *and* $\hat{\hom}(i, gf)$*, composition with the right square of the rectangle below determines a lifting problem against* $\hat{\hom}(i, g)$*, whose canonical solution is determined by the awfs* $(\mathbb{C}, \mathbb{F})$ *and the lifted functor. This solution d determines a lifting problem against* $\hat{\hom}(i, f)$*, which can again be solved by* $(\mathbb{C}, \mathbb{F})$ *and the lifted functor, and this solution e should be the chosen solution to the original lifting problem.*

$$\begin{tikzcd}
K \arrow[r, "a"] \arrow[d, "j"'] & X^B \arrow[r, equal] \arrow[d, "\hat{\hom}(i,f)"] & X^B \arrow[r, "f^B"] \arrow[d, "\hat{\hom}(i,gf)"] & Y^B \arrow[d, "\hat{\hom}(i,g)"] \\
L \arrow[ur, dashed, "e"] \arrow[urr, dashed, "e"] \arrow[urrr, dashed, "d"] \arrow[r, dashed, "d\times c"'] \arrow[rr, bend right, "b\times c"'] & Y^B \times_{Y^A} X^A \arrow[r, "g^B\times_{g^A}1"'] & Z^B \times_{Z^A} X^A \arrow[r, "1\times_1 f^A"'] & Z^B \times_{Z^A} Y^A
\end{tikzcd} \tag{II.5.13}$$

Proof. By Lemmas II.2.9 and II.5.9, the lifted functor $\mathbb{R}$-**alg** → $\mathbb{F}$-**alg** determines a lax morphism of awfs if and only if its characterizing natural transformation $\rho(i)$ is such that the left-hand pentagon

$$\begin{tikzcd}
& Q\hat{\hom}(i,f) \arrow[r, "\rho'(i)_f"] \arrow[dl, "\delta_{\hat{\hom}(i,f)}"'] & Ef^B \times_{Ef^A} X^A \arrow[dr, "\delta_f^B\times_{\delta_f^A}1"] & \\
QC\hat{\hom}(i,f) \arrow[dr, "Q(1,\rho'(i)_f)"'] & & & ELf^B \times_{ELf^A} X^A \\
& Q\hat{\hom}(i,Lf) \arrow[urr, "\rho'(i)_{Lf}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Q\hat{\hom}(i,f) \arrow[r, "\rho(i)_f"] \arrow[dl, "\delta_{\hat{\hom}(i,f)}"'] & Ef^B \arrow[dr, "\delta_f^B"] & \\
QC\hat{\hom}(i,f) \arrow[dr, "Q(1,\rho'(i)_f)"'] & & & ELf^B \\
& Q\hat{\hom}(i,Lf) \arrow[urr, "\rho(i)_{Lf}"'] & &
\end{tikzcd}$$

commutes. Projecting to one leg of the pullbacks, the left pentagon implies the right one, but an easy diagram chase—the essential point being that the other leg of $\rho'(i)_f$ is defined to be a leg of $F\mathrm{h\hat{o}m}(i,f)$—shows that the right pentagon also implies the left. Thus, it suffices to prove that the lifted functor satisfies the composition criterion if and only if this right-hand pentagon commutes.

Suppose $\mathrm{h\hat{o}m}(i,-)\colon \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg}$ is a lax morphism of awfs and consider composable $\mathbb{R}$-algebras (f,s) and (g,t). The lifted functor assigns the image of their composite $(gf, t \bullet s)$ the $\mathbb{F}$-algebra structure

$$\begin{gathered} Q\mathrm{h\hat{o}m}(i,gf) \xrightarrow[\rho(i)_{gf}]{} E(gf)^B \xrightarrow[(t\bullet s)^B]{} X^B = \\ Q\mathrm{h\hat{o}m}(i,gf) \xrightarrow[\rho(i)_{gf}]{} E(gf)^B \xrightarrow[\delta^B_{gf}]{} EL(gf)^B \xrightarrow[E(1,E(f,1))^B]{} E(Lg\cdot f)^B \xrightarrow[E(1,t)^B]{} Ef^B \xrightarrow[s^B]{} X^B \end{gathered} \tag{II.5.14}$$

As for any $\mathbb{F}$-algebra structure, this map is the canonical solution assigned to the lifting problem

$$\begin{tikzcd}
X^B \arrow[r, equal] \arrow[d, "C\mathrm{h\hat{o}m}(i,gf)"'] & X^B \arrow[d, "\mathrm{h\hat{o}m}(i,gf)"] \\
Q\mathrm{h\hat{o}m}(i,gf) \arrow[r, "F\mathrm{h\hat{o}m}(i,gf)"'] & Z^B \times_{Z^A} X^A
\end{tikzcd}$$

The composition criterion says that (II.5.14) should be obtained in the following manner. First, solve the composite lifting problem

$$\begin{tikzcd}
X^B \arrow[r, equal] \arrow[dd, "C\mathrm{h\hat{o}m}(i,gf)"'] & X^B \arrow[r, "f^B"] \arrow[dd] & Y^B \arrow[dd, "\mathrm{h\hat{o}m}(i,g)"] \\
{} \arrow[r, "Q(1{,}F\mathrm{h\hat{o}m}(i{,}gf))"] & {} \arrow[r, "Q(f^B{,}1\times_1 f^A)"'] & {} \arrow[u, "t^B\cdot\rho(i)_g"] \\
Q\mathrm{h\hat{o}m}(i,gf) \arrow[r, "F\mathrm{h\hat{o}m}(i,gf)"'] \arrow[u, "\delta"'] \arrow[ur, "1"'] & Z^B \times_{Z^A} X^A \arrow[r, "1\times_1 f^A"'] & Z^B \times_{Z^A} Y^A
\end{tikzcd}$$

in the manner displayed using the awfs $(\mathbb{C},\mathbb{F})$ and the $\mathbb{F}$-algebra structure assigned to $\mathrm{h\hat{o}m}(i,g)$. The first two arrows in this lift compose to the identity by a triangle identity for the comonad $\mathbb{C}$; hence, by naturality of $\rho(i)$, the canonical solution to this lifting problem is

$$Q\mathrm{h\hat{o}m}(i,gf) \xrightarrow{\rho(i)_{gf}} E(gf)^B \xrightarrow{E(f,1)^B} Eg^B \xrightarrow{t^B} Y^B\,.$$

This arrow defines the other leg of the lifting problem

$$\begin{tikzcd}
X^B \arrow[r, equal] \arrow[d, "C\widehat{\hom}(i{,}gf)"'] & X^B \arrow[d, "\widehat{\hom}(i{,}f)"] \\
Q\widehat{\hom}(i, gf) \arrow[r] & Y^B \times_{Y^A} X^A
\end{tikzcd}$$

whose canonical solution must agree with the $\mathbb{F}$-algebra structure of $\widehat{\hom}(i, gf)$. This lifting problem factors as

$$\begin{tikzcd}
X^B \arrow[r, equal] \arrow[d, "C\widehat{\hom}(i{,}gf)" description] & X^B \arrow[r, equal] \arrow[d, "\widehat{\hom}(i{,}L(gf))" description] & X^B \arrow[r, equal] \arrow[d, "\widehat{\hom}(i{,}Lg\cdot f)" description] & X^B \arrow[d, "\widehat{\hom}(i{,}f)" description] \\
Q\widehat{\hom}(i, gf) \arrow[r, "\rho'(i)_{gf}"'] & E(gf)^B \times_{E(gf)^A} X^A \arrow[r, "E(f{,}1)^B \times_{E(f{,}1)^A} 1"'] & Eg^B \times_{Eg^A} X^A \arrow[r, "t^B \times_{t^A} 1"'] & Z^B \times_{Z^A} X^A
\end{tikzcd}$$

so its canonical solution, by naturality of $\rho(i)$, is the composite

$$Q\widehat{\hom}(i, gf) \xrightarrow{\delta_{\widehat{\hom}(i,gf)}} QC\widehat{\hom}(i, gf) \xrightarrow{Q(1,\rho'(i)_{gf})} Q\widehat{\hom}(i, L(gf)) \xrightarrow{\rho(i)_{L(gf)}} EL(gf)^B \cdots$$

$$\cdots \xrightarrow{E(1,E(f,1))^B} E(Lg \cdot f)^B \xrightarrow{E(1,t)^B} Ef^B \xrightarrow{s^B} X^B$$

which agrees with (II.5.14) if the pentagon for gf is satisfied.

Conversely, suppose the lifted functor satisfies the composition criterion. Consider the composable pair of free $\mathbb{R}$-algebras $ELf \xrightarrow{RLf} Ef \xrightarrow{Rf} Y$. Employing the definition (II.3.10) of δ, the upper right composite of the pentagon is the top composite of

$$\begin{tikzcd}
 & Ef^B \arrow[rr, "E(L^2f{,}1)^B"] & & E(Rf \cdot RLf)^B \arrow[dr, "(\mu_f \cdot \mu_{Lf})^B"] & \\
Q\hat{\mathrm{hom}}(i,f) \arrow[ur, "\rho(i)_f"] \arrow[rr, "E((L^2f)^B{,}1\times_1(L^2f)^A)"'] \arrow[d, "\delta_{\hat{\mathrm{hom}}(i,f)}"'] & & Q\hat{\mathrm{hom}}(i,Rf \cdot RLf) \arrow[ur, "\rho(i)_{Rf\cdot RLf}"'] \arrow[d, "\delta_{\hat{\mathrm{hom}}(i,Rf\cdot RLf)}"] & & ELf^B \\
QC\hat{\mathrm{hom}}(i,f) \arrow[rr] \arrow[d, "Q(1{,}\rho'(i)_f)"'] & & QC\hat{\mathrm{hom}}(i,Rf \cdot RLf) \arrow[d, "Q(1{,}\rho'(i)_{Rf\cdot RLf})"] & & ERLf^B \arrow[u, "\mu^B_{Lf}"] \\
Q\hat{\mathrm{hom}}(i,Lf) \arrow[rr] \arrow[dr, "\rho(i)_{Lf}"'] & & Q\hat{\mathrm{hom}}(i,L(Rf \cdot RLf)) \arrow[dr, "\rho(i)_{L(Rf\cdot RLf)}"] & & E(LRf \cdot RLf)^B \arrow[u, "E(1{,}\mu_f)^B"] \\
 & ELf^B \arrow[rr, "E(L^2f{,}E(L^2f{,}1))^B"'] & & EL(Rf \cdot RLf)^B \arrow[ur, "E(1{,}E(RLf{,}1))^B"'] &
\end{tikzcd}$$

The squares commute by naturality of $\rho(i)$, δ, $\rho'(i)$, and $\rho(i)$; names for the unlabeled arrows can be deduced from this. The octagon is exactly the composition criterion, in the form just deduced, so the outer decagon commutes. The composite of the last four arrows along the bottom right is $(-)^B$ applied to an identity

$$\begin{tikzcd}
EL(Rf \cdot RLf) \arrow[r, "E(1{,}E(RLf{,}1))"] & E(LRf \cdot RLf) \arrow[r, "E(1{,}\mu_f)"] & ERLf \arrow[d, "\mu_{Lf}"] \\
ELf \arrow[u, "E(L^2f{,}E(L^2f{,}1))"] \arrow[r, "E(L^2f{,}1)"] \arrow[rr, bend right, "1"'] & ERLf \arrow[u, "E(1{,}E(Lf{,}1))"'] \arrow[ur, "1"'] & ELf
\end{tikzcd}$$

using functoriality of E and two applications of a monad triangle identity. So the exterior of our decagon is desired pentagon. □

A dual theorem describes those lifts of $i\hat{\otimes}-$ which determine colax morphisms of awfs. While the composition criterion of Theorem II.5.12 is a bit of a mess to state, it enables us to prove a cellularity theorem.

