Journal of Topology 13 (2020) 59–76 doi:10.1112/topo.12123

Lifting accessible model structures

Richard Garner, Magdalena Kędziorek and Emily Riehl

Abstract

A Quillen model structure is presented by an interacting pair of weak factorization systems. We prove that in the world of locally presentable categories, any weak factorization system with *accessible* functorial factorizations can be lifted along either a left or a right adjoint. It follows that *accessible model structures* on locally presentable categories – ones admitting accessible functorial factorizations, a class that includes all combinatorial model structures but others besides – can be lifted along either a left or a right adjoint if and only if an essential 'acyclicity' condition holds. A similar result was claimed in a paper of Hess–Kędziorek–Riehl–Shipley, but the proof given there was incorrect. In this note, we explain this error and give a correction, and also provide a new statement and a different proof of the theorem which is more tractable for homotopy-theoretic applications.

Contents

1. *Introduction*

In abstract homotopy theory, one often works with categories endowed with a class $\mathcal{W}$ of *weak equivalences* which, though not necessarily isomorphisms themselves, satisfy closure properties resembling those of the isomorphisms†. In many cases, the category M at issue is complete, cocomplete, and endowed with further classes of maps $\mathcal{C}$ and $\mathcal{F}$, called *cofibrations* and *fibrations*, for which the pairs

$$(\mathcal{C} \cap \mathcal{W}, \mathcal{F}) \quad \text{and} \quad (\mathcal{C}, \mathcal{F} \cap \mathcal{W}) \tag{1.1}$$

satisfy the factorization and lifting properties axiomatized by the notion of a *weak factorization system*; see Definition 2.1. One then has a *Quillen model category*: a setting rich enough to perform many of the classical constructions of homotopy theory.

Although model structures are convenient to have, they can be difficult to construct. One of the most useful tools for building model structures is that of 'lifting' an existing model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ on a category M along an adjoint functor in either one of the following situations:

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & \mathsf{M} \arrow[l, shift left, "L"] \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathsf{K} \arrow[r, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathsf{M}. \arrow[l, shift left, "R"] \end{tikzcd} \tag{1.2}$$

Received 4 November 2018; published online 17 August 2019.

2010 *Mathematics Subject Classification* 18C35, 18G55, 55U35 (primary).

The first author gratefully acknowledges the support of Australian Research Council grants DP160101519 and FT160100393. The second author would like to thank the Max Planck Institute for Mathematics in Bonn for its hospitality during work on that project. The third author received support from the National Science Foundation via grant DMS-1652600.

†More precisely, $\mathcal{W}$ should contain all identities and satisfy the 2-out-of-6 property.

On the one hand, if $U\colon \mathsf{C} \to \mathsf{M}$ is a right adjoint functor, we may attempt to define a model structure on C by taking the classes of weak equivalences and fibrations to be $U^{-1}(\mathcal{W})$ and $U^{-1}(\mathcal{F})$, respectively; the model category axioms then force the definition of the cofibrations in C, since they are supposed to provide the left class of a weak factorization system with right class $U^{-1}(\mathcal{F} \cap \mathcal{W})$. When these classes determine a model structure on C, we call it a *right-lifting* of $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ along U. On the other hand, if $V\colon \mathsf{K} \to \mathsf{M}$ is a left adjoint functor, we may define weak equivalences and cofibrations in K as the classes $V^{-1}(\mathcal{W})$ and $V^{-1}(\mathcal{C})$, and define the fibrations in the only way allowed by the model category axioms. When these classes determine a model structure on K, we call it a *left-lifting* of $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ along V.

It is not always the case that right or left lifting will determine a model structure. First, there is an essential 'acyclicity condition' which must be satisfied, which ensures that the left and right classes of the weak factorization systems are compatible with the cofibrations, fibrations, and weak equivalences in the sense of (1.1). In the right-lifted case, the acyclicity condition asserts that the left class of the weak factorization system determined by $U^{-1}(\mathcal{F})$ (that is, the class of maps which are supposed to be acyclic cofibrations) is contained in the class $U^{-1}(\mathcal{W})$ of lifted weak equivalences. This condition is non-trivial to check, and typically requires some genuine insight into the homotopy theory at issue.

The other precondition for existence of the lifted model structure is existence of the lifted weak factorization systems: and while the lifting axiom is satisfied by construction, the existence of factorizations is not automatic. However, there are a range of results available which verify this existence using only general properties of the categories involved and of the model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ – thus reducing the question of lifting model structures to the essential acyclicity condition.

One situation in which lifted weak factorization systems always exist is the combinatorial setting; here, the categories involved are *locally presentable* [**10**] – an assumption which will remain in place for the rest of the introduction – and the model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ is *cofibrantly generated* [**16**, Definition 2.1.17]. In this context, it has been understood for several decades that right-lifted factorizations can be constructed explicitly using Quillen's small object argument. Very recently, [**22**] showed that in this same setting, left-lifted factorizations also exist; this breakthrough result was put into the model-categorical context in [**5**, Theorem 2.23], and has since been used to construct interesting new model categories [**9**, **15**].

These results were generalized in [**14**] to obtain left and right liftings of factorizations for what the authors term *accessible* model structures. The simplest formulation of what this means is that given in [**25**]: a model structure is accessible if its factorizations into the classes (1.1) can be realized by accessible functors – ones which preserve λ-filtered colimits for some regular cardinal λ. In particular, any cofibrantly generated model structure is accessible, but others besides: for example, the model structures on dg-modules considered in [**4**]. The main Corollary 3.3.4 of [**14**] asserts that, if $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ is accessible, then both left- and right-lifted factorizations always exist; this has already found practical application to the construction of new model structures in [**13**, **23**].

Although the main result of [**14**] is correct, the proof given there turns out to contains a subtle error: in some cases, it exhibits 'lifted factorizations' which are not those of the desired left- or right-lifted weak factorization systems, but of slightly different ones. The purpose of this note is to fix this error. In fact, we do so in two ways: once by correcting the argument of [**14**], and once by a different argument which sidesteps the difficulties at issue. For good measure, we also give some concrete examples in which the previous argument does indeed break down.

We now retrace the reasoning of [**14**] with a view to explaining what goes wrong. First let us note that the authors of *ibid.* express accessibility of a model structure in a different way to [**25**] – taking it to mean that the two weak factorization systems of the model structure can be made into *accessible algebraic weak factorization systems* [**7**]. This means that, as well as accessible functors

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y \tag{1.3}$$

that realize the factorizations in each case, there should also be provided fillers:

$$\begin{tikzcd} X \arrow[r, "LLf"] \arrow[d, "Lf"'] & ELf \arrow[d, "RLf"] \\ Ef \arrow[r, equal] \arrow[ur, dashed, "\delta_f"'] & Ef \end{tikzcd} \qquad \begin{tikzcd} Ef \arrow[r, equal] \arrow[d, "LRf"'] & Ef \arrow[d, "Rf"] \\ ERf \arrow[r, "RRf"'] \arrow[ur, dashed, "\mu_f"'] & Ef \end{tikzcd} \tag{1.4}$$

subject to axioms which, among other things, cause these data to endow the functors $L\colon \mathsf{M}^{\mathbf{2}} \to \mathsf{M}^{\mathbf{2}}$ and $R\colon \mathsf{M}^{\mathbf{2}} \to \mathsf{M}^{\mathbf{2}}$ with the structure of a comonad $\mathbb{L}$ and a monad $\mathbb{R}$, respectively; see Definition 3.2. While this is apparently stronger than the notion of accessible model category used in [**25**], it turns out that, starting from the less elaborate definition, one can always derive the data required for the more elaborate one; this is the content of [**14**, Remark 3.1.8].

The motivation for adopting the more involved definition of accessibility is that it allows application of [**7**, Proposition 13], which says that accessible algebraic weak factorization systems can always be left- and right-lifted. The intended approach is thus the following. With $(\mathcal{L}, \mathcal{R})$ taken successively to be $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, one first algebraizes $(\mathcal{L}, \mathcal{R})$ to $(\mathbb{L}, \mathbb{R})$; then right- or left-lifts this along $U\colon \mathsf{C} \to \mathsf{M}$ or $V\colon \mathsf{K} \to \mathsf{M}$ to an algebraic weak factorization system $(\mathbb{L}', \mathbb{R}')$; and then takes the *underlying weak factorization system* $(\mathcal{L}', \mathcal{R}')$, whose classes comprise the retracts of maps of the form $L'f$ or $R'f$, respectively.

This is the argument of [**14**, Corollary 3.3.4]; for it to work, one must be sure that the $(\mathcal{L}', \mathcal{R}')$ produced above is indeed left- or right-lifted from $(\mathcal{L}, \mathcal{R})$, meaning in the left case that $\mathcal{L}' = V^{-1}(\mathcal{L})$, and dually in the right. This is claimed to be the case in [**14**, Theorems 3.3.1, 3.3.2], but the claim is incorrect. The reason is subtle, and has to do with what exactly is lifted in applying [**7**, Proposition 13].

Concentrating on the left case, one lifts not the $\mathcal{L}$-maps of the underlying weak factorization system, but rather the $\mathbb{L}$-*maps*: the coalgebras for the comonad $\mathbb{L}$. While every $\mathbb{L}$-map has the property of being an $\mathcal{L}$-map, a given $\mathcal{L}$-map may not be the underlying map of any $\mathbb{L}$-map; it may be necessary to take a retract[†]. The upshot of this is that, if one applies the above procedure to $(\mathcal{L}, \mathcal{R})$, one finds that $\mathcal{L}'$ comprises the retract-closure of $V^{-1}(\mathbb{L}\text{-map})$ rather than $V^{-1}(\mathcal{L}) = V^{-1}(\text{retract-closure of } \mathbb{L}\text{-map})$; and while the former is always included in the latter, the inclusion may be strict, as shown in Section 3.3.

