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The Yoneda lemma in the category of Matrices

Dedicated to Fred E. J. Linton ($1938^{-\epsilon}$–2017)

ACT 2020 Tutorial Day

The category of Matrices

A category has objects and arrows and a composition law satisfying identity and associativity axioms.

The category of matrices Mat has

- natural numbers $0, 1, 2, \ldots, j, k, \ell, m, n, \ldots$ as its objects,
- matrices as its arrows: an arrow $m \xleftarrow{A} n$ is an $m \times n$ matrix A,
- the composite of a $\ell \times m$ and a $m \times n$ matrix defined by matrix multiplication

$$\begin{tikzcd}
& m \arrow[dl, "B"'] & \\
\ell & & n \arrow[ul, "A"'] \arrow[ll, dashed, "B\cdot A"]
\end{tikzcd}
\qquad
\ell\left\{\overbrace{\begin{pmatrix} \quad & B & \quad \end{pmatrix}}^{m}\right. \cdot\, m\left\{\overbrace{\begin{pmatrix} \quad & A & \quad \end{pmatrix}}^{n}\right.$$

- with the identity arrow $n \xleftarrow{I_n} n$ given by the identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Column functors

In the category of matrices Mat

- objects are natural numbers $\ldots, j, k, \ell, m, n, \ldots$ and
- an arrow $m \xleftarrow{A} n$ is an $m \times n$ matrix A.

For each k, the set of all matrices with k columns is organized by the data of the k-column functor $h_k\colon \mathrm{Mat} \to \mathrm{Set}$, which is given by:

- a set $h_k(n) = \{n \times k \text{ matrices}\} = \{n \xleftarrow{C} k\}$ for each n
- a function $h(m) \xleftarrow{A} h(n)$ for each matrix $m \xleftarrow{A} n$ given by left multiplication:

$$m\left\{\overbrace{\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix}}^{n}\right. \cdot\, n\left\{\overbrace{\begin{pmatrix} & & \\ & C & \\ & & \end{pmatrix}}^{k}\right. \mapsfrom n\left\{\overbrace{\begin{pmatrix} & & \\ & C & \\ & & \end{pmatrix}}^{k}\right.$$

This data satisfies the axioms of functoriality.

Column functors are functorial

The set of all matrices with k-columns defines the k-column functor:

- a set $h_k(n) = \{n \times k - \text{matrices}\} = \{n \xleftarrow{C} k\}$
- a function $h_k(m) \xleftarrow{A} h_k(n)$ given by left multiplication by a matrix $m \xleftarrow{A} n$

This data satisfies the axioms of functoriality:

- $h_k(n) \xleftarrow{I_n} h_k(n)$ is the identity function
- for any matrices $m \xleftarrow{A} n$ and $\ell \xleftarrow{B} m$:

$$\begin{tikzcd} & h_k(m) \arrow[dl, "B"'] & \\ h_k(\ell) & & h_k(n). \arrow[ul, "A"'] \arrow[ll, dashed, "B \cdot A"] \end{tikzcd}$$

For any $n \times k$ matrix C,

$$I_n \cdot C = C \qquad \text{and} \qquad B \cdot (A \cdot C) = (B \cdot A) \cdot C.$$

Naturally-defined column operations on column functors

The set of all matrices with k-columns defines the k-column functor:

- a set $h_k(n) = \{n \times k - \text{matrices}\} = \{n \xleftarrow{C} k\}$
- a function $h_k(m) \xleftarrow{A} h_k(n)$ given by left multiplication by a matrix $m \xleftarrow{A} n$

A natural transformation $\begin{tikzcd} h_k \arrow[d, "\alpha"'] \\ h_j \end{tikzcd}$ is given by a function $\begin{tikzcd} h_k(n) \arrow[d, "\alpha_n"'] \\ h_j(n) \end{tikzcd}$ for each n

so that for each matrix $m \xleftarrow{A} n$ the diagram of functions commutes:

$$\begin{tikzcd} h_k(m) \arrow[d, "\alpha_m"'] & h_k(n) \arrow[l, "A"'] \arrow[d, "\alpha_n"] \\ h_j(m) & h_j(n) \arrow[l, "A"] \end{tikzcd}$$

This says that "α is a naturally-defined operation on column functors."

Projection operations on column functors

A natural transformation $\begin{array}{c} h_k \\ {\scriptstyle\alpha}\downarrow \\ h_j \end{array}$ is given by a function $\begin{array}{c} h_k(n) \\ {\scriptstyle\alpha_n}\downarrow \\ h_j(n) \end{array}$ for each n that encodes a naturally-defined operation on column functors.