Theorem II.5.15. *Suppose $\mathcal{M}$ has an awfs $(\mathbb{C},\mathbb{F})$ generated by $\mathcal{J}$ and $\mathcal{K}$ has an awfs $(\mathbb{L},\mathbb{R})$, not necessarily cofibrantly generated. Then $i\hat{\otimes}- \dashv \hat{\mathrm{hom}}(i,-)$ is an adjunction of awfs $(\mathbb{C},\mathbb{F}) \to (\mathbb{L},\mathbb{R})$*

if and only if there is a lift

$$\begin{tikzcd}
\mathcal{J} \arrow[r, dashed] \arrow[d] & \mathbb{L}\textbf{-coalg} \arrow[d] \\
\mathcal{M}^{\mathbf{2}} \arrow[r, "i\hat{\otimes}-"'] & \mathcal{N}^{\mathbf{2}}
\end{tikzcd}$$

That is to say, these functors form an adjunction of awfs if and only if $i\hat{\otimes}\mathcal{J}$ is cellular.

Proof. As in Theorem II.3.18, we define the lift $\mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg} \cong \mathcal{J}^{\boxslash}$ of $\widehat{\hom}(i,-)$ to be the composite

$$\mathbb{R}\textbf{-alg} \xrightarrow{\text{lift}} \mathbb{L}\textbf{-coalg}^{\boxslash} \xrightarrow{\text{res}} (i\hat{\otimes}\mathcal{J})^{\boxslash} \xrightarrow{\text{adj}} \mathcal{J}^{\boxslash}$$

Explicitly, the image of an $\mathbb{R}$-algebra f in $\mathcal{J}^{\boxslash}$ is the arrow $\widehat{\hom}(i,f)$ equipped with a lifting function $\phi_{\widehat{\hom}(i,f)}$ defined so that the chosen solution $\phi_{\widehat{\hom}(i,f)}(j,a,d\times c)$ to a lifting problem of the form displayed in the left-hand square of (II.5.13) is adjunct to the solution constructed via the awfs $(\mathbb{L},\mathbb{R})$ and the functor $\mathcal{J} \to \mathbb{L}\textbf{-coalg}$.

By Lemma II.3.8, the functor $\mathbb{R}\textbf{-alg} \to (i\hat{\otimes}\mathcal{J})^{\boxslash}$ preserves composition, so it suffices to show that $\text{adj}\colon (i\hat{\otimes}\mathcal{J})^{\boxslash} \to \mathcal{J}^{\boxslash}$ satisfies the criterion of Theorem II.5.12. Given $(f,\phi_f^{\sharp}),(g,\phi_g^{\sharp}) \in (i\hat{\otimes}\mathcal{J})^{\boxslash}$ with $\text{cod} f = \text{dom} g$, their composite is $(gf, \phi_g^{\sharp} \bullet \phi_f^{\sharp})$ where

$$\phi_g^{\sharp} \bullet \phi_f^{\sharp}(i\hat{\otimes}j, c^{\sharp} \sqcup a^{\sharp}, b^{\sharp}) := \phi_f^{\sharp}(i\hat{\otimes}j, c^{\sharp} \sqcup a^{\sharp}, \phi_g^{\sharp}(i\hat{\otimes}j, fc^{\sharp} \sqcup fa^{\sharp}, b^{\sharp}))$$

Transposing across the adjunction, we get the formula

$$\phi_{\widehat{\hom}(i,gf)}(j,a,b\times c) := \phi_{\widehat{\hom}(i,f)}(j,a,\phi_{\widehat{\hom}(i,g)}(j,f^B a, b\times f^A c)\times c)$$

which says that algebra structure for $\widehat{\hom}(i,gf)$ is obtained precisely as described in the statement of Theorem II.5.12. Indeed, this is how that condition was deduced. □

II.5.3 Extending the universal property of the small object argument

In the remainder of this section, whenever we refer to an adjunction between arrow categories we always mean either an adjunction of the form $T^{\mathbf{2}} \dashv S^{\mathbf{2}}$ defined pointwise via an adjunction on the underlying categories, an adjunction of the form $i\hat{\otimes}- \dashv \widehat{\hom}(i,-)$ arising from a two-variable adjunction on the underlying categories, or a composite of the two.

We extend Theorem I.6.22 to the newly defined adjunctions of awfs. The uniqueness theorem, Theorem II.4.9, is a corollary of this result.

Theorem II.5.16. *If $\mathcal{M}$ permits the small object argument and if $\mathcal{J} \to \mathcal{M}^{\mathbf{2}}$ is a small category of arrows, then the unit functor* (I.2.26) *is universal among adjunctions of awfs.*

Proof. Our argument extends the proof given for I.6.22. We broaden our interpretation of the categories

$$\mathcal{G}^{\mathrm{ladj}} = \mathbf{AWFS}_{\mathrm{ladj}} \xrightarrow{\mathcal{G}_1^{\mathrm{ladj}}} \mathbf{LAWFS}_{\mathrm{ladj}} \xrightarrow{\mathcal{G}_2^{\mathrm{ladj}}} \mathbf{Cmd}(-)^{\mathbf{2}}_{\mathrm{ladj}} \xrightarrow{\mathcal{G}_3^{\mathrm{ladj}}} \mathbf{CAT}/(-)^{\mathbf{2}}_{\mathrm{ladj}} \tag{II.5.17}$$

of (I.6.21). In each case the objects are the same, but we extend the class of morphisms to include those involving the sorts of adjunctions detailed above, always pointing in the direction of the left adjoint. A morphism in $\mathbf{CAT}/(-)^{\mathbf{2}}_{\mathrm{ladj}}$ is an adjunction between arrow categories together with a specified lift of the left adjoint to the fibers. A morphism in $\mathbf{Cmd}(-)^{\mathbf{2}}_{\mathrm{ladj}}$ is an adjunction between the arrow categories together with a specified colax comonad morphism over the left adjoint. $\mathbf{LAWFS}_{\mathrm{ladj}}$ is the full subcategory on comonads over the domain functor. $\mathbf{AWFS}_{\mathrm{ladj}}$ is the category of awfs and adjunctions of awfs, newly defined.

We must show that Garner's small object argument constructs a reflection along each functor $\mathcal{G}_i^{\mathrm{ladj}}$. The first of these, the reflection along $\mathcal{G}_3^{\mathrm{ladj}}$, proceeds exactly as in I.6.22: left adjoints preserve left Kan extensions, regardless of how the adjunctions are defined.

The argument given in I.6.22 to establish the reflection along $\mathcal{G}_2^{\mathrm{ladj}}$ requires that the functor $i\hat{\otimes}-$ preserve morphisms in the arrow category that are pushout squares in the underlying category. This follows from Lemma I.5.6 and the fact that the left adjoints $A\otimes-$ and $B\otimes-$ necessarily preserve pushouts. The rest of the argument is unchanged.

Thus, it is only for the final reflection along $\mathcal{G}_1^{\mathrm{ladj}}$ that we must do a bit of work. The context for the argument of I.6.22 is the category $\mathbf{FF}_{\mathrm{ladj}}$ whose objects are functorial factorizations and whose morphisms are colax morphisms of functorial factorizations lifting left adjoints. Because functors of the form $i\hat{\otimes}-$ preserve neither domains nor composability, a colax morphism of functorial factorizations $\lambda(i)\colon \vec{Q} \to \vec{E}$ lifting $i\hat{\otimes}-$ now has the form displayed in the left-hand diagram (II.5.7). Because colax morphisms of functorial factorizations are composable, it suffices to consider only those colax morphisms lifting functors the form $i\hat{\otimes}-$, the other case completed in the original proof.

The argument of I.6.22 applies once we establish that the category $\mathbf{FF}_{\mathrm{ladj}}$ has the following two properties. Each fiber, that is, each category of functorial factorizations on a fixed category, has two monoidal structures $\otimes$ and $\odot$, given by re-factoring the right or the left factor, respectively. We must show

- a pair of morphisms ϕ, ψ lifting the same left adjoint $i\hat{\otimes}-$ can be combined to give $\phi \otimes \psi$ and $\phi \odot \psi$
- the distributive law α, defined in each fiber using the functorial factorizations, is natural with respect to colax morphisms lifting $i\hat{\otimes}-$, i.e., the following diagram commutes:

$$\begin{tikzcd}
(\vec{X} \odot \vec{X}') \otimes (\vec{Z} \odot \vec{Z}') \arrow[r, "\alpha"] \arrow[d, "(\phi\odot\phi')\otimes(\psi\odot\psi')"'] & (\vec{X} \otimes \vec{Z}) \odot (\vec{X}' \otimes \vec{Z}') \arrow[d, "(\phi\otimes\psi)\odot(\phi'\otimes\psi')"'] \\
(\vec{Y} \odot \vec{Y}') \otimes (\vec{W} \odot \vec{W}') \arrow[r, "\alpha"'] & (\vec{Y} \otimes \vec{W}) \odot (\vec{Y}' \otimes \vec{W}')
\end{tikzcd}$$

It follows that if ϕ and ψ are $\odot$-comonoid morphisms, that is, if ϕ and ψ are morphisms in $\mathbf{LAWFS}_{\mathrm{ladj}}$, then so is $\phi \otimes \psi$.