In this way, the above procedure may produce factorizations for an incorrect lifting of one of the original weak factorization systems. The author who is responsible for this error was well aware of this issue when she proved [**24**, Theorem 3.10] – indeed, an important part of that argument explains why it does not arise in the cofibrantly generated and right-lifting context – but had fallen out of touch with that awareness when writing [**14**, Section 3].

Note that the problem we have described would not arise if the maps in M admitting $\mathbb{L}$-map structure were already closed under retracts. This observation suggests a fix: we adjust the algebraization $(\mathbb{L}, \mathbb{R})$ appropriately before lifting. Indeed, in Proposition 4.5, we will see that any algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ may be 'shifted' to one $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ whose underlying weak factorization system is the same, but whose $\mathbb{L}^\sharp$-maps *are* closed under retracts. Now, to correct the above procedure for left-lifting, we need only interpolate the step of replacing $(\mathbb{L}, \mathbb{R})$ by $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$. Of course, exactly the same issues arise in the case of right-lifting, and exactly

[†]A good intuition is that, if the $\mathcal{L}$-maps are the 'cofibrations', then the $\mathbb{L}$-maps are the 'relative cell complexes'. In many cases, this is literally true: see [**2**].

the same fix is possible, this time involving a dual shifting $(\mathbb{L}^\flat, \mathbb{R}^\flat)$; all of this is detailed in Section 4.

In addition to correcting the argument that proves [**14**, Corollary 3.3.4], the remaining aspect of this paper is a new proof of the result which proceeds directly from the simpler definition of accessible model structure given in [**25**]. In particular, this argument avoids the use of algebraic weak factorization systems entirely, since these are beside the point for the homotopy-theoretic applications. It is with this more streamlined proof that we now begin the paper.

2. *The new proof*

2.1. *Background and statement of results*

Our terminology and approach will largely follow that of [**14**]; we begin by recalling the necessary background. Given a class of maps $\mathcal{X}$ in a category C, we write ${}^{\boxslash}\mathcal{X}$ and $\mathcal{X}^{\boxslash}$ for the classes of maps with the left, respectively, right, lifting property against each map in $\mathcal{X}$, and given a functor $H\colon \mathsf{C}' \to \mathsf{C}$, we write $H^{-1}\mathcal{X}$ for the class of morphisms in C' which are mapped into $\mathcal{X}$ by H.

Definition 2.1. A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category C is given by a *left* class of maps $\mathcal{L}$ and a *right* class of maps $\mathcal{R}$ such that:

(1) every morphism in C can be factored as a map in $\mathcal{L}$ followed by one in $\mathcal{R}$;
(2) the classes $\mathcal{L}$ and $\mathcal{R}$ are mutually determined by the equations:

$$\mathcal{L} = {}^{\boxslash}\mathcal{R} \quad \text{and} \quad \mathcal{R} = \mathcal{L}^{\boxslash};$$

in the presence of the first axiom, this is equally to ask that each $\mathcal{L}$-map has the left lifting property against each $\mathcal{R}$-map, and that both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

We have already discussed in the introduction what we mean by a left- or right-lifting of a model structure along a left or right adjoint functor; this is what [**14**, Definition 2.1.3] called the *left-induced* or *right-induced* model structure. More generally, we can speak of the *left-lifting* or *right-lifting* of a weak factorization system $(\mathcal{L}, \mathcal{R})$ along a left adjoint V or right adjoint U as in (1.2); when these exist, they are by definition the weak factorization systems on the domain category with respective classes

$$(\overleftarrow{\mathcal{L}}, \overleftarrow{\mathcal{R}}) = (V^{-1}\mathcal{L}, (V^{-1}\mathcal{L})^{\boxslash}) \quad \text{and} \quad (\vec{\mathcal{L}}, \vec{\mathcal{R}}) = ({}^{\boxslash}(U^{-1}\mathcal{R}), U^{-1}\mathcal{R}). \tag{2.2}$$

A necessary condition for the existence of a left- or right-lifted model structure is that both of its underlying weak factorization systems (1.1) should admit left- or right-liftings. Conversely, if we assume such liftings, then we have:

Proposition 2.3 (Proposition 2.1.4 of [**14**]). *Let $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ be a model structure on M, and suppose there are given adjunctions as in (1.2) for which the right-lifted weak factorization systems exist on C and the left-lifted weak factorization systems exist on K. In this situation:*

(1) *the right-lifted model structure exists on C if and only if the* right acyclicity condition ${}^{\boxslash}(U^{-1}\mathcal{F}) \subseteq U^{-1}\mathcal{W}$ *is satisfied;*
(2) *the left-lifted model structure exists on C if and only if the* left acyclicity condition $(V^{-1}\mathcal{C})^{\boxslash} \subseteq V^{-1}\mathcal{W}$ *is satisfied.*

As noted in the introduction, the satisfaction of the acyclicity condition typically depends on non-trivial homotopy-theoretic arguments; this is discussed at some length in [**14**, § 2.2].

In this paper, however, our sole interest will be in verifying the existence of the lifted weak factorization systems as in (2.2). The setting in which we do so is that of *accessible weak factorization systems*.

DEFINITION 2.4. A weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category M is called *accessible* if M is locally presentable, and there is given a functorial realization

$$A \xrightarrow{f} B \quad \mapsto \quad A \xrightarrow{Lf} Ef \xrightarrow{Rf} B \tag{2.5}$$

for $(\mathcal{L}, \mathcal{R})$ whose underlying functor $E \colon \mathsf{M}^2 \to \mathsf{M}$ is accessible[†]. A model structure on M is *accessible* if its underlying weak factorization systems (1.1) are so.

The key objective of this paper is to give a correct proof of:

THEOREM 2.6. *Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system on M, and suppose that there are given adjunctions (1.2) with C and K also locally presentable. In these circumstances, $(\mathcal{L}, \mathcal{R})$ admits a left-lifting along V and a right-lifting along U, and these are again accessible.*

Using this, we re-find the main Corollary 3.3.4 of [**14**]:

COROLLARY 2.7. *Let $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ be an accessible model structure on M, and suppose given adjunctions (1.2) with K and C also locally presentable.*

(1) *The right-lifted model structure exists on C if and only if the right acyclicity condition holds.*
(2) *The left-lifted model structure exists on K if and only if the left acyclicity condition holds.*

2.2. *Cloven $\mathcal{L}$- and $\mathcal{R}$-maps*

The first proof we give of Theorem 2.6 will still employ ideas derived from [**7**], but will be given in a fully self-contained manner with the minimum of additional machinery. The main notion we require is:

DEFINITION 2.8. Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system on M. A *cloven $\mathcal{L}$-map* $(f, s) \colon A \to B$ comprises a map $f \colon A \to B$ of M together with a lift of f against its own right factor, as to the left in:

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, equal] \arrow[ur, dashed, "s"'] & B \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, equal] \arrow[d, "Lg"'] & C \arrow[d, "g"] \\ Eg \arrow[r, "Rg"'] \arrow[ur, dashed, "p"] & D. \end{tikzcd} \tag{2.9}$$

Dually, a *cloven $\mathcal{R}$-map* $(g, p) \colon C \to D$ is a map $g \colon C \to D$ together with a lift of g against its own left factor, as above right. The cloven $\mathcal{L}$-maps are the objects of a category $\mathsf{Clov}(\mathcal{L})$, wherein a morphism $(f, s) \to (g, t)$ is a map $(h, k) \colon f \to g$ in M^2 as below left which also renders

[†]By the usual retract argument, a given functorial factorization provides factorizations for at most one weak factorization system: namely, that whose left and right classes comprise the retracts of the maps Lf and the maps Rf, respectively. This is the sense in which we refer to (L, R) as a *functorial realization* of the weak factorization system $(\mathcal{L}, \mathcal{R})$.

commutative the square to the right:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \qquad \begin{tikzcd} Ef \arrow[r, "E(h,k)"] & Eg \\ B \arrow[u, "s"] \arrow[r, "k"'] & D. \arrow[u, "t"'] \end{tikzcd}$$

Dually, the cloven $\mathcal{R}$-maps form a category $\mathsf{Clov}(\mathcal{R})$. We write $\mathrm{U}_{\mathcal{L}} \colon \mathsf{Clov}(\mathcal{L}) \to \mathsf{M}^{\mathbf{2}}$ and $\mathrm{U}_{\mathcal{R}} \colon \mathsf{Clov}(\mathcal{R}) \to \mathsf{M}^{\mathbf{2}}$ for the functors forgetting the liftings.

It will be useful to express the above definition differently. Any functorial factorization (1.3) yields endofunctors $L, R \colon \mathsf{M}^{\mathbf{2}} \to \mathsf{M}^{\mathbf{2}}$ and natural transformations $\vec{\eta} \colon \mathrm{id}_{\mathsf{M}^{\mathbf{2}}} \Rightarrow R$ and $\vec{\epsilon} \colon L \Rightarrow \mathrm{id}_{\mathsf{M}^{\mathbf{2}}}$ with components

$$\vec{\eta}_f = (Lf, 1) \colon f \to Rf \quad \text{and} \quad \vec{\epsilon}_f = (1, Rf) \colon Lf \to f. \tag{2.10}$$

In these terms, to endow $g \colon C \to D$ with cloven $\mathcal{R}$-map structure is to endow it with a choice of retraction $\vec{p} \colon Rg \to g$ for $\vec{\eta}_g$, or in other words, with $(R, \vec{\eta})$-*algebra* structure. We may thus identify $\mathrm{U}_{\mathcal{R}}$ with the forgetful functor $\mathsf{Alg}^{(R,\vec{\eta})} \to \mathsf{M}^{\mathbf{2}}$ from the category of $(R, \vec{\eta})$-algebras. Similarly, we may identify $\mathrm{U}_{\mathcal{L}}$ with the forgetful functor $\mathsf{Coalg}_{(L,\vec{\epsilon})} \to \mathsf{M}^{\mathbf{2}}$ from the category of $(L, \vec{\epsilon})$-coalgebras.