Example: There is an operation $\begin{array}{c} h_k \\ {\scriptstyle\pi}\downarrow \\ h_{k-1} \end{array}$ that deletes the kth column.

Naturality in $m \xleftarrow{A} n$:

$$\begin{array}{ccc}
m\Big\{\overbrace{\big((AC)_1 \cdots (AC)_k\big)}^{k} & \longmapsfrom & n\Big\{\overbrace{\big(C_1 \cdots C_k\big)}^{k} \\
\downarrow & & \downarrow \\
m\Big\{\overbrace{\big((AC)_1 \cdots (AC)_{k-1}\big)}^{k-1} = m\Big\{\overbrace{\big(A\big)\cdot\big(C_1 \cdots C_{k-1}\big)}^{k-1} & \longmapsfrom & n\Big\{\overbrace{\big(C_1 \cdots C_{k-1}\big)}^{k-1}
\end{array}$$

Inclusion operations on column functors

A natural transformation $\begin{array}{c} h_k \\ \alpha\big\downarrow \\ h_j \end{array}$ is given by a function $\begin{array}{c} h_k(n) \\ \alpha_n\big\downarrow \\ h_j(n) \end{array}$ for each n

that encodes a naturally-defined operation on column functors.

Example: There is an operation $\begin{array}{c} h_k \\ \iota\big\downarrow \\ h_{k+1} \end{array}$ that appends a column of zeros.

Naturality in $m \xleftarrow{A} n$:

$$\begin{array}{ccc}
m\Big\{\overbrace{\big((AC)_1 \cdots (AC)_k\big)}^{k} & \mapsfrom & n\Big\{\overbrace{\big(C_1 \cdots C_k\big)}^{k} \\
\big\downarrow & & \big\downarrow \\
m\Big\{\overbrace{\big((AC)_1 \cdots (AC)_k \vec{0}\big)}^{k+1} = m\Big\{\overbrace{\big(A\big) \cdot \big(C_1 \cdots C_k \vec{0}\big)}^{k+1} & \mapsfrom & n\Big\{\overbrace{\big(C_1 \cdots C_k \vec{0}\big)}^{k+1}
\end{array}$$

Classifying column operations

Challenge: Describe all natural transformations $\begin{array}{c} h_k \\ {\scriptstyle\alpha}\big\downarrow \\ h_j \end{array}$

In other words:

Challenge: Classify all naturally-defined column operations that transform matrices with k columns into matrices with j columns.

The Yoneda lemma in the category of Matrices

Yoneda Lemma:

1. Every naturally-defined column operation $\begin{array}{c} h_k \\ {\scriptstyle\alpha}\big\downarrow \\ h_j \end{array}$ is determined by a single $k \times j$ matrix.
2. This $k \times j$ matrix $k \xleftarrow{\alpha_k(I_k)} j$ is obtained by applying the column operation $\begin{array}{c} h_k(k) \\ {\scriptstyle\alpha_k}\big\downarrow \\ h_j(k) \end{array}$ to the identity matrix $k \xleftarrow{I_k} k$.
3. The column operation $\begin{array}{c} h_k(n) \\ {\scriptstyle\alpha_n}\big\downarrow \\ h_j(n) \end{array}$ is given by right multiplication by the matrix $k \xleftarrow{\alpha_k(I_k)} j$.
$$\alpha_n(C) := {}_n\Bigg\{ \overbrace{\Bigg(\quad C \quad \Bigg)}^{k} \cdot {}_k\Bigg\{ \overbrace{\Bigg(\quad \alpha_k(I_k) \quad \Bigg)}^{j}$$
4. Every matrix $k \xleftarrow{M} j$ determines a naturally-defined column operation defined by right multiplication.

Permutation operations on column functors

Yoneda Lemma: Every naturally-defined column operation $\begin{matrix} h_k \\ \alpha\downarrow \\ h_j \end{matrix}$ is given by right multiplication by the matrix $k \xleftarrow{\alpha_k(I_k)} j$ obtained by applying the column operation α_k to the identity matrix $k \xleftarrow{I_k} k$, and every $k \times j$ matrix determines a naturally-defined column operation in this way.

Example: The operation $\begin{matrix} h_k \\ \sigma\downarrow \\ h_k \end{matrix}$ that swaps the first two columns is defined by right multiplication by the matrix

$$\sigma_k(I_k) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Multiplication operations on column functors

Yoneda Lemma: Every naturally-defined column operation $\begin{matrix} h_k \\ {\scriptstyle\alpha}\big\downarrow \\ h_j \end{matrix}$ is given by right multiplication by the matrix $k \xleftarrow{\alpha_k(I_k)} j$ obtained by applying the column operation α_k to the identity matrix $k \xleftarrow{I_k} k$, and every $k \times j$ matrix determines a naturally-defined column operation in this way.