We define the products $\phi \otimes \psi$ and $\phi \odot \psi$ of colax morphisms lifting $i\hat{\otimes}-$ and leave the tedious but straightforward diagram chase exhibiting the distributive law to the reader. Given functorial factorizations $\vec{Q} = (C, F)$, $\vec{Q^*} = (C^*, F^*)$ on $\mathcal{M}$ and $\vec{E} = (L, R)$, $\vec{E^*} = (L^*, R^*)$ on $\mathcal{N}$ together with morphisms $\phi\colon \vec{Q} \to \vec{E}$ and $\psi\colon \vec{Q^*} \to \vec{E^*}$ lifting $i\hat{\otimes}-$, $\phi \otimes \psi$ is the composite $E(\psi', 1) \cdot \phi_{F'}$ displayed below

$$\begin{tikzcd}
B \otimes K \underset{A\otimes K}{\sqcup} A \otimes QF^* j \arrow[rrr, "1\sqcup A\otimes FF^* j"] \arrow[dr, "B\otimes C^* j\sqcup 1"] \arrow[dd, "i\hat{\otimes}(CF^*\cdot C^*)j"'] & & & B \otimes K \underset{A\otimes K}{\sqcup} A \otimes L \arrow[dl, "B\otimes C^* j\sqcup 1"'] \arrow[d, "L^*(i\hat{\otimes} j)"] \\
& B \otimes Q^* j \underset{A\otimes Q^* j}{\sqcup} A \otimes QF^* j \arrow[r] \arrow[dl, "i\hat{\otimes} CF^* j"] & B \otimes Q^* j \underset{A\otimes Q^* j}{\sqcup} A \otimes L \arrow[r, "\psi'_j"'] \arrow[d, "L(i\hat{\otimes} F^* j)"'] & E^*(i\hat{\otimes} j) \arrow[d, "LR^*(i\hat{\otimes} j)"] \\
B \otimes QF^* j \arrow[rr, "\phi_{F^* j}"] \arrow[d, "\iota"'] & & E(i\hat{\otimes} F^* j) \arrow[r, "E(\psi'_j{,}1)"] \arrow[dd, "R(i\hat{\otimes} F^* j)"'] & ER^*(i\hat{\otimes} j) \arrow[dd, "RR^*(i\hat{\otimes} j)"] \\
B \otimes QF^* j \underset{A\otimes QF^* j}{\sqcup} A \otimes L \arrow[urr, "\phi'_{F^* j}"'] \arrow[d, "i\hat{\otimes} FF^* j"'] & & & \\
B \otimes L \arrow[rr, equal] & & B \otimes L \arrow[r, equal] & B \otimes L
\end{tikzcd}$$

Similarly, $\phi \odot \psi$ is the composite $E(1 \sqcup A \otimes F^*, \psi) \cdot \phi_{C^*}$ displayed below

$$\begin{tikzcd}
B \otimes K \underset{A\otimes K}{\sqcup} A \otimes QC^*j \arrow[rr, "1\sqcup A\otimes(F^*\cdot FC^*)j"] \arrow[dr, "1\sqcup A\otimes FC^*j"'] \arrow[dd, "i\hat{\otimes}CC^*j"'] & & B \otimes K \underset{A\otimes K}{\sqcup} A \otimes L \arrow[dd, "LL^*(i\hat{\otimes}j)"] \\
& B \otimes K \underset{A\otimes K}{\sqcup} A \otimes Q^*j \arrow[ur, "1\sqcup A\otimes F^*j"'] \arrow[d, "L(i\hat{\otimes}C^*j)"] & \\
B \otimes QC^*j \arrow[r, "\phi_{C^*j}"] \arrow[d, "\iota"'] & E(i\hat{\otimes}C^*j) \arrow[r, "E(1\sqcup A\otimes F^*j{,}\psi)"'] \arrow[d, "R(i\hat{\otimes}C^*j)"] & EL^*(i\hat{\otimes}j) \arrow[d, "RL^*(i\hat{\otimes}j)"] \\
B \otimes QC^*j \underset{A\otimes QC^*j}{\sqcup} A \otimes L \arrow[ur, "\phi'_{C^*j}"] \arrow[r, "i\hat{\otimes}FC^*j"] \arrow[d, "i\hat{\otimes}(F^*\cdot FC^*)j"'] & B \otimes Q^*j \arrow[r, "\psi_j"'] \arrow[dl, "B\otimes F^*j"] & E^*(i\hat{\otimes}j) \arrow[d, "R^*(i\hat{\otimes}j)"] \\
B \otimes L \arrow[rr, equal] & & B \otimes L
\end{tikzcd}$$

□

Theorem II.4.9 follows as a corollary.

Theorem II.4.9. *There can be at most one two-variable adjunction of awfs* $(\mathbb{C}', \mathbb{F}') \times (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ *whose lifted left adjoint restricts along the unit functors to a given lifted functor* $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg**.

Proof. Suppose given a pair of two-variable adjunctions of awfs

$$\mathbb{C}'\text{-}\mathbf{coalg} \times \mathbb{C}\text{-}\mathbf{coalg} \rightrightarrows \mathbb{L}\text{-}\mathbf{coalg}$$

extending $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg**. On morphisms their behavior is completely specified by the condition that they lift $-\hat{\otimes}-$, so it suffices to consider whether these functors agree at each pair of objects. Restricting along the unit $\mathcal{I} \to \mathbb{C}'$-**coalg**, we obtain a pair of functors $\mathcal{I} \times \mathbb{C}$-**coalg** $\rightrightarrows$ $\mathbb{L}$-**coalg** necessarily distinct: if they agreed for each $j \in \mathbb{C}$-**coalg**, their extensions at each j, the adjunctions of awfs $\mathbb{C}'$-**coalg** $\rightrightarrows$ $\mathbb{L}$-**coalg**, would also agree by Theorem II.5.16. Now restricting these functors along the unit $\mathcal{J} \to \mathbb{C}$-**coalg** we obtain, in both cases, the original $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg**, by hypothesis. But this contradicts the argument just given: at each $i \in \mathcal{I}$, the extension to an adjunction of awfs $\mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg** is unique by Theorem II.5.16. Thus, there can be at most one functor $\mathcal{I} \times \mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg**, and hence at most one $\mathbb{C}'$-**coalg** $\times$ $\mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg** extending $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg**. □

II.6 Two-variable adjunctions of algebraic weak factorization systems

Using the technical material developed in the previous section, we now give the precise definition of a two-variable adjunction of awfs and prove the main cellularity theorem, Theorem II.4.8. As defined in §II.4, the data of a two-variable adjunction of awfs $\otimes\colon (\mathbb{C}', \mathbb{F}') \times (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ consists of lifted functors, characterized by the natural transformations, displayed below (abbreviating the codomain and domain functors from arrow categories to their base with "c" and "d")

$$\begin{tikzcd}
\mathbb{C}'\text{-}\mathbf{coalg} \times \mathbb{C}\text{-}\mathbf{coalg} \arrow[r, dashed] \arrow[d] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[d] & d \otimes Q \underset{d\otimes d}{\sqcup} Q' \otimes d \arrow[r, "1\otimes F \sqcup F'\otimes 1"] \arrow[d, swap, "C'\hat{\otimes}C"] & d \otimes c \underset{d\otimes d}{\sqcup} c \otimes d \arrow[d, "L(-\hat{\otimes}-)"] & =: \vec{\lambda} \\
\mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \arrow[r, "-\hat{\otimes}-"] & \mathcal{N}^{\mathbf{2}} & Q' \otimes Q \arrow[r, "\lambda"] & E(-\hat{\otimes}-) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}'\text{-}\mathbf{coalg}^{\mathrm{op}} \times \mathbb{R}\text{-}\mathbf{alg} \arrow[r, dashed] \arrow[d] & \mathbb{F}\text{-}\mathbf{alg} \arrow[d] & Q\hat{\mathrm{hom}}_\ell(-,-) \arrow[r, "\rho^\ell"] \arrow[d, swap, "F\hat{\mathrm{hom}}_\ell"] & \mathrm{hom}_\ell(Q', E) \arrow[d, "\hat{\mathrm{hom}}_\ell(C'{,}R)"] & =: \vec{\rho^\ell} \\
(\mathcal{K}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \arrow[r, "\hat{\mathrm{hom}}_\ell(-{,}-)"] & \mathcal{M}^{\mathbf{2}} & \mathrm{hom}_\ell(c,c) \underset{\mathrm{hom}_\ell(d,c)}{\times} \mathrm{hom}_\ell(d,d) \arrow[r, swap, "\mathrm{hom}_\ell(C'{,}1)\times\mathrm{hom}_\ell(1{,}L)"] & \mathrm{hom}_\ell(Q',c) \underset{\mathrm{hom}_\ell(d,c)}{\times} \mathrm{hom}_\ell(d,E) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}\text{-}\mathbf{coalg}^{\mathrm{op}} \times \mathbb{R}\text{-}\mathbf{alg} \arrow[r, dashed] \arrow[d] & \mathbb{F}'\text{-}\mathbf{alg} \arrow[d] & Q'\hat{\mathrm{hom}}_r(-,-) \arrow[r, "\rho^r"] \arrow[d, swap, "F'\hat{\mathrm{hom}}_r"] & \mathrm{hom}_r(Q, E) \arrow[d, "\hat{\mathrm{hom}}_r(C{,}R)"] & =: \vec{\rho^r} \\
(\mathcal{M}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \arrow[r, "\hat{\mathrm{hom}}_r(-{,}-)"] & \mathcal{K}^{\mathbf{2}} & \mathrm{hom}_r(c,c) \underset{\mathrm{hom}_r(d,c)}{\times} \mathrm{hom}_r(d,d) \arrow[r, swap, "\mathrm{hom}_r(C{,}1)\times\mathrm{hom}_r(1{,}L)"] & \mathrm{hom}_r(Q,c) \underset{\mathrm{hom}_r(d,c)}{\times} \mathrm{hom}_r(d,E) &
\end{tikzcd}$$

(II.6.1)