LEMMA 2.11. *Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system on* M.

(1) $\mathrm{U}_{\mathcal{L}} \colon \mathsf{Clov}(\mathcal{L}) \to \mathsf{M}^{\mathbf{2}}$ *is a left adjoint isofibration between locally presentable categories, and the objects in its image are precisely the $\mathcal{L}$-maps.*
(2) $\mathrm{U}_{\mathcal{R}} \colon \mathsf{Clov}(\mathcal{R}) \to \mathsf{M}^{\mathbf{2}}$ *is a right adjoint isofibration between locally presentable categories, and the objects in its image are precisely the $\mathcal{R}$-maps.*

Recall here that a functor $F \colon \mathcal{A} \to \mathcal{B}$ is an *isofibration* when, for every isomorphism $f \colon b \to Fa$ in $\mathcal{B}$, there exists an isomorphism $f' \colon a' \to a$ in $\mathcal{A}$ with $Ff' = f$. These are the fibrations of the 'folk' model structure on CAT [**18**].

Proof. It follows from the identification of $\mathsf{Clov}(\mathcal{L})$ and $\mathsf{Clov}(\mathcal{R})$ with $\mathsf{Coalg}_{(L,\vec{\epsilon})}$ and $\mathsf{Alg}^{(R,\vec{\eta})}$ that $\mathrm{U}_{\mathcal{L}}$ and $\mathrm{U}_{\mathcal{R}}$ are isofibrations, that $\mathrm{U}_{\mathcal{L}}$ creates colimits and that $\mathrm{U}_{\mathcal{R}}$ creates limits (cf. [**3**, Theorem 3.4.2]). In particular, $\mathsf{Clov}(\mathcal{L})$ is complete and $\mathsf{Clov}(\mathcal{R})$ cocomplete, and so by [**1**, Theorem 2.47], both will be locally presentable so long as they are *accessible categories* [**21**]. We show this using Theorem 5.1.6 of *ibid.*, which states that the 2-category ACC of accessible categories and accessible functors is closed in CAT under bilimits. This implies the accessibility of $\mathsf{Clov}(\mathcal{L})$ and $\mathsf{Clov}(\mathcal{R})$, because the passage from a (co)pointed endofunctor to its category of (co)algebras can be realized using bilimits (cf. [**9**, Appendix A]), and because the accessibility of E implies that $L, R, \vec{\eta}$ and $\vec{\epsilon}$ all live in ACC.

Now, $\mathrm{U}_{\mathcal{L}}$ is a cocontinous functor between locally presentable categories, and so by [**20**, Theorem 5.33] has a right adjoint; while $\mathrm{U}_{\mathcal{R}}$ is a continuous and accessible functor between locally presentable categories – accessible due to its construction from bilimits in ACC – and so by [**10**, Satz 14.6] has a left adjoint.

For the final claim, if f is an $\mathcal{L}$-map, then it lifts against Rf, and so admits a cleavage; conversely, if f is endowed with a cleavage, then it is an $\mathcal{L}$-map as a retract of the $\mathcal{L}$-map Lf. So the image of $\mathrm{U}_{\mathcal{L}}$ comprises precisely the $\mathcal{L}$-maps, and dually the image of $\mathrm{U}_{\mathcal{R}}$ comprises the $\mathcal{R}$-maps. ∎

We will require one final result relating to cloven maps. We state it here only for $\mathcal{L}$-maps, and leave the dualization to the right case to the reader.

Lemma 2.12. *Let $(f, s)\colon A \to B$ be a cloven $\mathcal{L}$-map and $g\colon B \to C$ an $\mathcal{L}$-map. There is a cleavage t for $gf\colon A \to C$ such that $(1, g)\colon (f, s) \to (gf, t)$ in $\mathsf{Clov}(\mathcal{L})$.*

Proof. Take t be any filler for the square

$$\begin{tikzcd} B \arrow[r, "E(1,g).s"] \arrow[d, "g"'] & E(gf) \arrow[d, "R(gf)"] \\ C \arrow[r, equal] \arrow[ur, dashed, "t"] & C. \end{tikzcd}$$

We have $R(gf) \circ t = 1_C$ and $tgf = E(1, g) \circ sf = E(1, g) \circ Lf = L(gf)$, so t is a cleavage. Moreover, $(1, g)\colon (f, s) \to (gf, t)$ is a map in $\mathsf{Clov}(\mathcal{L})$ by commutativity of the top triangle above. $\square$

2.3. *Lifting accessible weak factorization systems*

We are now ready to give our first proof of Theorem 2.6. In order to exhibit the desired factorizations into the lifted classes, we consider the following pullback diagrams:

$$\begin{tikzcd} \mathsf{Clov}(\overleftarrow{\mathcal{L}}) \arrow[r] \arrow[d, "\mathrm{U}_{\overleftarrow{\mathcal{L}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Clov}(\mathcal{L}) \arrow[d, "\mathrm{U}_{\mathcal{L}}"] \\ \mathsf{K}^2 \arrow[r, "V^2"'] & \mathsf{M}^2 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Clov}(\overrightarrow{\mathcal{R}}) \arrow[r] \arrow[d, "\mathrm{U}_{\overrightarrow{\mathcal{R}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Clov}(\mathcal{R}) \arrow[d, "\mathrm{U}_{\mathcal{R}}"] \\ \mathsf{C}^2 \arrow[r, "U^2"'] & \mathsf{M}^2. \end{tikzcd} \tag{2.13}$$

The notation for the categories defined by these pullbacks is slightly abusive; the meaning cannot be the one asserted by Definition 2.8, since we do not yet have functorial factorizations for $(\overleftarrow{\mathcal{L}}, \overleftarrow{\mathcal{R}})$ or $(\overrightarrow{\mathcal{L}}, \overrightarrow{\mathcal{R}})$. Indeed, the whole point is to find such factorizations, and we will do this with the aid of the above pullbacks.

The abuse of notation is justified by the observation that an object of, say, $\mathsf{Clov}(\overleftarrow{\mathcal{L}})$ is a pair (f, s) where f is a map of K and s is a cleavage for Vf – thus, by Lemma 2.11, a witness that Vf is an $\mathcal{L}$-map and so equally, a witness that f is an $\overleftarrow{\mathcal{L}}$-map. This proves the final clauses in the two parts of the following result.

Lemma 2.14. *Let $(\mathcal{L}, \mathcal{R})$ be an accessible weak factorization system on M, and suppose given adjunctions (1.2) with C and K also locally presentable.*

(1) *$\mathrm{U}_{\overleftarrow{\mathcal{L}}}\colon \mathsf{Clov}(\overleftarrow{\mathcal{L}}) \to \mathsf{K}^2$ is a left adjoint isofibration between locally presentable categories, and the objects in its image are precisely the $\overleftarrow{\mathcal{L}}$-maps.*

(2) *$\mathrm{U}_{\overrightarrow{\mathcal{R}}}\colon \mathsf{Clov}(\overrightarrow{\mathcal{R}}) \to \mathsf{C}^2$ is a right adjoint isofibration between locally presentable categories, and the objects in its image are precisely the $\overrightarrow{\mathcal{R}}$-maps.*

Proof. It remains to prove the first clauses. By Lemma 2.11, $\mathrm{U}_{\mathcal{L}}$ is an isofibration; whence by [**17**], its pullback along V^2 is also a bipullback (= homotopy pullback in CAT). By [**6**, Theorem 3.15], the 2-category of locally presentable categories and left adjoint functors is closed under bilimits in CAT, so that, in particular, $\mathrm{U}_{\overleftarrow{\mathcal{L}}}$ is a left adjoint between locally presentable categories. Similarly, by [**6**, Theorem 2.18], the 2-category of locally presentable categories and *right* adjoint functors is closed under bilimits in CAT, and so $\mathrm{U}_{\overrightarrow{\mathcal{R}}}$ is a right adjoint between locally presentable categories. $\square$

We now show that the adjoints asserted by this lemma provide the desired functorial $(\overleftarrow{\mathcal{L}},\overleftarrow{\mathcal{R}})$- and $(\overrightarrow{\mathcal{L}},\overrightarrow{\mathcal{R}})$-factorizations. The argument from this point is completely dualizable, so we concentrate on the case of left-lifting.

Proposition 2.15. *Under the hypotheses of Theorem 2.6, the counit of the adjunction $\mathrm{U}_{\overleftarrow{\mathcal{L}}} \dashv G\colon \mathsf{K}^2 \to \mathsf{Clov}(\overleftarrow{\mathcal{L}})$ at an object f may be taken to be of the form:*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\overleftarrow{L}f"'] & X \arrow[d, "f"] \\ \overleftarrow{E}f \arrow[r, "\overleftarrow{R}f"'] & Y \end{tikzcd} \tag{2.16}$$

Proof. It suffices to prove that, for any right adjoint G for $\mathrm{U}_{\overleftarrow{\mathcal{L}}}$, the counit maps $\mathrm{U}_{\overleftarrow{\mathcal{L}}}Gf \to f$ have invertible domain-components; then we may transport the values of G along these invertible maps to get a right adjoint with counit as in (2.16).