Example: The operation $\begin{matrix} h_k \\ {\scriptstyle\mu}\big\downarrow \\ h_k \end{matrix}$ that multiplies the first column by a scalar λ is defined by right multiplication by the matrix:

$$\mu_k(I_k) = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Column addition operations on column functors

Yoneda Lemma: Every naturally-defined column operation $\begin{array}{c} h_k \\ {\scriptstyle\alpha}\downarrow \\ h_j \end{array}$ is given by right multiplication by the matrix $k \xleftarrow{\alpha_k(I_k)} j$ obtained by applying the column operation α_k to the identity matrix $k \xleftarrow{I_k} k$, and every $k \times j$ matrix determines a naturally-defined column operation in this way.

Example: The operation $\begin{array}{c} h_k \\ {\scriptstyle\alpha}\downarrow \\ h_k \end{array}$ that adds the first column to the second column is defined by right multiplication by the matrix:

$$\alpha_k(I_k) = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Elementary column operations

Corollary of the Yoneda lemma: A naturally-defined column operation $\begin{array}{c} h_k \\ \alpha\big\downarrow \\ h_k \end{array}$ is invertible if and only if the matrix $k \xleftarrow{\alpha_k(I_k)} k$ is invertible.

The elementary column operations that

- swap two columns,
- multiply a column by a scalar, or
- add a scalar multiple of one column to another column

are invertible since the corresponding elementary matrices are invertible.

Composite column operations

Corollary of the Yoneda lemma: The composite of two naturally-defined column operations

$$\begin{tikzcd}
 & h_k \arrow[dl, "\alpha"'] \arrow[dd, dashed, "\beta\alpha"] \\
h_j \arrow[dr, "\beta"'] & \\
 & h_m
\end{tikzcd}$$

is defined by right multiplication by the product of the representing matrices:

$$k\left\{\overbrace{\begin{pmatrix} & (\beta\alpha)_k(I_k) & \end{pmatrix}}^{m}\right. := k\left\{\overbrace{\begin{pmatrix} & \alpha_k(I_k) & \end{pmatrix}}^{j}\right. \cdot j\left\{\overbrace{\begin{pmatrix} & \beta_j(I_j) & \end{pmatrix}}^{m}\right.$$

In this way the elementary column operations generate all invertible column operations.

Unnatural column operations

Non-Example: The operation that appends a column of ones is not natural. Applying this operation to the identity matrix yields:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

but then right multiplication defines a different column operation:

$$\begin{pmatrix} | & \cdots & | \\ C_1 & \cdots & C_k \\ | & \cdots & | \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} | & \cdots & | & | \\ C_1 & \cdots & C_k & C_1 + \cdots + C_k \\ | & \cdots & | & | \end{pmatrix}$$

$$\neq \begin{pmatrix} | & \cdots & | & 1 \\ C_1 & \cdots & C_k & \vdots \\ | & \cdots & | & 1 \end{pmatrix}$$

Category theory in context

Emily Riehl Dec 28, 2014, 4:52 PM
to categories

Hi all,

I am writing in hopes that I might pick the collective brain of the categories list. This spring, I will be teaching an undergraduate-level category theory course, entitled "Category theory in context":

http://www.math.harvard.edu/~eriehl/161

It has two aims:

(i) To provide a thorough "Cambridge-style" introduction to the basic concepts of category theory: representability, (co)limits, adjunctions, and monads.

(ii) To revisit as many topics as possible from the typical undergraduate curriculum, using category theory as a guide to deeper understanding.

For example, when I was an undergraduate, I could never remember whether the axioms for a group action required the elements of the group to act via *automorphisms*. But after learning what might be called the first lemma in category theory -- that functors preserve isomorphisms -- I never worried about this point again.

Over the past few months I have been collecting examples that I might use in the course, with the focus on topics that are the most "sociologically important" (to quote Tom Leinster's talk at CT2014) and also the most illustrative of the categorical concept in question. (After all, aim (i) is to help my students internalize the categorical way of thinking!)

Here are a few of my favorites:

Dedication and acknowledgment

Fred E.J. Linton

to Emily

Hi, Emily,

I suppose I would be remiss not to point out all the examples your fellow Cambridge co-citizen David Spivak offers in his recent text, Category Theory for the Sciences (MIT Press).

And then there's the Yoneda Lemma embodied in the classical Gaussian row reduction observation, that a given row reduction operation (on matrices with say k rows) being a "natural" operation (in the sense of natural transformations) is just multiplication (on the appropriate side) by the effect of that operation on the k-by-k identity matrix.

And dually for column-reduction operations :-) .

Cheers, -- Fred

Thank you!