Given the lifted functors displayed above left, the natural transformations displayed above right are defined by composing the (co)algebra structures assigned free (co)algebras with the appropriate (co)unit maps. Conversely, by Lemma II.2.9 and the observation that a product of monads is a monad on a product category, e.g., the natural transformation $\vec{\rho^r}$ corresponding to the lift of $\widehat{\hom}_r$ satisfies

$$\begin{tikzcd}[column sep=small]
 & \widehat{\hom}_r \arrow[dl, "\vec{\eta}_{\widehat{\hom}_r}"'] \arrow[dr, "\widehat{\hom}_r(\vec{\epsilon},\vec{\eta})"] & \\
F'\widehat{\hom}_r \arrow[rr, "\vec{\rho}^r"'] & & \widehat{\hom}_r(C,R)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & F'\widehat{\hom}_r(C,R) \arrow[rr, "\vec{\rho}^r_{C,R}"] & & \widehat{\hom}_r(C^2,R^2) \arrow[dl, "\widehat{\hom}_r(\vec{\delta},\vec{\mu})"] \\
F'^2\widehat{\hom}_r \arrow[ur, "F'\vec{\rho}^r"] \arrow[r, "\vec{\mu}_{\widehat{\hom}_r}"'] & F'\widehat{\hom}_r \arrow[rr, "\vec{\rho}^r"'] & & \widehat{\hom}_r(C,R)
\end{tikzcd}$$

The maps $\vec{\lambda}, \vec{\rho^\ell}, \vec{\rho^r}$ are determined by the components $\lambda = \mathrm{cod}\vec{\lambda}, \rho^\ell = \mathrm{dom}\vec{\rho^\ell}, \rho^r = \mathrm{dom}\vec{\rho^r}$. As in §II.5, the unit condition determines $\mathrm{cod}\vec{\rho^r}$ and stipulates that $\rho^r = \mathrm{dom}\vec{\rho^r}$ satisfy

$$\begin{tikzcd}
\hom_r(\mathrm{cod},\mathrm{dom}) \arrow[r, equal] \arrow[d, "C'\widehat{\hom}_r(-,-)"'] & \hom_r(\mathrm{cod},\mathrm{dom}) \arrow[d, "\hom_r(F,L)"] \\
Q'\widehat{\hom}_r(-,-) \arrow[r, "\rho^r"] \arrow[d, "F'\widehat{\hom}_r(-,-)"'] & \hom_r(Q,E) \arrow[d, "\widehat{\hom}_r(C,R)"] \\
\hom_r(\mathrm{cod},\mathrm{cod}) \underset{\hom_r(\mathrm{dom},\mathrm{cod})}{\times} \hom_r(\mathrm{dom},\mathrm{dom}) \arrow[r, "\hom_r(F,1)\times\hom_r(1,L)"'] & \hom_r(Q,\mathrm{cod}) \underset{\hom_r(\mathrm{dom},\mathrm{cod})}{\times} \hom_r(\mathrm{dom},E)
\end{tikzcd} \tag{II.6.2}$$

The diagram (II.6.2) asserts that ρ^r defines a bilax morphism of functorial factorizations; compare with (II.5.7) in light of Remark II.5.4.

In order that each lifted functor interact with the entire awfs, we ask that the natural transformations $\lambda, \rho^\ell, \rho^r$ specifying the lifted functors are parameterized mates

$$\begin{tikzcd}
\mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \arrow[r, "Q'\times Q"] \arrow[d, "\hat{\otimes}"'] \arrow[dr, phantom, "\lambda\Swarrow"] & \mathcal{K} \times \mathcal{M} \arrow[d, "\otimes"] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "E"'] & \mathcal{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "Q"] \arrow[dr, phantom, "\Searrow\rho^\ell"] & \mathcal{M} \\
(\mathcal{K}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \arrow[r, "Q'\times E"'] \arrow[u, "\widehat{\hom}_\ell"] & \mathcal{K}^{\mathrm{op}} \times \mathcal{N} \arrow[u, "\hom_\ell"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}^{\mathbf{2}} \arrow[r, "Q'"] \arrow[dr, phantom, "\Searrow\rho^r"] & \mathcal{K} \\
(\mathcal{M}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \arrow[r, "Q\times E"'] \arrow[u, "\widehat{\hom}_r"] & \mathcal{M}^{\mathrm{op}} \times \mathcal{N} \arrow[u, "\hom_r"']
\end{tikzcd}$$

as defined in §II.2.2. It follows that, any of the lifted functors (II.6.1) determines the others, capturing an important feature of the classical setting, where the criteria for a two-variable Quillen adjunction can be stated in terms of the pushout-product or one of the pullback-homs alone.

We can now give the precise version of Definition II.4.4.

Definition II.6.3. Given a two-variable adjunction $(\otimes, \hom_\ell, \hom_r)\colon \mathcal{K}\times\mathcal{M} \to \mathcal{N}$, a *two-variable adjunction of awfs* $\otimes\colon (\mathbb{C}', \mathbb{F}') \times (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ consists of lifted functors

$$\begin{array}{ll} -\hat{\otimes}-\colon \mathbb{C}'\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg} & Q'i \otimes Qj \xrightarrow{\lambda_{i,j}} E(i\hat{\otimes}j) \\ \hat{\hom}_\ell(-,-)\colon \mathbb{C}'\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg} & Q\hat{\hom}_\ell(i,f) \xrightarrow{\rho^\ell_{i,f}} \hom_\ell(Q'i, Ef) \\ \hat{\hom}_r(-,-)\colon \mathbb{C}\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}\textbf{-alg} \to \mathbb{F}'\textbf{-alg} & Q'\hat{\hom}_r(j,f) \xrightarrow{\rho^r_{j,f}} \hom_r(Qj, Ef) \end{array}$$

characterized by natural transformations $\vec{\lambda}$, $\vec{\rho^\ell}$, and $\vec{\rho^r}$ whose components λ, ρ^ℓ, ρ^r are parameterized mates.

Without an abundance of examples, this definition would be too strong. However, as in Part I, we can use a composition criterion, extending Theorem II.5.12, to recognize two-variable adjunctions of awfs.

Theorem II.6.4 (Composition criterion)**.** *Let* $(\otimes, \hom_\ell, \hom_r)\colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$ *be a two-variable adjunction between categories equipped with awfs* $(\mathbb{C}', \mathbb{F}'), (\mathbb{C}, \mathbb{F}), (\mathbb{L}, \mathbb{R})$*. A single lifted functor*

$$\mathbb{C}'\textbf{-coalg}\times\mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg}, \quad \mathbb{C}'\textbf{-coalg}^{\mathrm{op}}\times\mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg}, \quad \text{or} \quad \mathbb{C}\textbf{-coalg}^{\mathrm{op}}\times\mathbb{R}\textbf{-alg} \to \mathbb{F}'\textbf{-alg}$$

specifies a two-variable adjunction of awfs if and only if it satisfies the criterion of Theorem II.5.12 or its dual, as appropriate, in each variable.

Proof. This follows easily from the calculus of parameterized mates. Without loss of generality, suppose given $\mathbb{C}'\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg}$. Evaluating at $i \in \mathbb{C}'\textbf{-coalg}$ defines a lifted functor $i\hat{\otimes}-\colon \mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg}$ characterized by a natural transformation $\lambda(i)\colon \mathrm{cod}\, i \times Q- \Rightarrow E(i\hat{\otimes}-)$. By Theorem II.5.12, the mates $\rho^\ell(i)\colon Q\hat{\hom}_\ell(i,-) \Rightarrow \hom_\ell(\mathrm{cod}\, i, E-)$ specify lifted functors $\hat{\hom}_\ell(i,-)\colon \mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg}$. Because the $\lambda(i)$ are natural in $\mathbb{C}'\textbf{-coalg}$, so are the $\rho^\ell(i)$, applying Lemma II.2.11. By definition of the lifted functor $\hat{\hom}_\ell(i,-)$ in terms of $\rho^\ell(i)$ and an easy diagram chase, these functors assemble into a lifted bifunctor

$$\hat{\hom}_\ell(-,-)\colon \mathbb{C}'\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg}.$$

It remains only to show that the characterizing natural transformation of this functor is a parameterized mate of the natural transformation characterizing the original $\mathbb{C}'$-**coalg** $\times$ $\mathbb{C}$-**coalg** $\to$ $\mathbb{L}$-**coalg**. By definition $\lambda\colon Q' \otimes Q \Rightarrow E(-\hat{\otimes}-)$ and $\rho^\ell\colon Q\widehat{\hom}_\ell(-,-) \Rightarrow \hom_\ell(Q', E)$ are obtained by composing $\lambda(C'-)$ and $\rho^\ell(C'-)$ with the comonad counit. Explicitly, $\lambda_{i,-}$ and $\rho^\ell_{i,-}$ are the pasted composites displayed below in the squares involving the left and right adjoints respectively

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathcal{M}^{\mathbf{2}} \arrow[r, "1"] \arrow[d, shift right=1.5ex, "i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\Searrow(\vec{e}_i)^*", pos=0.3] \arrow[dr, phantom, "(\vec{\epsilon}_i)_*\Swarrow", pos=0.7] & \mathcal{M}^{\mathbf{2}} \arrow[r, "Q"] \arrow[d, shift right=1.5ex, "C'i\hat{\otimes}-"'] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\Searrow\rho^\ell(C'i)", pos=0.3] \arrow[dr, phantom, "\lambda(C'i)\Swarrow", pos=0.7] & \mathcal{M} \arrow[d, shift right=1.5ex, "Q'i\otimes-"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{N}^{\mathbf{2}} \arrow[r, "1"'] \arrow[u, shift right=1.5ex, "\widehat{\hom}_\ell(i,-)"'] & \mathcal{N}^{\mathbf{2}} \arrow[r, "E"'] \arrow[u, shift right=1.5ex, "\widehat{\hom}_\ell(C'i,-)"'] & \mathcal{N} \arrow[u, shift right=1.5ex, "\hom_\ell(Q'i,-)"']
\end{tikzcd}$$

Hence, $\lambda_{i,-}$ and $\rho^\ell_{i,-}$ are pointwise mates by Theorem II.2.7 and thus parameterized mates by Lemma II.2.12. $\square$

In particular, this allows us to prove the now-familiar cellularity condition, classifying all two-variable adjunctions of awfs $(\mathbb{C}', \mathbb{F}') \times (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ whenever the first two awfs are cofibrantly generated.