So let $(g, s) \in \mathsf{Clov}(\overleftarrow{\mathcal{L}})$ be the value at f of some right adjoint for $\mathrm{U}_{\overleftarrow{\mathcal{L}}}$, with the corresponding counit map given by the square in K left below.

$$\begin{tikzcd} X' \arrow[r, "x"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} 1_{X'} \arrow[r, "{(1,g)}"] \arrow[d, "{(x,x)}"'] & g \arrow[d, "{(x,y)}"] \\ 1_X \arrow[r, "{(1,f)}"'] & f \end{tikzcd} \qquad \begin{tikzcd} \mathrm{U}_{\overleftarrow{\mathcal{L}}}(1_{X'}, L1_{FX'}) \arrow[r, "{\mathrm{U}_{\overleftarrow{\mathcal{L}}}(1,g)}"] \arrow[d, "{\mathrm{U}_{\overleftarrow{\mathcal{L}}}(x,x)}"'] & \mathrm{U}_{\overleftarrow{\mathcal{L}}}(g,s) \arrow[d, "{(x,y)}"] \\ \mathrm{U}_{\overleftarrow{\mathcal{L}}}(1_X, L1_{FX}) \arrow[r, "{(1,f)}"'] \arrow[ur, dashed, "{\mathrm{U}_{\overleftarrow{\mathcal{L}}}(z,w)}"] & f \end{tikzcd}$$

This square in K yields one in K^2 as center above, and a short calculation shows that we can lift its top and left sides to $\mathsf{Clov}(\overleftarrow{\mathcal{L}})$ as in the solid square right above. Since the counit (x, y) is, by definition, terminal in the comma category $\mathrm{U}_{\overleftarrow{\mathcal{L}}} \downarrow f$, we induce a unique diagonal filler as displayed making both triangles commute. In particular, both $zx = 1$ and $xz = 1$ so x is invertible as desired. ∎

The naturality of the counit means that the factorization $f \mapsto (\overleftarrow{L}f, \overleftarrow{R}f)$ in (2.16) is functorial; and in fact, as the notation suggests, we have:

Proposition 2.17. *Under the hypotheses of Theorem 2.6, the factorization (2.16) is a functorial $(\overleftarrow{\mathcal{L}},\overleftarrow{\mathcal{R}})$-factorization.*

Proof. The diagram (2.16) is the counit at $f\colon X \to Y$ of an adjunction $\mathrm{U}_{\overleftarrow{\mathcal{L}}} \dashv G\colon \mathsf{K}^2 \to \mathsf{Clov}(\overleftarrow{\mathcal{L}})$. In particular, each $\overleftarrow{L}f = \mathrm{U}_{\overleftarrow{\mathcal{L}}}Gf$ is in the image of $\mathrm{U}_{\overleftarrow{\mathcal{L}}}$ and so is an $\overleftarrow{\mathcal{L}}$-map by Lemma 2.14. It remains to show each $\overleftarrow{R}f \in \overleftarrow{\mathcal{R}}$.

We can write $Gf = (\overleftarrow{L}f, s)$, where s is a cleavage for $V\overleftarrow{L}f\colon VX \to V\overleftarrow{E}f$. Now, since $V\overleftarrow{L}\overleftarrow{R}f\colon V\overleftarrow{E}f \to V\overleftarrow{E}\overleftarrow{R}f$ is an $\mathcal{L}$-map, there is by Lemma 2.12 a cleavage t for $V(\overleftarrow{L}\overleftarrow{R}f.\overleftarrow{L}f)$ such that $(1, \overleftarrow{L}\overleftarrow{R}f)\colon (\overleftarrow{L}f, s) \to (\overleftarrow{L}\overleftarrow{R}f.\overleftarrow{L}f, t)$ in $\mathsf{Clov}(\overleftarrow{\mathcal{L}})$. This gives a square as to the left of:

$$\begin{tikzcd} \mathrm{U}_{\overleftarrow{\mathcal{L}}}(\overleftarrow{L}f, s) \arrow[r, "{\mathrm{U}_{\overleftarrow{\mathcal{L}}}(1,1)}"] \arrow[d, "{\mathrm{U}_{\overleftarrow{\mathcal{L}}}(1,\overleftarrow{L}\overleftarrow{R}f)}"'] & \mathrm{U}_{\overleftarrow{\mathcal{L}}}(\overleftarrow{L}f, s) \arrow[d, "{(1,\overleftarrow{R}f)}"] \\ \mathrm{U}_{\overleftarrow{\mathcal{L}}}(\overleftarrow{L}\overleftarrow{R}f.\overleftarrow{L}f, t) \arrow[r, "{(1,\overleftarrow{R}\overleftarrow{R}f)}"'] \arrow[ur, dashed, "{\mathrm{U}_{\overleftarrow{\mathcal{L}}}(1,\mu)}"] & f \end{tikzcd} \qquad \begin{tikzcd} \overleftarrow{E}f \arrow[r, equal] \arrow[d, "\overleftarrow{L}\overleftarrow{R}f"'] & \overleftarrow{E}f \arrow[d, "\overleftarrow{R}f"] \\ \overleftarrow{E}\overleftarrow{R}f \arrow[r, "\overleftarrow{R}\overleftarrow{R}f"'] \arrow[ur, dashed, "\mu"] & Y \end{tikzcd}$$

in K^2. Since the counit is terminal in $\mathrm{U}_{\overleftarrow{\mathcal{L}}} \downarrow f$, we induce a unique diagonal filler as displayed making both triangles commute; and on taking the codomain projection, we obtain the commuting diagram above right. We are now ready to show that $\overleftarrow{R}f \in \overleftarrow{\mathcal{R}} = \overleftarrow{\mathcal{L}}^{\boxslash}$. So suppose $\ell \in \overleftarrow{\mathcal{L}}$, and we are given a square as to the left in:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\ell"'] & \overleftarrow{E}g \arrow[d, "\overleftarrow{R}g"] \\ B \arrow[r, "k"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\ell"'] & \overleftarrow{E}g \arrow[r, equal] \arrow[d, "\overleftarrow{L}\overleftarrow{R}f"] & \overleftarrow{E}f \arrow[d, "\overleftarrow{R}f"] \\ B \arrow[r, "k'"'] & \overleftarrow{E}\overleftarrow{R}f \arrow[r, "\overleftarrow{R}\overleftarrow{R}f"'] & Y. \end{tikzcd}$$

Choose a cleavage r for $V\ell$. By terminality of the counit $(1, \overleftarrow{R}\overleftarrow{R}f)\colon \overleftarrow{L}\overleftarrow{R}f \to \overleftarrow{R}f$ in $\mathrm{U}_{\overleftarrow{\mathcal{L}}} \downarrow \overleftarrow{R}f$, we may now factor $(h,k)\colon \mathrm{U}_{\overleftarrow{\mathcal{L}}}(\ell, r) \to \overleftarrow{R}f$ as right above, and so obtain the desired filler as $\mu k'\colon B \to \overleftarrow{E}f$. □

Putting the above results together, we obtain:

Proof of Theorem 2.6. In the left-lifted case, Proposition 2.17 exhibits (2.16) as a functorial $(\overleftarrow{\mathcal{L}}, \overleftarrow{\mathcal{R}})$-factorization, so that the lifted weak factorization system exists. To show accessibility, it suffices to show that $\overleftarrow{E}$ in (2.16) is an accessible functor; but this is so since it is the composite of accessible functors:

$$\mathsf{K}^{\mathbb{2}} \xrightarrow{G} \mathsf{Clov}(\overleftarrow{\mathcal{L}}) \xrightarrow{\mathrm{U}_{\overleftarrow{\mathcal{L}}}} \mathsf{K}^{\mathbb{2}} \xrightarrow{\mathrm{cod}} \mathsf{K}.$$

The case of right-lifting is entirely dual. □

3. *The previous proof*

In the rest of the paper, we revisit the proof of Theorem 2.6 given in [**14**] in order to explain where it goes wrong, and to suggest a way of fixing it. This proof starts from a different, though equivalent, formulation of accessibility for a model structure, given in terms of the *algebraic weak factorization systems* of [**12**], and we begin by explaining this.

3.1. *Accessible algebraic weak factorization systems*

Lemma 2.11 tells us that we can recapture a weak factorization system $(\mathcal{L}, \mathcal{R})$ from any of its functorial realizations (L, R): indeed, $\mathcal{L}$ and $\mathcal{R}$ are the classes of maps admitting $(L, \vec{\epsilon})$-coalgebra, respectively, $(R, \vec{\eta})$-algebra, structure. However, not every functorial factorization realizes a weak factorization system; the additional structure required to ensure this was identified in [**26**, Theorem 2.4]:

Lemma 3.1. *A functorial factorization (L, R) realises a weak factorization system $(\mathcal{L}, \mathcal{R})$ if and only if each Lf admits $(L, \vec{\epsilon})$-coalgebra structure and each Rf admits $(R, \vec{\eta})$-algebra structure.*

Choosing such coalgebra and algebra structures amounts to choosing sections $\vec{\delta}_f\colon Lf \to LLf$ for each $\vec{\epsilon}_f$, and retractions $\vec{\mu}_f\colon RRf \to Rf$ for each $\vec{\eta}_f$; or, in more elementary terms, to choosing fillers $\delta_f\colon Ef \to ELf$ and $\mu_f\colon ERf \to Rf$ for all squares as in (1.4). If this is done carefully enough, we may obtain an instance of the following structure.

Definition 3.2. An *algebraic weak factorization system* $(\mathbb{L}, \mathbb{R})$ on a category M comprises a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ and a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ on M^2 such that $L, R, \vec{\epsilon}$ and $\vec{\eta}$ arise from a functorial factorization (2.5) in the manner of (2.10), and such that the canonical map $(\delta, \mu)\colon LR \Rightarrow RL$ is a distributive law.