Theorem II.4.8. *Suppose* $\mathcal{I}$ *generates* $(\mathbb{C}', \mathbb{F}')$ *on* $\mathcal{K}$ *and* $\mathcal{J}$ *generates* $(\mathbb{C}, \mathbb{F})$ *on* $\mathcal{M}$ *and* $\mathcal{N}$ *has an awfs* $(\mathbb{L}, \mathbb{R})$*. Then* $(\otimes, \hom_\ell, \hom_r)$ *gives rise to a two-variable adjunction of awfs if and only if* $\mathcal{I}\hat{\otimes}\mathcal{J}$ *is cellular, that is, if and only if there is a lift*

$$\begin{tikzcd}
\mathcal{I} \times \mathcal{J} \arrow[r, dashed] \arrow[d] & \mathbb{L}\text{-}\mathbf{coalg} \arrow[d] \\
\mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \arrow[r, "-\hat{\otimes}-"] & \mathcal{N}^{\mathbf{2}}
\end{tikzcd}$$

Proof. By Theorem II.5.15, for each fixed $i \in \mathcal{I}$, the functor $i\hat{\otimes}-\colon \mathcal{J} \to \mathbb{L}$-**coalg** determines an adjunction of awfs $(i\hat{\otimes}-, \widehat{\hom}_\ell(i,-))\colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$. A morphism $(a, b)\colon i' \Rightarrow i$ in $\mathcal{I}$ determines a natural transformation $\widehat{\hom}_\ell(i,-) \Rightarrow \widehat{\hom}_\ell(i',-)$ on the arrow categories. The lifted functors $\widehat{\hom}_\ell(i,-)\colon \mathbb{R}$-**alg** $\to$ $\mathbb{F}$-**alg** assemble into a functor

$$\widehat{\hom}_\ell(-,-)\colon \mathcal{I}^{\mathrm{op}} \times \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$$

if and only if each component $\widehat{\hom}_\ell(i, f) \Rightarrow \widehat{\hom}_\ell(i', f) \in \mathcal{M}^{\mathbf{2}}$ lifts to a morphism in $\mathcal{J}^{\boxslash}$. If this is the case, it follows that the natural transformations $\rho^\ell(i)$ characterizing each lifted functor $\widehat{\hom}_\ell(i, -)$ are natural in $\mathcal{I}$. By Lemma II.2.11, their mates are then also natural in $\mathcal{I}$, and so the lifts of the left adjoints will assemble into a functor $\mathcal{I} \times \mathbb{C}$-**coalg** $\to \mathbb{L}$-**coalg**, as we saw in the proof of Theorem II.6.4.

In other words, we must show that each lifted functor $\widehat{\hom}_\ell(i, -)$ assigns, to each $\mathbb{R}$-algebra f, solutions to all lifting problems between $j \in \mathcal{J}$ and $\widehat{\hom}_\ell(i, f)$ that are natural with respect to morphisms in $\mathcal{J}$ (so that this defines an object of $\mathcal{J}^{\boxslash}$), $\mathbb{R}$-**alg** (so that this defines a functor), and $\mathcal{I}$ (so that the functors assemble into a bifunctor). The construction of Theorem II.5.15, which solves the adjunct lifting problem and using the functor $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg** and the awfs $(\mathbb{L}, \mathbb{R})$, has all of these properties.

To see this, note that the top composite below specifies the chosen solution to any lifting problem; in other words, this defines $\widehat{\hom}_\ell(i, f)$ as an element of $\mathcal{J}^{\boxslash}$.

$$\begin{tikzcd}
\mathcal{N}^{\mathbf{2}}(j, \widehat{\hom}_\ell(i, f)) \arrow[r, "\cong"] \arrow[d, "\widehat{\hom}_\ell((a,b),f)_*"'] & \mathcal{N}^{\mathbf{2}}(i \hat{\otimes} j, f) \arrow[r, "\text{solve}"] \arrow[d, "((a,b)\hat{\otimes} j)^*"'] & \mathcal{N}(B \otimes L, X) \arrow[r, "\cong"] \arrow[d, "(b\otimes L)^*"] & \mathcal{M}(L, \hom_\ell(B, X)) \arrow[d, "\hom_\ell(b,X)_*"] \\
\mathcal{N}^{\mathbf{2}}(j, \widehat{\hom}_\ell(i', f)) \arrow[r, "\cong"'] & \mathcal{N}^{\mathbf{2}}(i' \hat{\otimes} j, f) \arrow[r, "\text{solve}"'] & \mathcal{N}(B' \otimes L, X) \arrow[r, "\cong"'] & \mathcal{M}(L, \hom_\ell(B', X))
\end{tikzcd} \tag{II.6.5}$$

Given $(a, b)\colon i' \Rightarrow i$ in $\mathcal{I}$, the left square and right squares commute by naturality of the parameterized adjunctions in $\mathcal{K}^{\mathbf{2}}$ and $\mathcal{K}$. The essential point is that the middle square, whose horizontal arrows use the awfs $(\mathbb{L}, \mathbb{R})$ to solve the lifting problem, also commutes, by functoriality of $\mathcal{I} \times \mathcal{J} \to \mathbb{L}$-**coalg** in the first variable and the fact that morphisms of $\mathbb{L}$-coalgebras preserve the chosen solutions to lifting problems against $\mathbb{R}$-algebras. The left bottom composite of the rectangle choses solutions to lifting problems against $\widehat{\hom}_\ell(i', f)$ that factor through lifting problems against $\widehat{\hom}_\ell(i, f)$. Commutativity of (II.6.5) asserts that these are the same lifts obtained by solving the lifting problem against $\widehat{\hom}_\ell(i, f)$ and then composing. This says exactly that the arrow in $\mathcal{M}^{\mathbf{2}}$ induced from $(a, b)\colon i' \Rightarrow i$ lifts to $\mathcal{J}^{\boxslash}$, as desired.

We now use the lifted functor

$$\mathcal{I} \times \mathbb{C}\text{-}\mathbf{coalg} \to \mathbb{L}\text{-}\mathbf{coalg} \tag{II.6.6}$$

and repeat the argument just given. For each fixed $j \in \mathbb{C}$-**coalg**, the functor $-\hat{\otimes} j\colon \mathcal{I} \to \mathbb{L}$-**coalg** determines an adjunction of awfs

$$(-\hat{\otimes} j, \widehat{\mathrm{hom}}_r(j,-))\colon (\mathbb{C}', \mathbb{F}') \to (\mathbb{L}, \mathbb{R})$$

that depends also on the $\mathbb{C}$-coalgebra structure for j. As above, the characterizing maps are also natural in $\mathbb{C}$-**coalg** and so the lifted functors assemble into functors

$$-\hat{\otimes}-\colon \mathbb{C}'\text{-}\mathbf{coalg} \times \mathbb{C}\text{-}\mathbf{coalg} \to \mathbb{L}\text{-}\mathbf{coalg} \qquad \widehat{\mathrm{hom}}_r(-,-)\colon \mathbb{C}\text{-}\mathbf{coalg}^{\mathrm{op}} \times \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}'\text{-}\mathbf{alg}. \tag{II.6.7}$$

Furthermore, their characterizing natural transformations are parameterized mates by the second half of the proof of Theorem II.6.4.

The last step is subtle. We use the dual of the composition criterion of Theorem II.5.12 to show that for each $f \in \mathbb{R}$-**alg**, the lift

$$\widehat{\mathrm{hom}}_r(-,f)\colon \mathbb{C}\text{-}\mathbf{coalg}^{\mathrm{op}} \to \mathbb{F}'\text{-}\mathbf{alg}$$

obtained by restricting the right-hand functor of (II.6.7) is a lax morphism of awfs. It follows from Theorem II.6.4 that the parameterized mates of the natural transformations characterizing the lifted functors (II.6.7) define the final lifted functor

$$\widehat{\mathrm{hom}}_\ell(-,-)\colon \mathbb{C}'\text{-}\mathbf{coalg}^{\mathrm{op}} \times \mathbb{R}\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg},$$

completing the desired two-variable adjunction of awfs.

In order to apply Theorem II.5.12, we must show that given $j\colon J \to K$, $k\colon K \to L \in \mathbb{C}$-**coalg**, the unlabeled solutions $\widehat{\mathrm{hom}}_r(-,f)\colon \mathbb{C}'\text{-}\mathbf{coalg}^{\mathrm{op}} \to \mathbb{F}'\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$ assigns to lifting problems below (where we have abbreviated hom_r using exponential notation) agree.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\mathcal{J}\ni i"'] & X^L \arrow[r, equal] \arrow[d, "\widehat{\mathrm{hom}}_r(k,f)"] & X^L \arrow[r, "X^k"] \arrow[d, "\widehat{\mathrm{hom}}_r(kj,f)"] & X^K \arrow[d, "\widehat{\mathrm{hom}}_r(j,f)"] \\
B \arrow[ur, dashed] \arrow[urr, dashed] \arrow[urrr, dashed, "d"' near end] \arrow[r, dashed, "b\times d"'] \arrow[rr, bend right, "b\times c"'] & Y^L \underset{Y^K}{\times} X^K \arrow[r, "1\times_{Y^j} X^j"'] & Y^L \underset{Y^J}{\times} X^J \arrow[r, "Y^k\times 1"'] & Y^K \underset{Y^J}{\times} X^J
\end{tikzcd} \tag{II.6.8}$$

The chosen lifts are defined by solving the adjunct lifting problems using the awfs $(\mathbb{L},\mathbb{R})$ and (II.6.6); transposing across the adjunction, it suffices to show that the unlabeled chosen solutions in the diagram

$$\begin{tikzcd}
A\otimes K \underset{A\otimes J}{\sqcup} B\otimes J \arrow[r, "A\otimes k\sqcup 1"] \arrow[d, "i\hat{\otimes} j"'] & A\otimes L \underset{A\otimes J}{\sqcup} B\otimes J \arrow[r, "1\sqcup_{A\otimes j}B\otimes j"] \arrow[d, "i\hat{\otimes}(kj)"'] \arrow[rr, bend left, "a^\sharp\sqcup c^\sharp"] & A\otimes L \underset{A\otimes K}{\sqcup} B\otimes K \arrow[r, "a^\sharp\sqcup d^\sharp"] \arrow[d, "i\hat{\otimes} k"'] & X \arrow[d, "f"] \\
B\otimes K \arrow[r, "B\otimes k"'] \arrow[urrr, dashed, "d"] & B\otimes L \arrow[r, equal] \arrow[urr, dashed] & B\otimes L \arrow[r, "b^\sharp"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

agree.