By Lemma 3.1, any algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ has an underlying weak factorization $(\mathcal{L}, \mathcal{R})$ whose classes are the maps admitting $(L, \vec{\epsilon})$-coalgebra or $(R, \vec{\eta})$-algebra structure. However, equally important in this context are the $\mathbb{L}$-*maps* and $\mathbb{R}$-*maps*: the coalgebras for the comonad $\mathbb{L}$ and the algebras for the monad $\mathbb{R}$. The *data* for $\mathbb{L}$- or $\mathbb{R}$-map structure is the same as that for $(L, \vec{\epsilon})$-coalgebra or $(R, \vec{\eta})$-algebra structure – a choice of filler as to the left or right in (2.9) – but an additional (co)associativity axiom is required; so not every $\mathcal{L}$- or $\mathcal{R}$-map need admit $\mathbb{L}$- or $\mathbb{R}$-map structure. The general situation is that:

Lemma 3.3. *If $(\mathbb{L}, \mathbb{R})$ is an algebraic weak factorization system, then its underlying weak factorization system has classes $\mathcal{L} = \mathrm{Retr}(\exists\mathbb{L})$ and $\mathcal{R} = \mathrm{Retr}(\exists\mathbb{R})$, where we write $\mathrm{Retr}(\text{–})$ for the operation of retract-closure, and write*

$$\exists\mathbb{L} = \{f \in \mathsf{M}^2 : f \text{ admits } \mathbb{L}\text{-map structure}\}$$
$$\text{and}\quad \exists\mathbb{R} = \{g \in \mathsf{M}^2 : g \text{ admits } \mathbb{R}\text{-map structure}\}.$$

Proof. Each $\mathbb{L}$-map is *a fortiori* an $(L, \vec{\epsilon})$-coalgebra and so has underlying map in $\mathcal{L}$; whence $\mathrm{Retr}(\exists\mathbb{L}) \subseteq \mathrm{Retr}(\mathcal{L}) = \mathcal{L}$. Conversely, each $\mathcal{L}$-map admits by Lemma 2.11 a cloven $\mathcal{L}$-map structure exhibiting it as a retract of Lf; as Lf underlies the $\mathbb{L}$-map $(Lf, \vec{\delta}_f)$, we thus have $\mathcal{L} \subseteq \mathrm{Retr}(\exists\mathbb{L})$. The right case is dual. □

Remark 3.4. It was shown in [**11**] that, in the locally presentable setting, each weak factorization system $(\mathcal{L}, \mathcal{R})$ generated by a set of maps J has an algebraic realization $(\mathbb{L}, \mathbb{R})$, in which the $\mathbb{R}$-maps are morphisms $f\colon X \to Y$ equipped with chosen lifts against each map in J. In this case, we have $\exists\mathbb{R} = \mathcal{R}$, but typically $\exists\mathbb{L} \subsetneq \mathcal{L}$; in fact, $\exists\mathbb{L}$ often comprises precisely the *J-cell complexes* of which the $\mathcal{L}$-maps are retracts (cf. [**2**]). On the other hand, [**7**, Proposition 17] gives an example of an algebraic weak factorization system on CAT for which $\exists\mathbb{R} \subsetneq \mathcal{R}$.

We say that an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ is *accessible* if M is locally presentable and the functor $E\colon \mathsf{M}^2 \to \mathsf{M}$ underlying the functorial factorization is accessible. In this circumstance, the underlying weak factorization system is clearly accessible. In the other direction, we have the following result; for the proof, see [**14**, § 3.1], in particular Remark 3.1.8.

Proposition 3.5. *Every accessible weak factorization system is the underlying weak factorization system of an accessible algebraic weak factorization system.*

In light of this, we can equally define an accessible model structure on a locally presentable category M to be one whose underlying weak factorization systems admit accessible algebraic realizations. This is the choice made in [**14**, Definition 3.1.6], in order to exploit known results on lifting accessible algebraic weak factorization systems; it is to these that we now turn.

3.2. *Lifting algebraic weak factorization systems*

To explain left- and right-lifting of algebraic weak factorization systems, we first need to recall the manner in which $\mathbb{L}$- and $\mathbb{R}$-maps *compose*. This is governed by certain functors into the categories $\mathsf{Coalg}_{\mathbb{L}}$ and $\mathsf{Alg}^{\mathbb{R}}$ of $\mathbb{L}$- and $\mathbb{R}$-maps, as in the dotted parts of:

$$
\begin{tikzcd}
\mathsf{Coalg}_{\mathbb{L}} \times_{\mathsf{M}} \mathsf{Coalg}_{\mathbb{L}} \arrow[r, dashed, "\circ"] \arrow[d, "\mathrm{U}_{\mathbb{L}} \times_{\mathsf{M}} \mathrm{U}_{\mathbb{L}}"'] & \mathsf{Coalg}_{\mathbb{L}} \arrow[r, shift left=2, "s\mathrm{U}_{\mathbb{L}}"] \arrow[r, shift right=2, "t\mathrm{U}_{\mathbb{L}}"'] \arrow[d, "\mathrm{U}_{\mathbb{L}}"] & \mathsf{M} \arrow[l, dashed, "i" description] \arrow[d, "\mathrm{id}"] & \mathbb{C}\mathsf{oalg}_{\mathbb{L}} \arrow[d, "\mathbb{U}_{\mathbb{L}}"] \\
\mathsf{M}^3 \arrow[r, "\circ"] & \mathsf{M}^2 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathsf{M} \arrow[l, "i" description] & \mathbb{S}\mathsf{q}(\mathsf{M}) \\
\mathsf{Alg}^{\mathbb{R}} \times_{\mathsf{M}} \mathsf{Alg}^{\mathbb{R}} \arrow[r, dashed, "\circ"] \arrow[u, "\mathrm{U}_{\mathbb{R}} \times_{\mathsf{M}} \mathrm{U}_{\mathbb{R}}"] & \mathsf{Alg}^{\mathbb{R}} \arrow[r, shift left=2, "s\mathrm{U}_{\mathbb{R}}"] \arrow[r, shift right=2, "t\mathrm{U}_{\mathbb{R}}"'] \arrow[u, "\mathrm{U}_{\mathbb{R}}"'] & \mathsf{M} \arrow[l, dashed, "i" description] \arrow[u, "\mathrm{id}"'] & \mathbb{A}\mathsf{lg}^{\mathbb{R}}. \arrow[u, "\mathbb{U}_{\mathbb{R}}"']
\end{tikzcd}
$$

These functors exhibit the top and bottom rows as *double categories* – that is, internal categories in CAT – over the double category $\mathbb{S}\mathsf{q}(\mathsf{M})$ of objects, morphisms, morphisms and commutating squares in M. We display this to the right above. In more detail, objects and horizontal morphisms of these double categories $\mathbb{C}\mathsf{oalg}_{\mathbb{L}}$ and $\mathbb{A}\mathsf{lg}^{\mathbb{R}}$ are just objects and arrows of M; vertical arrows are $\mathbb{L}$-coalgebras (respectively, $\mathbb{R}$-algebras); while squares are commutative squares – maps in M^2 – that lift to maps of $\mathbb{L}$-coalgebras (respectively, $\mathbb{R}$-algebras).

This is relevant due to a powerful and slightly surprising result: an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ is completely determined by either of the double categories $\mathbb{U}_{\mathbb{L}} \colon \mathbb{C}\mathsf{oalg}_{\mathbb{L}} \to \mathbb{S}\mathsf{q}(\mathsf{M})$ or $\mathbb{U}_{\mathbb{R}} \colon \mathbb{A}\mathsf{lg}^{\mathbb{R}} \to \mathbb{S}\mathsf{q}(\mathsf{M})$ over $\mathbb{S}\mathsf{q}(\mathsf{M})$; see [**24**, Theorem 2.24]. This result was strengthened in [**7**] to give a complete characterization of when a double category over $\mathbb{S}\mathsf{q}(\mathsf{M})$ is isomorphic to the double category of left or right maps for an algebraic weak factorization system.

THEOREM 3.6 [**7**, Theorem 6]. *A double category $\mathbb{U} \colon \mathbb{A} \to \mathbb{S}\mathsf{q}(\mathsf{M})$ over $\mathbb{S}\mathsf{q}(\mathsf{M})$ is isomorphic to the double category of left (respectively, right) maps for an algebraic weak factorization system on M if and only if:*

(i) *the object-level functor $U \colon \mathcal{A}_0 \to \mathsf{M}$ is an isomorphism, and the arrow-level functor $U \colon \mathcal{A}_1 \to \mathsf{M}^2$ is strictly comonadic (respectively, monadic); and*
(ii) *for every $f \in \mathcal{A}_1$, the square left below (respectively, right below) in $\mathbb{S}\mathsf{q}(\mathsf{M})$ is in the image of $\mathbb{U}$:*

$$
\begin{tikzcd}
a \arrow[r, "1"] \arrow[d, "1"'] & a \arrow[d, "Uf"] \\
a \arrow[r, "Uf"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "Uf"] \arrow[d, "Uf"'] & b \arrow[d, "1"] \\
b \arrow[r, "1"'] & b.
\end{tikzcd}
$$

This result allows for a straightforward definition and a straightforward construction of left- and right-liftings for algebraic weak factorization systems.