If j and k have $\mathbb{C}$-coalgebra structures s and t and f has $\mathbb{R}$-algebra structure r, the left-most unlabeled solution is defined to be

$$B\otimes L \xrightarrow{B\otimes(t\bullet s)} B\otimes Q(kj) \xrightarrow{\lambda(i)_{kj}} E(i\hat{\otimes}(kj)) \xrightarrow{E(a^\sharp\sqcup c^\sharp,b^\sharp)} Ef \xrightarrow{r} X \tag{II.6.9}$$

while the right-most is defined to be

$$B\otimes L \xrightarrow{B\otimes t} B\otimes Qk \xrightarrow{\lambda(i)_{k}} E(i\hat{\otimes}k) \xrightarrow{E(a^\sharp\sqcup d^\sharp,b^\sharp)} Ef \xrightarrow{r} X \tag{II.6.10}$$

where d, by naturality of $\lambda(i)$ with respect to the morphism $(1,k)\colon j \Rightarrow kj$ of $\mathcal{M}^{\mathbf{2}}$, is

$$B\otimes K \xrightarrow{B\otimes s} B\otimes Qj \xrightarrow{B\otimes Q(1,k)} B\otimes Q(kj) \xrightarrow{\lambda(i)_{kj}} E(i\hat{\otimes}(kj)) \xrightarrow{E(a^\sharp\sqcup c^\sharp,b^\sharp)} Ef \xrightarrow{r} X$$

We use this factorization of d to factor the morphism $(a^\sharp \sqcup d^\sharp, b^\sharp)\colon i\hat{\otimes}k \Rightarrow f$ of $\mathcal{N}^{\mathbf{2}}$ as

$$\begin{tikzcd}
A\otimes L \underset{A\otimes K}{\sqcup} B\otimes K \arrow[r, "1\sqcup B\otimes[Q(1{,}k)s]"] \arrow[d, "i\hat{\otimes}k"'] & A\otimes L \underset{A\otimes Q(kj)}{\sqcup} B\otimes Q(kj) \arrow[r, "\lambda'(i)_{kj}"] \arrow[d, "i\hat{\otimes}F(kj)"'] & E(i\hat{\otimes}(kj)) \arrow[r, "E(a^\sharp\sqcup c^\sharp{,}b^\sharp)"] \arrow[d, "R(i\hat{\otimes}(kj))"'] & Ef \arrow[r, "r"] \arrow[d, "Rf"'] & X \arrow[d, "f"'] \\
B\otimes L \arrow[r, equal] & B\otimes L \arrow[r, equal] & B\otimes L \arrow[r, "b^\sharp"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Applying the functor E and substituting this factorization for $E(a^\sharp \sqcup d^\sharp, b^\sharp)$ in (II.6.10),

$$\begin{aligned}
&r{\cdot}E(r,1)\cdot E(E(a^\sharp \sqcup c^\sharp, b^\sharp), b^\sharp)\cdot E(\lambda'(i)_{kj},1)\cdot E(1\sqcup B\otimes Q(1,k)s,1)\cdot \lambda(i)_k\cdot B\otimes t\\
&\quad = r\cdot E(r,1)\cdot E(E(a^\sharp \sqcup c^\sharp, b^\sharp), b^\sharp)\cdot E(\lambda'(i)_{kj},1)\cdot \lambda(i)_{F(kj)}\cdot B\otimes Q(Q(1,k)s,1)\cdot B\otimes t\\
&\quad = r\cdot \mu_f\cdot E(E(a^\sharp \sqcup c^\sharp, b^\sharp), b^\sharp)\cdot E(\lambda'(i)_{kj},1)\cdot \lambda(i)_{F(kj)}\cdot B\otimes Q(Q(1,k)s,1)\cdot B\otimes t\\
&\quad = r\cdot E(a^\sharp \sqcup c^\sharp, b^\sharp)\cdot \mu_{i\hat{\otimes}(kj)}\cdot E(\lambda'(i)_{kj},1)\cdot \lambda(i)_{F(kj)}\cdot B\otimes Q(Q(1,k)s,1)\cdot B\otimes t\\
&\quad = r\cdot E(a^\sharp \sqcup c^\sharp, b^\sharp)\cdot \lambda(i)_{kj}\cdot B\otimes \mu_{kj}\cdot B\otimes Q(Q(1,k)s,1)\cdot B\otimes t\\
&\quad = r\cdot E(a^\sharp \sqcup c^\sharp, b^\sharp)\cdot \lambda(i)_{kj}\cdot B\otimes (t\bullet s)
\end{aligned}$$

by naturality of $\lambda(i)$, associativity of r, naturality of μ, the monad pentagon for $\lambda(i)_{kj}$ which holds because $\lambda(i)$ defines a colax morphism of awfs, and the definition of $t \bullet s$. This last line is (II.6.9), completing the proof. □

An advantage of the calculus of parameterized mates is that the following proof requires no diagram chasing.

Lemma II.6.11. *Two-variable adjunctions of awfs can be composed with adjunctions of awfs (pointing in the correct direction) in any of the variables to obtain another two-variable adjunction of awfs.*

Proof. The functors lifting the left adjoints can clearly be composed; unpacking Lemma II.2.9, the natural transformation characterizing the composite is a pasted composite of the natural transformations characterizing each piece. By Lemma II.2.13 and the calculus of parameterized mates, the parameterized mates of this composite natural transformation are obtained by pasting the mates of characterizing natural transformations, and hence characterizes the functor obtained by composing the appropriate right adjoints. So we see that the composite is again a two-variable adjunction of awfs. □

Corollary II.6.12. *The lifted functors (II.4.6) commute if and only if the lifts*

$$\begin{tikzcd}
& \mathbb{C}'_t\text{-}\mathbf{coalg}\times\mathbb{R}_t\text{-}\mathbf{alg} \arrow[dl, "1\times\xi^{\mathcal{N}}"'] \arrow[ddr, "\xi^{\mathcal{K}}\times 1"] \arrow[drr, dashed, two heads] & & & \\
\mathbb{C}'_t\text{-}\mathbf{coalg}\times\mathbb{R}\text{-}\mathbf{alg} \arrow[rrr, two heads] \arrow[ddr, "\xi^{\mathcal{K}}\times 1"'] & & & \mathbb{F}_t\text{-}\mathbf{alg} \arrow[ddr, "\xi^{\mathcal{M}}"] & \\
& & \mathbb{C}'\text{-}\mathbf{coalg}\times\mathbb{R}_t\text{-}\mathbf{alg} \arrow[ur, two heads] \arrow[dl, "1\times\xi^{\mathcal{N}}"] & & \\
& \mathbb{C}'\text{-}\mathbf{coalg}\times\mathbb{R}\text{-}\mathbf{alg} \arrow[rrr] & & & \mathbb{F}\text{-}\mathbf{alg}
\end{tikzcd} \tag{II.6.13}$$

of $\hat{\hom}_\ell$ *commute, and similarly for* $\hat{\hom}_r$.

Proof. The mate of the composite two-variable adjunction of awfs defined by each commuting square of (II.4.6) characterizes the corresponding commuting square of (II.6.13). □

II.7 Monoidal algebraic model structures

Finally, we define

Definition II.7.1. A *monoidal algebraic model structure* on a closed monoidal category

$$(\otimes, \hom_\ell, \hom_r) \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

with monoidal unit I is an algebraic model structure $\xi \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ such that

(i) $(\otimes, \hom_\ell, \hom_r)$ is an algebraic Quillen two-variable adjunction

(ii) tensoring on either side with the cofibrant replacement comonad counit $\epsilon_I : QI \to I$ sends (algebraic) cofibrant objects to weak equivalences.

Monoidal algebraic model categories are in particular monoidal model categories in the sense of [Hov99]. It makes no difference whether condition (ii) is stated for algebraic cofibrant objects or ordinary cofibrant objects. If the unit is cofibrant, (ii) is automatic from (i) and Ken Brown's lemma.

In the case where the monoidal structure is symmetric, a two-variable adjunction of awfs $(\mathbb{C}_t, \mathbb{F}) \times (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{C}_t, \mathbb{F})$ gives rise to a two-variable adjunction of awfs $(\mathbb{C}, \mathbb{F}_t) \times (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}_t, \mathbb{F})$ by composing with the symmetry isomorphism. When $(\mathbb{C}, \mathbb{F}_t)$ is generated by $\mathcal{J}$, by Theorem II.4.9, the two-variable adjunction of awfs $(\mathbb{C}, \mathbb{F}_t) \times (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{C}, \mathbb{F}_t)$ commutes with the symmetry isomorphism if and only if the functor $\mathcal{J} \hat{\otimes} \mathcal{J} \to \mathbb{C}$-**coalg** is defined symmetrically. Thus,

Definition II.7.2. On a closed symmetric monoidal category

$$(\otimes, \hom, \hom) \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

with monoidal unit I, a *symmetric monoidal algebraic model structure* is an algebraic model structure such that

(i) $(\otimes, \hom, \hom)$ is an algebraic Quillen two-variable adjunction such that the lifted functors of algebraic (trivial) cofibrations commute up to isomorphism with the symmetry isomorphism

(ii) tensoring with $\epsilon_I : QI \to I$ sends (algebraic) cofibrant objects to weak equivalences

We now use Theorems II.4.8, II.4.9, and II.6.4 to find examples.

Theorem II.7.3. *The folk model structure on* **Cat** *is a maximally coherent (cartesian) monoidal algebraic model structure.*

Proof. The folk model structure on **Cat** is generated by the following sets of functors

$$\mathcal{I} = \left\{ \begin{tikzcd} \emptyset \arrow[dd, mapsto, "c"'] \\ \\ \bullet \end{tikzcd}, \begin{tikzcd} \bullet & {} \arrow[dd, mapsto, "d"'] & \bullet \\ & & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd}, \begin{tikzcd} \bullet \arrow[rr, shift left] \arrow[rr, shift right] & {} \arrow[dd, mapsto, "e"'] & \bullet \\ & & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd} \right\} \qquad \mathcal{J} = \left\{ \begin{tikzcd} & \bullet \arrow[dd, mapsto, "j"'] \\ & \\ \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd} \right\}$$

Write $\mathbf{I}$ for the codomain of j, that is, the free-standing isomorphism

By Corollary II.3.12, **Cat** has an algebraic model structure generated by $\mathcal{I}$ and $\mathcal{J}$ if and only if j is $\mathcal{I}$-cellular, i.e., if and only if there is a functor $\mathcal{J} \to \mathbb{C}$-**coalg**, where $\mathbb{C}$ is the comonad of the awfs generated by $\mathcal{I}$. The comonad $\mathbb{C}$ is particularly simple to describe. By a dimension argument, it can be constructed by running Garner's small object argument first using the generator c, then using d, and then using e. Each process converges after a single step, which means that the comonad $\mathbb{C}$ is constructed in three steps: each of which forms a single pushout of the coproduct over squares of the generator in question. Verifying these assertions is an easy exercise once one understands the small object argument; see [Gar09, §4] or §I.2.5.