DEFINITION 3.7. Given an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ on M, its *left-lifting* along a left adjoint V or its *right-lifting* along a right adjoint U as in (1.2) are, when they exist, the algebraic weak factorization systems $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ on K and $(\overrightarrow{\mathbb{L}}, \overrightarrow{\mathbb{R}})$ on C characterized by the following pullbacks of double categories:

$$
\begin{tikzcd}
\mathbb{C}\mathsf{oalg}_{\overleftarrow{\mathbb{L}}} \arrow[r] \arrow[d, "\mathbb{U}_{\overleftarrow{\mathbb{L}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}\mathsf{oalg}_{\mathbb{L}} \arrow[d, "\mathbb{U}_{\mathbb{L}}"] \\
\mathbb{S}\mathsf{q}(\mathsf{K}) \arrow[r, "\mathbb{S}\mathsf{q}(V)"'] & \mathbb{S}\mathsf{q}(\mathsf{M})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{A}\mathsf{lg}^{\overrightarrow{\mathbb{R}}} \arrow[r] \arrow[d, "\mathbb{U}_{\overrightarrow{\mathbb{R}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}\mathsf{lg}^{\mathbb{R}} \arrow[d, "\mathbb{U}_{\mathbb{R}}"] \\
\mathbb{S}\mathsf{q}(\mathsf{C}) \arrow[r, "\mathbb{S}\mathsf{q}(U)"'] & \mathbb{S}\mathsf{q}(\mathsf{M}).
\end{tikzcd}
\tag{3.8}
$$

Proposition 3.9 [7, Proposition 13]. *Let $(\mathbb{L}, \mathbb{R})$ be an accessible algebraic weak factorization system on M, and suppose there are given adjunctions (1.2). If C and K are also locally presentable, then $(\mathbb{L}, \mathbb{R})$ admits both an accessible left-lifting $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ along V and an accessible right-lifting $(\overrightarrow{\mathbb{L}}, \overrightarrow{\mathbb{R}})$ along U, in the sense of Definition 3.7.*

The proof is an application of Theorem 3.6: in the left-lifted case, say, we first pull back $\mathbb{U}_{\mathbb{L}}$ along $\mathbb{S}\mathsf{q}(V)$ to obtain a double functor $\mathbb{U}\colon \mathbb{A} \to \mathbb{S}\mathsf{q}(\mathsf{K})$, and obtain the desired $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ from this by showing that $\mathbb{U}$ satisfies the hypotheses of Theorem 3.6. The only hypothesis which is non-trivial to verify is that $U_1\colon \mathcal{A}_1 \to \mathsf{K}^{\mathbf{2}}$ is a left adjoint, and for this, we exploit local presentability and argue exactly as in the proof of Lemma 2.11.

3.3. *The flaw in the previous proof*

We are now in a position to explain the error made in [14] in proving Theorem 2.6. The authors state Proposition 3.9 above as Theorems 3.3.1 (for the left case) and Theorem 3.3.2 (for the right), but add clauses which amount to the following:

Claim 3.10. In the situation of Definition 3.7, if the stated left- and right-liftings $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ and $(\overrightarrow{\mathbb{L}}, \overrightarrow{\mathbb{R}})$ of $(\mathbb{L}, \mathbb{R})$ exist, then:

(i) the underlying weak factorization system of $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ is the left-lifting of the underlying weak factorization system of $(\mathbb{L}, \mathbb{R})$; and
(ii) the underlying weak factorization system of $(\overrightarrow{\mathbb{L}}, \overrightarrow{\mathbb{R}})$ is the right-lifting of the underlying weak factorization system of $(\mathbb{L}, \mathbb{R})$.

This claim would legitimize the following means of constructing the left- or right-liftings of an accessible weak factorization system as in Theorem 2.6. One first chooses an accessible algebraic realization; then lifts that; and then takes the underlying weak factorization system. The problem with this is that:

Proposition 3.11. *Claim 3.10 is false.*

As left-lifting along a left adjoint is the same as right-lifting along its opposite, it suffices to disprove either (i) or (ii). So let us concentrate on (i), the case of left-lifting $(\mathbb{L}, \mathbb{R})$ along a left adjoint $V\colon \mathsf{K} \to \mathsf{M}$. On the one hand, the underlying weak factorization system of $(\mathbb{L}, \mathbb{R})$ has left class $\mathrm{Retr}(\exists\mathbb{L})$, and so the left-lifting of this underlying weak factorization system along V has left class $V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$. On the other hand, the left-lifting of $(\mathbb{L}, \mathbb{R})$ along V is characterized by a pullback of double categories as in (3.8); so in particular, we have a pullback of categories as follows; compare with the situation of (2.13):

$$\begin{tikzcd}
\mathsf{Coalg}_{\overleftarrow{\mathbb{L}}} \arrow[r] \arrow[d, "\mathrm{U}_{\overleftarrow{\mathbb{L}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Coalg}_{\mathbb{L}} \arrow[d, "\mathrm{U}_{\mathbb{L}}"] \\
\mathsf{K}^{\mathbf{2}} \arrow[r, "V^{\mathbf{2}}"'] & \mathsf{M}^{\mathbf{2}}.
\end{tikzcd}$$

Inspecting the images of the vertical functors, we see that a map f of K admits $\overleftarrow{\mathbb{L}}$-map structure if and only if Vf admits $\mathbb{L}$-map structure. So, $\exists(\overleftarrow{\mathbb{L}}) = V^{-1}(\exists\mathbb{L})$ and the underlying weak factorization system of $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ has left class $\mathrm{Retr}(V^{-1}(\exists\mathbb{L}))$. This analysis shows that Claim 3.10(i) is equally the claim that

$$\mathrm{Retr}(V^{-1}(\exists\mathbb{L})) = V^{-1}(\mathrm{Retr}(\exists\mathbb{L})). \tag{3.12}$$

Since functors preserve retracts, it will always be the case that $\mathrm{Retr}(V^{-1}(\exists\mathbb{L})) \subseteq V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$; however, the two examples that we give below show that, in certain cases, this inclusion is *strict*. Both of these examples exploit the following general construction of an algebraic weak factorization system which originates in [**8**, § 4.1]; we refer the reader to there for more details.

Example 3.13. Let M be a category with finite coproducts, and let $\mathbb{P}$ be a comonad on M with counit $\upsilon\colon P \Rightarrow 1$ and comultiplication $\Delta\colon P \Rightarrow PP$. There is an algebraic weak factorization system on M with functorial factorization:

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\iota_1} X + PY \xrightarrow{\langle f, \upsilon_Y \rangle} Y$$

and with the fillers δ_f and μ_f of (1.4) given by the respective composites

$$X + PY \xrightarrow{1_X + P\iota_2 \Delta_Y} X + P(X + PY) \text{ and } X + PY + PY \xrightarrow{1_X + \nabla_{PY}} X + PY.$$

The $\mathbb{R}$-maps of this algebraic weak factorization system are the $\mathbb{P}$-*split epis* $(p, i)\colon X \to Y$, comprising a map $p\colon X \to Y$ together with a '$\mathbb{P}$-section': a map $i\colon PY \to X$ such that $pi = \upsilon_Y\colon PY \to Y$. The $\mathbb{L}$-maps do not in general admit a direct description, but the 'algebraically cofibrant objects' – the $\mathbb{L}$-maps with domain 0 – are precisely the coalgebras for the comonad $\mathbb{P}$.

We now give the first of our examples disproving the equality (3.12).

Example 3.14. If A is any commutative ring, then there is a weak factorization system $(\mathcal{L}, \mathcal{R})$ on Mod_A cofibrantly generated by the single map $0 \to A$. The class $\mathcal{L}$ comprises the monomorphisms with projective cokernel – so in particular, the $\mathcal{L}$-cofibrant objects are the projective modules – while $\mathcal{R}$ comprises the epimorphisms; see [**16**, Lemma 2.2.6, Proposition 2.2.9].

We obtain an algebraic realization $(\mathbb{L}, \mathbb{R})$ for $(\mathcal{L}, \mathcal{R})$ using Example 3.13, where we take the comonad $\mathbb{P}$ therein to be the one generated by the forgetful-free adjunction $U\colon \mathsf{Mod}_A \leftrightarrows \mathsf{Set}\colon F$. In this case, the $\mathbb{R}$-maps are A-module morphisms $f\colon M \to N$ endowed with a section at the level of underlying sets; while an algebraically cofibrant object – a $\mathbb{P}$-coalgebra – is easily seen to be a free A-module endowed with a choice of generators.

We now specialize to the case $A = \mathbb{Z}/6$, so that Mod_A is the category of abelian groups in which every element is 6-torsion. We will disprove the equality $\mathrm{Retr}(V^{-1}(\exists\mathbb{L})) = V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$ in (3.12) when V is taken to be the left adjoint:

$$V\colon \mathsf{Mod}_{\mathbb{Z}/6} \xrightarrow{\mathbb{Z}/2 \otimes_{\mathbb{Z}/6} (-)} \mathsf{Mod}_{\mathbb{Z}/6}.$$

On the one hand, $0 \to M$ lies in $\exists\mathbb{L}$ just when M is a free $\mathbb{Z}/6$-module. Since the objects in the image of V are all 2-torsion, and the only $\mathbb{Z}/6$-module which is 2-torsion and free is 0, it follows that $0 \to M$ lies in $V^{-1}(\exists\mathbb{L})$ just when M contains no 2-torsion elements. Since such M can be identified with the $\mathbb{Z}/3$-modules, they are retract-closed and so, finally, $0 \to M$ lies in $\mathrm{Retr}(V^{-1}(\exists\mathbb{L}))$ just when M contains no 2-torsion elements.

On the other hand, a map $0 \to M$ is in $\mathrm{Retr}(\exists\mathbb{L})$ just when M is projective; it now follows that $0 \to \mathbb{Z}/6$ lies in $V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$, since $V(\mathbb{Z}/6) = \mathbb{Z}/2$ is projective as a direct summand $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$. We have thus shown that $0 \to \mathbb{Z}/6$ is in $V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$ but not in $\mathrm{Retr}(V^{-1}(\exists\mathbb{L}))$, as desired.

The second example is built on the same principle.

EXAMPLE 3.15. Let M be the monoid $\{1, e\}$ with $e^2 = e$ and consider the category M-Set of M-sets endowed with the weak factorization system $(\mathcal{L}, \mathcal{R})$ cofibrantly generated by the single map $\emptyset \to M$. The $\mathcal{R}$-maps are the epimorphisms, and as each M-set is a retract of a coproduct of copies of M, each object is cofibrant.

We obtain an algebraic realization $(\mathbb{L}, \mathbb{R})$ for $(\mathcal{L}, \mathcal{R})$ using Example 3.13, where we take the comonad $\mathbb{P}$ to be the one generated by the free-forgetful adjunction $U: M\text{-}\mathsf{Set} \leftrightarrows \mathsf{Set}: F$. Now, $\mathbb{R}$-maps are maps of M-sets endowed with a section of the underlying function; while an algebraically cofibrant object is one with free M-action (the coalgebra structure is, in this case, uniquely determined).