The resulting functorial factorization is equivalent to the usual mapping cylinder construction:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d] \\ A \arrow[r, "i_0"'] & A \times \mathbf{I} \arrow[r] & A \times \mathbf{I} \coprod_A B \end{tikzcd}$$

Concretely, $A \times \mathbf{I} \coprod_A B$ is the unique category with objects $A_0 \coprod B_0$ such that the functor (f, id) to B is fully faithful, and hence a trivial fibration. The bottom composite above is used to define the functorial factorization

$$A \xrightarrow{f} B \quad \mapsto \quad A \xrightarrow{Cf := i_0} A \times \mathbf{I} \coprod_A B \xrightarrow{F_t f := (f, \mathrm{id})} B$$

Here A does not necessarily inject into the mapping cylinder, because arrows in A that become equal in B get identified, but it is injective on objects; hence i_0 is a cofibration.

On morphisms, the functor $C : \mathbf{Cat}^{\mathbf{2}} \to \mathbf{Cat}^{\mathbf{2}}$ sends

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "v"'] & B' \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} A \arrow[rr, "u"] \arrow[d, "i_0"'] & & A' \arrow[d, "i_0"] \\ A \times \mathbf{I} \coprod_A B \arrow[rr, "(u\times\mathrm{id},v)"'] & & A' \times \mathbf{I} \coprod_{A'} B' \end{tikzcd}$$

The counit and comultiplication natural transformations have components

$$\vec{\epsilon_f} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, "i_0"'] & A \arrow[d, "f"] \\ A \times \mathbf{I} \coprod_A B \arrow[r, "(f\times\mathrm{id},\mathrm{id})"'] & B \end{tikzcd} \quad \text{and} \quad \vec{\delta_f} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, "i_0"'] & A \arrow[d, "i_0"] \\ A \times \mathbf{I} \coprod_A B \arrow[r, "(i_0,\mathrm{id})"'] & A \times \mathbf{I} \coprod_A (A \times \mathbf{I} \coprod_A B) \end{tikzcd}$$

In particular, δ_f includes $A \times \mathbf{I}$ into the first copy of this object in the coproduct; the second copy is not in the image of this map.

It turns out that every cofibration in **Cat** admits a unique $\mathbb{C}$-coalgebra structure. Suppose f is injective on objects. Then there is a unique arrow from its codomain to the mapping cylinder so that

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "f"'] & A \times \mathbf{I} \coprod_A B \arrow[d, "(f,\mathrm{id})"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

commutes. Objects $b \in B$ of the form $b = f(a)$ necessarily map to $(a, 0) \in A \times \mathbf{I}$ while objects not in the image of A necessarily map to themselves in B. Because (f, id) is full and faithful, there is no need to define the functor $B \to A \times \mathbf{I} \coprod_A B$ on morphisms. It is easy to check that this arrow makes f a $\mathbb{C}$-coalgebra; its clear that there is no other possibility. In particular, the cofibration j is automatically $\mathfrak{I}$-cellular, and $\mathfrak{I}$ and $\mathfrak{J}$ give **Cat** an algebraic model structure.

To show that it is monoidal, we apply Theorem II.4.8 and examine pushout-products of generating (trivial) cofibrations. must check the cellularity and compatibility conditions. $\mathfrak{I}$-cellularity is automatic from the fact that **Cat** is a monoidal model category in the ordinary sense [Lac07], so we must only check that the pushout-product of elements of $\mathfrak{I}$ with elements of $\mathfrak{J}$ is $\mathfrak{J}$-cellular.

By an easy computation

$$c\hat{\times}j = j \quad \text{and} \quad d\hat{\times}j = e\hat{\times}j = \mathrm{id}_{\mathbf{2}\times\mathbf{I}}.$$

The first of these has a canonical and the second a unique $\mathbb{C}_t$-coalgebra structure. This defines

$$-\hat{\times}-\colon \mathbb{C}\text{-}\mathbf{coalg} \times \mathbb{C}_t\text{-}\mathbf{coalg} \to \mathbb{C}_t\text{-}\mathbf{coalg}.$$

Because $\mathbb{C}$-coalgebra structures are unique, the lifted functors with codomain $\mathbb{C}$-**coalg** automatically commute. Because the functors $\mathbb{C}$-**coalg** × $\mathbb{C}_t$-**coalg** → $\mathbb{C}_t$-**coalg** and $\mathbb{C}_t$-**coalg** × $\mathbb{C}$-**coalg** → $\mathbb{C}_t$-**coalg** are defined symmetrically and $\mathcal{J}$ consists of a single generator, we can apply Theorem II.4.9 to conclude that

$$\begin{tikzcd}
\mathbb{C}_t\text{-}\mathbf{coalg} \times \mathbb{C}_t\text{-}\mathbf{coalg} \arrow[r] \arrow[d] & \mathbb{C}_t\text{-}\mathbf{coalg} \times \mathbb{C}\text{-}\mathbf{coalg} \arrow[d] \\
\mathbb{C}\text{-}\mathbf{coalg} \times \mathbb{C}_t\text{-}\mathbf{coalg} \arrow[r] & \mathbb{C}_t\text{-}\mathbf{coalg}
\end{tikzcd}$$

commutes. □

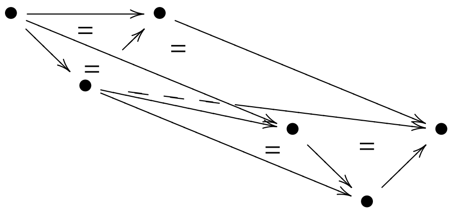
Theorem II.7.4. *Quillen's original model structure on simplicial sets is a monoidal algebraic model structure with the usual generating (trivial) cofibrations.*

Proof. It is well-known that simplicial sets form a symmetric monoidal model category generated by the usual sets $\mathcal{I}$ and $\mathcal{J}$ of sphere and horn inclusions. As with **Cat**, a dimension argument can be used to give an inductive description of the comonad $\mathbb{C}$ in such a way that it is clear that all cofibrations admit unique $\mathbb{C}$-coalgebra structures. In particular, the generators $\mathcal{J}$ are $\mathcal{I}$-cellular—we give an explicit description of their $\mathcal{I}$-cellular structures below—defining an algebraic model structure.

Because all cofibrations are uniquely cellular, to show that the cartesian product forms an algebraic Quillen two-variable adjunction, we need only worry about the algebraic trivial cofibrations. By work of André Joyal in the appendices to [Joy08], elements of the pushout-product $\mathcal{J}\hat{\times}\mathcal{I}$ is cellular. This is to say, the maps in $\mathcal{J}\hat{\times}\mathcal{I}$ can be factored as composites of pushouts of the generating horn inclusions $\mathcal{J}$. By Theorem II.4.8, we are done. □

The monoidal algebraic model structure on simplicial sets is *mostly* but not *maximally* coherent. It is instructive to see why. It is mostly coherent because all monomorphisms of simplicial sets have a unique $\mathbb{C}$-coalgebra structure, so the lifted functors with codomain $\mathbb{C}$-**coalg** commute because the functors they are lifting commute.

However, the pushout-product of a pair of generating trivial cofibrations is assigned two different $\mathbb{C}_t$-coalgebra structures, depending on which generator is regarded as a $\mathbb{C}$-coalgebra. We illustrate with an example. Write $h^1_0 \colon \Lambda^1_0 \to \Delta^1$ and $h^2_1 \colon \Lambda^2_1 \to \Delta^2$ for the inclusions of 1- and 2-dimensional horns missing the 0th and 1st faces, respectively. The pushout-product $h^1_0 \hat{\times} h^2_1$ has codomain the solid cylinder $\Delta^1 \times \Delta^2$ and domain a hollow "trough" with one of the end triangles and the two $\Delta^1 \times \Delta^1$ missing.



(II.7.5)

For simplicial sets, $\mathbb{C}$-coalgebra structures are precisely $\mathcal{J}$-cellular structures, that is, factorizations of a given monomorphism into pushouts of coproducts of elements of $\mathcal{J}$ filtered by attaching degree; see the preface. The $\mathcal{J}$-cellular structure assigned the horn inclusion h^n_k is given by the factorization

$$\Lambda^n_k \longrightarrow \partial\Delta^n \longrightarrow \Delta^n \tag{II.7.6}$$

The first map is a pushout of $\partial\Delta^{n-1} \to \Delta^{n-1}$ and attaches the "missing face" to the horn; the second map fills the resulting sphere.

The following general lemma, stated using the notation relevant to this example, will facilitate our computation. This is an application of the converse of the composition criterion of Theorem II.6.4.

Lemma II.7.7. *Given* $i \colon A \to B \in \mathbb{C}_t$-**coalg** *and* $j \colon J \to K, k \colon K \to L \in \mathbb{C}$-**coalg**, *the lifted functor* $-\hat{\times}- \colon \mathbb{C}_t$-**coalg** $\times\, \mathbb{C}$-**coalg** $\to \mathbb{C}_t$-**coalg** *of a two-variable adjunction of awfs assigns* $i\hat{\times}(kj)$ *the* $\mathbb{C}_t$*-coalgebra structure of the composite of the following pushout of the* $\mathbb{C}_t$*-coalgebra*

$i\hat{\times}j$ with the $\mathbb{C}_t$-coalgebra $i\hat{\times}k$

$$\begin{tikzcd}
A\times K \underset{A\times J}{\sqcup} B\times J \arrow[r, "A\times k\sqcup 1"] \arrow[d, "i\hat{\times}j"'] & A\times L \underset{A\times J}{\sqcup} B\times J \arrow[d, "p=1\sqcup_{A\times j}B\times j"'] \arrow[rdd, bend left, "i\hat{\times}(kj)"] & \\
B\times K \arrow[r, "\iota"] \arrow[rrd, bend right=15, "B\times k"'] & A\times L \underset{A\times K}{\sqcup} B\times K \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[rd, "i\hat{\times}k"] & \\
 & & B\times L
\end{tikzcd} \tag{II.7.8}$$

Proof. It is straightforward to check that (II.7.8) makes sense, i.e., that the square is a pushout and gives the described factorization of $i\hat{\times}(kj)$. We compute the canonical $\mathbb{C}_t$-coalgebra structure assigned $i\hat{\times}(kj)$ as the composite of these maps and show that it agrees with that assigned $i\hat{\times}(kj)$ by the composition criterion. Write p for the pushout of $i\hat{\times}j$, and write z_j, z_k, z_p respectively for the $\mathbb{C}_t$-coalgebra structures assigned to $i\hat{\times}j$, $i\hat{\times}k$, and p. Because p is assigned the coalgebra structure of a pushout, z_p equals

$$A\times L \underset{A\times K}{\sqcup} B\times K \cong \left(A\times L \underset{A\times J}{\sqcup} B\times J\right) \underset{\sim}{\sqcup} B\times K \xrightarrow{C_t p\sqcup R(A\times k\sqcup 1,\iota)\cdot z_j} Rp.$$

The coalgebra structure assigned the composite is

$$B\times L \xrightarrow{z_k} R(i\hat{\times}k) \xrightarrow{R(R(1,i\hat{\times}k)\cdot z_p),1)} RF(i\hat{\times}(kj)) \xrightarrow{\mu_{i\hat{\times}(kj)}} R(i\hat{\times}(kj)) \tag{II.7.9}$$

By definition, $R(1, i\hat{\times}k)\cdot z_p$ is the top arrow of the lifting problem

$$\begin{tikzcd}
A\times L \underset{A\times K}{\sqcup} B\times K \arrow[r, "C_t(i\hat{\times}(kj))\sqcup R(A\times k\sqcup 1,B\times k)\cdot z_j"] \arrow[d, "i\hat{\times}k"'] & R(i\hat{\times}(kj)) \arrow[d, "F(i\hat{\times}(kj))"] \\
B\times L \arrow[r, equal] & B\times L
\end{tikzcd}$$

whose canonical solution is the composite (II.7.9). But this is precisely what is required by the composition criterion of Theorem II.5.12, which holds for the lifted functor $i\hat{\times}-$ obtained from a two-variable adjunction of awfs. □

By a similar dimension argument, $\mathbb{C}_t$-coalgebra structures on **sSet** are $\mathcal{J}$-cellular structures, that is sequences of monomorphisms which attach fillers for all previously unexamined horns. We use this intuition and the above lemma to compute the coalgebra structures assigned to $h^1_0 \hat{\times} h^2_1$ by the two lifted functors.

We first apply Lemma II.7.7 to the $\mathcal{I}$-cellular decomposition (II.7.6) of h^1_0. The pushout-product of h^2_1 with the inclusion $\emptyset \to \Delta^1$ is simply h^1_0. Hence, the $\mathcal{J}$-coalgebra structure assigned its pushouts, including in particular the first factor of $h^1_0 \hat{\times} h^2_1$ defined in Lemma II.7.7, first fills the Λ^2_1-horn on the front edges of (II.7.5) to obtain a "trough," before filling the "trough" in the way specified by the lifted functor $\mathcal{I} \times \mathcal{J} \to \mathbb{C}_t$-**coalg**.

On the other hand, the pushout-product of h^1_0 with $\partial\Delta^1 \to \Delta^1$ is the monomorphism

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[d] & & \bullet \\
\bullet \arrow[rr] & {} \arrow[d, mapsto] & \bullet \\
\bullet \arrow[rr] \arrow[d] \arrow[drr, "=", "="'] & {} & \bullet \arrow[d] \\
\bullet \arrow[rr] & & \bullet
\end{tikzcd}$$

This map has $\mathcal{J}$-cellular structure given by first filling the Λ^2_1-horn formed by the right and bottom edges and then filling the resulting Λ^2_0-horn formed by the top edge and the diagonal. Pushouts of this map inherit a similar $\mathcal{J}$-cellular structure. In particular the $\mathcal{J}$-cellular structure assigned $h^1_0 \hat{\times} h^2_1$ by this method first fills the top of the trough (II.7.5), at which point it must fill the end triangle very last, using a 3-dimensional horn, not a 2-dimensional one. So this $\mathbb{C}_t$-coalgebra structure can't possibly agree with the one assigned via the other lifted functor.

Remark II.7.10. We expect this sort of argument to apply to many situations, which is why we did not require monoidal algebraic model structures to be maximally coherent.

If $(\mathcal{M}, \times, *)$ is a monoidal category such that the monoidal unit is terminal, then there is a monoidal product $\wedge$ on $\mathcal{M}_*$ defined as follows. Write $\vee$ for the coproduct in $\mathcal{M}_*$. Given $x\colon * \to X, y\colon * \to Y$ in $\mathcal{M}_*$, the pushout

$$\begin{tikzcd}
X \vee Y \arrow[r, "(1\times y)\vee(x\times 1)"] \arrow[d] & X \times Y \arrow[d] \\
* \arrow[r] & X \wedge Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

defines a bifunctor $- \wedge -\colon \mathcal{M}_* \times \mathcal{M}_* \to \mathcal{M}_*$ that we call the smash product. The monoidal unit denoted $S^0 = (*)_+ = * \sqcup *$. See [Hov99, 4.2.9].

Theorem II.7.11. *If $\mathcal{M}$ is a monoidal algebraic model category and the monoidal unit $*$ is terminal and cofibrant, then $\mathcal{M}_*$ is also a monoidal algebraic model category, symmetric if $\mathcal{M}$ is.*

Proof. By what one might call the "hyper-cube pushout lemma," which is an application of the fact that colimits commute with each other, the top square in the cube below is a pushout.

$$\begin{tikzcd}
 & (A \vee L) \underset{A\vee K}{\sqcup} (B \vee K) \arrow[rr] \arrow[dl] \arrow[dd, "1"] & & A \times L \underset{A\times K}{\sqcup} B \times K \arrow[dl] \arrow[dd, "i \hat{\times} j"] \\
* \underset{*}{\sqcup} * \arrow[rr] \arrow[dd, "1"'] & & A \wedge L \underset{A\wedge K}{\sqcup} B \wedge K \arrow[dd, dashed, "i \hat{\wedge} j"] \arrow[ur, phantom, "\ulcorner", very near start] & \\
 & B \vee L \arrow[rr] \arrow[dl] & & B \times L \arrow[dl] \\
* \arrow[rr] & & B \wedge L \arrow[ur, phantom, "\urcorner", very near start] &
\end{tikzcd} \tag{II.7.12}$$

The left and bottom faces are pushouts tautologically and definitionally. It follows that the composite rectangle from the top left edge to the bottom right edge is a pushout, and hence that the right face is a pushout. This says that the pushout-smash-product $i \hat{\wedge} j$ is a pushout of the pushout-product $i \hat{\times} j$.

On account of the pullbacks

$$\begin{tikzcd}
(\mathbb{C}_t)_*\textbf{-coalg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_t\textbf{-coalg} \arrow[d] & \mathbb{C}_*\textbf{-coalg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}\textbf{-coalg} \arrow[d] \\
(\mathcal{M}_*)^{\mathbf{2}} \arrow[r, "U^{\mathbf{2}}"'] & \mathcal{M}^{\mathbf{2}} & (\mathcal{M}_*)^{\mathbf{2}} \arrow[r, "U^{\mathbf{2}}"'] & \mathcal{M}^{\mathbf{2}}
\end{tikzcd}$$

$(\mathbb{C}_t)_*$-coalgebra or $\mathbb{C}_*$-coalgebra structures for based maps are given by $\mathbb{C}_t$-coalgebra or $\mathbb{C}$-coalgebra structures for the underlying arrows. Hence, we define, for instance, the lifted functor $-\hat{\wedge}-\colon (\mathbb{C}_t)_*\textbf{-coalg} \times \mathbb{C}_*\textbf{-coalg} \to (\mathbb{C}_t)_*\textbf{-coalg}$ by assigning $i \hat{\wedge} j$ the $\mathbb{C}_t$-coalgebra structure created by the pushout of the $\mathbb{C}_t$-coalgebra $i \hat{\times} j$.

To see that this defines a two-variable adjunction of awfs, we appeal to Theorem II.6.4 and show that this functor satisfies the composition criterion in both variables. This follows easily from the fact that the coalgebra structures assigned to the pushout-smash-products displayed in the front of the diagram below are determined by the coalgebra structures assigned to the pushout-products displayed at the back. By the universal property of the pushouts, the canonical solutions to lifting problems against the front arrows will behave analogously to those against the back arrows; and these, by hypothesis, satisfy the composition criterion.

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[dr] \arrow[dd, "i\hat{\times}j"'] & & \cdot \arrow[rr] \arrow[dr] \arrow[dd, "i\hat{\times}kj"'] & & \cdot \arrow[dr] \arrow[dd, "i\hat{\times}k"'] & & \\
& \cdot \arrow[rr] \arrow[dd, "i\hat{\wedge}j"'] & & \cdot \arrow[rr] \arrow[dd, "i\hat{\wedge}kj"'] & & \cdot \arrow[r, dashed] \arrow[dd, "i\hat{\wedge}k"'] & \cdot \arrow[dd, "f"] \\
\cdot \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\ulcorner", very near end] & & \cdot \arrow[rr, equal] \arrow[dr] \arrow[dr, phantom, "\ulcorner", very near end] & & \cdot \arrow[dr] \arrow[dr, phantom, "\ulcorner", very near end] & & \\
& \cdot \arrow[rr] & & \cdot \arrow[rr, equal] & & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Because the monoidal unit $*$ is assumed to be cofibrant and $(-)_+$ is left Quillen, the unit S^0 for the monoidal structure on $\mathcal{M}_*$ is cofibrant, and the second condition of Definition II.7.1 is automatic. It remains only to see that the algebraic Quillen two-variable adjunction is maximally coherent whenever the original monoidal algebraic model structure is. Because the algebraic model structure on $\mathcal{M}_*$ was defined by pullback, the left-hand square of lifted functors commutes.

$$\begin{tikzcd}
(\mathbb{C}_t)_*\textbf{-coalg} \times \mathbb{C}_*\textbf{-coalg} \arrow[r, "U\times U"] \arrow[d, "\xi_*\times 1"'] & \mathbb{C}_t\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \arrow[r, "\hat{\times}"] \arrow[d, "\xi\times 1"] & \mathbb{C}_t\textbf{-coalg} \arrow[d, "\xi"] \\
\mathbb{C}_*\textbf{-coalg} \times \mathbb{C}_*\textbf{-coalg} \arrow[r, "U\times U"] & \mathbb{C}\textbf{-coalg} \times \mathbb{C}\textbf{-coalg} \arrow[r, "\hat{\times}"] & \mathbb{C}\textbf{-coalg}
\end{tikzcd}$$

The right-hand square commutes by hypothesis. At each pair of coalgebras in $(\mathcal{M}_*)^{\mathbf{2}}$, the $(\mathbb{C}_t)_*$-coalgebra structure assigned their pushout-smash-product is determined by the $\mathbb{C}_t$-coalgebra structure assigned the pushout of the arrow in their image along the top row of this diagram; its $\mathbb{C}_*$-coalgebra structure is similarly determined by the $\mathbb{C}$-coalgebra structure assigned the pushout of the map in the image at the bottom right. Writing down explicit formulae (I.5.4), it is easy to see that the process of assigning coalgebra structures to pushouts commutes with the comparison map for $\mathcal{M}$. $\square$

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