In this situation, we will show $\mathrm{Retr}(V^{-1}(\exists\mathbb{L})) \subsetneq V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$ with V taken to be the left adjoint functor $\mathsf{Set} \to M\text{-}\mathsf{Set}$ which endows each set with its trivial M-action. On the one hand, $\emptyset \to X$ lies in $\exists\mathbb{L}$ just when X is a free M-set. Since the trivial action is only free on the empty set, we see that $\emptyset \to X$ lies in $V^{-1}(\exists\mathbb{L})$, or equally in $\mathrm{Retr}(V^{-1}(\exists\mathbb{L}))$, only when $X = \emptyset$. On the other hand, every map of the form $\emptyset \to X$ is in $\mathrm{Retr}(\exists\mathbb{L})$, and so every map $\emptyset \to X$ lies in $V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$. So $\mathrm{Retr}(V^{-1}(\exists\mathbb{L})) \subsetneq V^{-1}(\mathrm{Retr}(\exists\mathbb{L}))$ as desired.

These examples are concerned with lifting factorizations for a single weak factorization system. If desired, they can be enhanced to examples concerning lifting factorizations for an accessible model category by taking $\mathcal{C} = \mathcal{L}$ and $\mathcal{F} = \mathcal{R}$ and $\mathcal{W} =$ all maps. Of course, the model categories so arising are homotopically rather uninteresting, but in particular cases we may be able to do better. For instance, a dg version of Example 3.14 occurs in lifting the (cofibration, acyclic fibration) weak factorization system of the standard model structure on $\mathsf{Ch}(\mathsf{Mod}_{\mathbb{Z}/6})$.

4. *Fixing the previous proof*

In this final section, we describe how the erroneous Claim 3.10 can be corrected by adding extra hypotheses, and then show that this revised claim allows for a correct proof of the algebraic version of Theorem 2.6. Toward our first goal, let us define an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ to be *left-retract-closed* (respectively, *right-retract-closed*) if the class of maps $\exists\mathbb{L}$ (respectively, $\exists\mathbb{R}$) is closed under retracts.

PROPOSITION 4.1. *Claim* 3.10(i) *holds for any* $(\mathbb{L}, \mathbb{R})$ *which is left-retract-closed, while Claim* 3.10(ii) *holds for any right-retract-closed* $(\mathbb{L}, \mathbb{R})$.

Proof. The two cases are dual, so it suffices to consider a left-retract-closed $(\mathbb{L}, \mathbb{R})$ on M and a left adjoint $V: \mathsf{K} \to \mathsf{M}$ along which the left-lifting $(\overleftarrow{\mathbb{L}}, \overleftarrow{\mathbb{R}})$ exists. We must prove the equality (3.12). We already noted that

$$\mathrm{Retr}(V^{-1}(\exists\mathbb{L})) \subseteq V^{-1}(\mathrm{Retr}(\exists\mathbb{L})),$$

since functors preserve retracts. Conversely, because $\exists\mathbb{L} = \mathrm{Retr}(\exists\mathbb{L})$, we have

$$V^{-1}(\mathrm{Retr}(\exists\mathbb{L})) = V^{-1}(\exists\mathbb{L}) \subseteq \mathrm{Retr}(V^{-1}(\exists\mathbb{L})). \qquad \square$$

This suggests the following legitimate construction of the left- or right-liftings of an accessible weak factorization system $(\mathcal{L}, \mathcal{R})$. One first chooses a *left-retract-closed* (respectively, right-retract-closed) accessible algebraic realization; then lifts that; and then takes the underlying weak factorization system.

In order for this to work, the required left- and right-retract-closed algebraic realizations of $(\mathcal{L}, \mathcal{R})$ must exist. Since we already know that at least one accessible algebraic realization $(\mathbb{L}, \mathbb{R})$ exists, it suffices to show that this can be adjusted to a left-retract-closed one $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ and a right-retract-closed one $(\mathbb{L}^\flat, \mathbb{R}^\flat)$ with the same underlying weak factorization system.

The idea is to construct the adjustment $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ in such a way that the $\mathbb{L}^\sharp$-maps are precisely the cloven $\mathcal{L}$-maps: for then, by Lemma 2.11, $\exists(\mathbb{L}^\sharp) = \mathcal{L}$, which is indeed closed under retracts; moreover, the underlying weak factorization system is clearly the same. Dually, we will construct $(\mathbb{L}^\flat, \mathbb{R}^\flat)$ such that the $\mathbb{R}^\flat$-maps are the cloven $\mathcal{R}$-maps. In fact, by Theorem 2.6, these motivating *descriptions* of $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ and $(\mathbb{L}^\flat, \mathbb{R}^\flat)$ are nearly sufficient for their *construction*. The only additional aspect that is required is:

Proposition 4.2. *Let $(\mathbb{L}, \mathbb{R})$ be an accessible algebraic weak factorization system. The cloven $\mathcal{L}$-maps admit a composition law $\mathsf{Clov}(\mathcal{L}) \times_{\mathsf{M}} \mathsf{Clov}(\mathcal{L}) \to \mathsf{Clov}(\mathcal{L})$ making them the vertical morphisms and squares of a double category $\mathbb{C}\mathsf{lov}(\mathcal{L}) \to \mathbb{S}\mathsf{q}(\mathsf{M})$ over $\mathbb{S}\mathsf{q}(\mathsf{M})$ whose objects and horizontal morphisms are those of M. Dually, the cloven $\mathcal{R}$-maps constitute a double category $\mathbb{C}\mathsf{lov}(\mathcal{R}) \to \mathbb{S}\mathsf{q}(\mathsf{M})$.*

Proof. By duality, we need only consider the left case. Our proof follows [**7**, § 2.7]. To begin with, we define an *algebra lifting operation* for a map $f\colon A \to B$ to be the choice, for each $\mathbb{R}$-algebra $(g,p)\colon C \to D$ and each map $(h,k)\colon f \to g$ in M^2 of a filler $\varphi_{(g,p)}(h,k)\colon B \to C$:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] \arrow[ur, dashed, "\varphi(h,k)"] & D, \end{tikzcd} \tag{4.3}$$

subject to the naturality condition that, for any map $(u,v)\colon (g,p) \to (h,r)$ of $\mathbb{R}$-algebras, we have $u\varphi_{(g,p)}(h,k) = \varphi_{(h,r)}(uh, vk)$.

Now, a square like (4.3) is equally an object of the comma category $f \downarrow \mathrm{U}_\mathbb{R}$, and the unit map $(Lf, 1)\colon f \to \mathrm{U}_\mathbb{R}(Rf, \mu_f)$ is initial in this comma category; so to give φ is equally to give a single map $\varphi_{(Rf,\mu_f)}(Lf,1)\colon B \to Ef$ filling the left square of (2.9). In this way, we obtain an isomorphism $\mathsf{Clov}(\mathcal{L}) \cong \mathsf{Lift}_\mathbb{R}$ over M^2, where $\mathsf{Lift}_\mathbb{R}$ is the category of maps endowed with algebra lifting operations and squares commuting with the lifting operations.

We may now exploit this isomorphism to define the desired composition law on algebra lifting operations rather than on cloven $\mathcal{L}$-maps. Given maps $f\colon A \to B$ and $g\colon B \to C$ endowed with lifting operations φ and ψ, we obtain a composite lifting operation $\psi\varphi$ on gf by first lifting against f and then against g:

$$\psi\varphi_{(h,p)}(u,v) = \psi_{(h,p)}(\varphi_{(h,p)}(u,vg),v) \qquad \begin{tikzcd} A \arrow[rr, "u"] \arrow[d, "f"'] & & D \arrow[dd, "h"] \\ B \arrow[d, "g"'] \arrow[urr, dashed, "\varphi(u,vg)"] & & \\ C \arrow[rr, "v"'] \arrow[uurr, dashed, "\psi(\varphi(u,vg),v)"'] & & E\, . \end{tikzcd}$$

This assignation is easily functorial with respect to maps of lifting operations, thus yielding a functor $\mathsf{Lift}_\mathbb{R} \times_{\mathsf{M}} \mathsf{Lift}_\mathbb{R} \to \mathsf{Lift}_\mathbb{R}$. To see that this gives rise to the desired double category, we must check associativity and unitality of this composition law. Associativity is immediate on comparing the formulae for $\xi(\psi\varphi)$ and $(\xi\psi)\varphi$; while an identity at A is easily seen to be given by the lifting structure $(1_A, \iota_A)\colon A \to A$ with $(\iota_A)_{(g,p)}(u,v) = u$. ∎

Remark 4.4. The double categories $\mathbb{C}\mathsf{lov}(\mathcal{L})$ and $\mathbb{C}\mathsf{lov}(\mathcal{R})$ are in fact expansions of the double categories $\mathbb{C}\mathsf{oalg}_{\mathbb{L}}$ and $\mathbb{A}\mathsf{lg}^{\mathbb{R}}$ of $\mathbb{L}$- and $\mathbb{R}$-maps: the above proof simply repeats the construction of the composition laws on the latter in the broader context. In particular, this means that there are canonical inclusion double functors $\mathbb{C}\mathsf{oalg}_{\mathbb{L}} \hookrightarrow \mathbb{C}\mathsf{lov}(\mathcal{L})$ and $\mathbb{A}\mathsf{lg}^{\mathbb{R}} \hookrightarrow \mathbb{C}\mathsf{lov}(\mathcal{R})$ over $\mathbb{S}\mathsf{q}(\mathsf{M})$.

We now use these double categories to build the desired left- and right-shifted algebraic weak factorization systems.

Proposition 4.5. *Let $(\mathbb{L}, \mathbb{R})$ be an accessible algebraic weak factorization system on a locally presentable category M. There exist accessible algebraic weak factorization systems $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ and $(\mathbb{L}^\flat, \mathbb{R}^\flat)$ characterized by isomorphisms of double categories*

$$\mathbb{C}\mathsf{oalg}_{\mathbb{L}^\sharp} \cong \mathbb{C}\mathsf{lov}(\mathcal{L}) \quad \textit{and} \quad \mathbb{A}\mathsf{lg}^{\mathbb{R}^\flat} \cong \mathbb{C}\mathsf{lov}(\mathcal{R}) \tag{4.6}$$

over $\mathbb{S}\mathsf{q}(\mathsf{M})$. Furthermore, $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ is left-retract-closed, $(\mathbb{L}^\flat, \mathbb{R}^\flat)$ is right-retract-closed, and both have the same underlying weak factorization system as $(\mathbb{L}, \mathbb{R})$.

Proof. We verified the final sentence above; as for the existence of $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ and $(\mathbb{L}^\flat, \mathbb{R}^\flat)$, the arguments involve applying Theorem 3.6 to the double categories $\mathbb{C}\mathsf{lov}(\mathcal{L})$ and $\mathbb{C}\mathsf{lov}(\mathcal{R})$ over $\mathbb{S}\mathsf{q}(\mathsf{M})$. We give details only in the left case.

For hypothesis (i) of Theorem 2.6, the object-level functor $1_{\mathsf{M}} \colon \mathsf{M} \to \mathsf{M}$ is clearly an isomorphism, while the arrow-level functor $U_{\mathcal{L}} \colon \mathsf{Clov}(\mathcal{L}) \to \mathsf{M}^{\mathbf{2}}$ has a right adjoint by Lemma 2.11, and is therefore comonadic because it is the forgetful functor from the category of coalgebras for a copointed endofunctor; see [**19**, § 5.1], for example.

For hypothesis (ii), note first that the *unique* cloven $\mathcal{L}$-map structure on an identity map $1_A \colon A \to A$ is given by $(1_A, L1_A) \colon A \to A$. To verify (ii) therefore, we must show that any cloven $\mathcal{L}$-map $(f, s) \colon A \to B$, the square left below lifts to a map $(1_A, L1_A) \to (f, s)$ of cloven $\mathcal{L}$-maps.

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \qquad \begin{tikzcd} E1_A \arrow[r, "E(1_A, f)"] & Ef \\ A \arrow[u, "L1_A"] \arrow[r, "f"'] & B \arrow[u, "s"'] \end{tikzcd}$$

This is equally to show the commutativity of the square above right; for which we calculate that $E(1_A, f) \circ L1_A = Lf = sf$. □

Remark 4.7. The inclusion double functors of Remark 4.4 compose with the isomorphisms (4.6) to yield double functors $\mathbb{C}\mathsf{oalg}_{\mathbb{L}} \to \mathbb{C}\mathsf{oalg}_{\mathbb{L}^\sharp}$ and $\mathbb{A}\mathsf{lg}^{\mathbb{R}} \to \mathbb{A}\mathsf{lg}^{\mathbb{R}^\sharp}$ over $\mathbb{S}\mathsf{q}(\mathsf{M})$. The existence of these double functors can be equivalently expressed as saying that we have *oplax* (= 'left Quillen') morphisms of algebraic weak factorization systems $(\mathbb{L}^\flat, \mathbb{R}^\flat) \to (\mathbb{L}, \mathbb{R}) \to (\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ with underlying functor the identity; see [**24**, Lemma 6.9].

Using the preceding proposition, we can finally give:

Proof of Theorem 2.6 (*bis*). Given the accessible weak factorization system $(\mathcal{L}, \mathcal{R})$ on M, we first choose an accessible algebraic realization $(\mathbb{L}, \mathbb{R})$. In the left-lifted case, we then replace this with the left-retract-closed realization $(\mathbb{L}^\sharp, \mathbb{R}^\sharp)$ given by Proposition 4.5. Now, by Proposition 3.9, this admits a left-lifting along $V \colon \mathsf{K} \to \mathsf{M}$ to an accessible algebraic weak factorization system $(\overleftarrow{\mathbb{L}}^\sharp, \overleftarrow{\mathbb{R}}^\sharp)$ on K. Since we are in the left-retract-closed situation, Proposition 4.1 ensures that the underlying weak factorization system of $(\overleftarrow{\mathbb{L}}^\sharp, \overleftarrow{\mathbb{R}}^\sharp)$ is the desired accessible left-lifting of $(\mathcal{L}, \mathcal{R})$ along V. The case of right-lifting is entirely dual. □

REMARK 4.8. In giving the preceding proof, we treated the left- and right-lifted cases entirely symmetrically; however, in practice there is an asymmetry. The proof of Proposition 3.5 above, which we omitted, involves the construction of a particular accessible algebraic realization $(\mathbb{L}, \mathbb{R})$ for each given accessible $(\mathcal{L}, \mathcal{R})$. It turns out that this particular $(\mathbb{L}, \mathbb{R})$ is always right-retract-closed, since its category of $\mathbb{R}$-maps is *cofibrantly generated by a small category* in the sense of [**11**]. Thus, so long as this particular algebraic realization is chosen, there is no need to make an adjustment in the right-lifted case. This point was already spelt out by the third author in [**24**, Theorem 2.10].

References

1. J. ADÁMEK and J. ROSICKÝ, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series 189 (Cambridge University Press, Cambridge, 1994).
2. T. ATHORNE, 'Coalgebraic cell complexes', PhD Thesis, University of Sheffield, 2013.
3. M. BARR and C. WELLS, *Toposes, triples and theories*, Grundlehren der Mathematischen Wissenschaften 278 (Springer, 1985).
4. T. BARTHEL, J. MAY and E. RIEHL, 'Six model structures for dg-modules over dgas: Model category theory in homological action', *New York J. Math.* 20 (2014) 1077–1160.
5. M. BAYEH, K. HESS, V. KARPOVA, M. KĘDZIOREK, E. RIEHL and B. SHIPLEY, 'Left-induced model structures and diagram categories', *Women in topology: collaborations in homotopy theory*, Contemporary Mathematics 641 (eds M. Basterra, K. Bauer, K. Hess and B. Johnson; American Mathematical Society, Providence, RI, 2015) 49–81.
6. G. BIRD, 'Limits in 2-categories of locally-presented categories', PhD Thesis, University of Sydney, 1984.
7. J. BOURKE and R. GARNER, 'Algebraic weak factorisation systems I: accessible AWFS', *J. Pure Appl. Algebra* 220 (2016) 108–147.
8. J. BOURKE and R. GARNER, 'Algebraic weak factorisation systems II: categories of weak maps', *J. Pure Appl. Algebra* 220 (2016) 148–174.
9. M. CHING and E. RIEHL, 'Coalgebraic models for combinatorial model categories', *Homology Homotopy Appl.* 16 (2014) 171–184.
10. P. GABRIEL and F. ULMER, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics 221 (Springer, Berlin–Heidelberg, 1971).
11. R. GARNER, 'Understanding the small object argument', *Appl. Categ. Structures* 17 (2009) 247–285.
12. M. GRANDIS and W. THOLEN, 'Natural weak factorization systems', *Arch. Math.* (*Brno*) 42 (2006) 397–408.
13. K. HESS and M. KĘDZIOREK, 'The homotopy theory of coalgebras over simplicial comonads', *Homology Homotopy Appl.* 21 (2019) 247–268.
14. K. HESS, M. KĘDZIOREK, E. RIEHL and B. SHIPLEY, 'A necessary and sufficient condition for induced model structures', *J. Topol.* 10 (2017) 324–369.
15. K. HESS and B. SHIPLEY, 'The homotopy theory of coalgebras over a comonad', *Proc. Lond. Math. Soc.* 108 (2014) 484–516.
16. M. HOVEY, *Model categories*, Mathematical Surveys and Monographs 63 (American Mathematical Society, Providence, RI, 1999).
17. A. JOYAL and R. STREET, 'Pullbacks equivalent to pseudopullbacks', *Cah. Topol. Géom. Différ. Catég.* 34 (1993) 153–156.
18. A. JOYAL and M. TIERNEY, 'Strong stacks and classifying spaces', *Category theory* (*Como*, 1990), Lecture Notes in Mathematics 1488 (eds A. Carboni, M. C. Pedicchio and G. Rosolini; Springer, Berlin, 1991) 213–236.
19. G. M. KELLY, 'A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on', *Bull. Aust. Math. Soc.* 22 (1980) 1–83.
20. G. M. KELLY, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series 64 (Cambridge University Press, Cambridge, 1982). (Republished as: *Reprints in Theory and Applications of Categories* 10 (2005).)
21. M. MAKKAI and R. PARÉ, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics 104 (American Mathematical Society, Providence, RI, 1989).
22. M. MAKKAI and J. ROSICKÝ, 'Cellular categories', *J. Pure Appl. Algebra* 218 (2014) 1652–1664.
23. L. MOSER, 'Injective and projective model structures on enriched diagram categories', *Homology Homotopy Appl.* 21 (2019) 279–300.
24. E. RIEHL, 'Algebraic model structures', *New York J. Math.* 17 (2011) 173–231.
25. J. ROSICKÝ, 'Accessible model categories', *Appl. Categ. Structures* 25 (2017) 187–196.
26. J. ROSICKÝ and W. THOLEN, 'Lax factorization algebras', *J. Pure Appl. Algebra* 175 (2002) 355–382.

Richard Garner
Centre of Australian Category Theory
Macquarie University
Sydney, NSW 2109
Australia

richard.garner@mq.edu.au

Magdalena Kędziorek
Universiteit Utrecht
Hans Freudenthalgebouw
Budapestlaan 6
3584 CD Utrecht
The Netherlands

m.kedziorek@uu.nl

Emily Riehl
Department of Mathematics
Johns Hopkins University
3400 N. Charles St.
Baltimore, MD 21218
USA

eriehl@math.jhu.edu