HOMOTOPY (LIMITS AND) COLIMITS

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ABSTRACT. These notes were written to accompany two talks given in the Algebraic Topology and Category Theory Proseminar at the University of Chicago in Winter 2009. When a category has some notion of limits and colimits associated to it, its ordinary limits and colimits are not necessarily homotopically meaningful. We describe a notion of a "homotopy colimit" for two sorts of categories with a homotopy theory: categories enriched in simplicial sets and model categories. For simplicial categories, we define an object with a "homotopical universal property" using the well-known bar construction. For model categories, we define a homotopy colimit functor to be a derived functor of the usual colimit functor. Finally, we note that in the setting of a simplicial model category, these two approaches coincide and refer the reader to appropriate sources.

1. Introduction

Motivated by topological spaces, we call a category *homotopical* if it is equipped with some specified notion of *weak equivalence*, a class of morphisms with the property that if any two of a composable pair and their composite is a weak equivalence then so is the third. In such a setting, one is often interested in how objects behave in the *homotopy category*, formed by formally inverting this class of arrows. But because this localization process can be a bit unwieldy, it it preferable to work at the "point-set level," i.e., in the original category, and ask whether particular constructions are invariant under weak equivalence.

Frequently, limits and colimits do not have this property: so-called *homotopy* limits and colimits do. There are two approaches to the definition. A mathematician blessed with sufficient intuition might define a particular homotopy colimit directly by "fattening up" the ordinary colimit to produce a new object with room to "hang a homotopy." This process can be formalized using functor tensor products (secretly weighted colimits) or the bar construction. We give these formulas in §3 below, though regrettably do not fully explain the intuition. Instead, we hope the reader will compute a few examples and see that the results are familiar.

The second approach defines all homotopy colimits for diagrams of shape $\mathcal{D}$ simultaneously by taking the left derived functor of the colimit functor. There are several settings in which this derived functor is guaranteed to exist. In §4 below, we employ the theory of model categories, though weaker settings suffice.

Finally, in §5, we note that these two approaches are consistent in the most common setting in which both make sense. A proof is given in [9], which is a much better source for this material than these hastily written notes.

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2. Kan Extensions and Coends

Before discussing homotopy colimits, we begin with some categorical preliminaries – Kan extensions and coends – that will appear frequently in what follows. Derived functors are examples of Kan extensions and the bar construction is defined using a coend.

2.1. **Kan Extensions.** Given functors $T\colon \mathcal{M} \to \mathcal{A}$ and $K\colon \mathcal{M} \to \mathcal{C}$, the left Kan extension of T along K, when it exists, will consist of a functor $\mathrm{Lan}_K T\colon \mathcal{C} \to \mathcal{A}$ and a natural transformation $\eta\colon T \Rightarrow \mathrm{Lan}_K T \circ K$ that is universal from T to functors $U \circ K$. Dually, the right Kan extension, when it exists, consists of a functor $\mathrm{Ran}_K T\colon \mathcal{C} \to \mathcal{A}$ and a natural transformation $\epsilon\colon \mathrm{Ran}_K T \circ K \Rightarrow T$ with a dual universal property.

$$
\begin{tikzcd}
\mathcal{M} \arrow[rr, "T"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{A} \\
& \mathcal{C} \arrow[ur, dashed, "\mathrm{Lan}_K T"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M} \arrow[rr, "T"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathcal{A} \\
& \mathcal{C} \arrow[ur, dashed, "\mathrm{Ran}_K T"'] &
\end{tikzcd}
$$

When the left and right Kan extensions exist for all $T \in \mathcal{A}^{\mathcal{M}}$, they will form left and right adjoints, respectively, to the functor $- \circ K\colon \mathcal{A}^{\mathcal{C}} \to \mathcal{A}^{\mathcal{M}}$, i.e., we have natural bijections

$$\mathcal{A}^{\mathcal{C}}(\mathrm{Lan}_K T, S) \cong \mathcal{A}^{\mathcal{M}}(T, S \circ K) \quad \text{and} \quad \mathcal{A}^{\mathcal{M}}(S \circ K, T) \cong \mathcal{A}^{\mathcal{C}}(S, \mathrm{Ran}_K T).$$

The natural transformations η and ϵ above are components of the unit and counit for these respective adjunctions.

Example 2.1. Let $F\colon \mathcal{M} \to \mathcal{K}$ be a functor between two model categories with localizations $\gamma\colon \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ and $\delta\colon \mathcal{K} \to \mathrm{Ho}\,\mathcal{K}$, respectively. Immediately from the definitions, a right derived functor $\mathbb{R}F\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{K}$ is a left Kan extension of δF along γ. Dually, a left derived functor $\mathbb{L}F\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{K}$ is a right Kan extension of δF along γ.

By the universal properties, left and right Kan extensions are unique up to unique isomorphism. Left Kan extensions will arise more frequently in what follows, so we will focus on them in particular, but all of the results hold dually for right Kan extensions.

Given K and T as above and assuming the colimits that appear below exist, we can define $\mathrm{Lan}_K Tc$ for any $c \in \mathcal{C}$ to be

$$(\mathrm{Lan}_K T)c\colon = \mathrm{colim}(K/c \xrightarrow{U} \mathcal{M} \xrightarrow{T} \mathcal{A}), \tag{2.2}$$

where U denotes the forgetful functor and K/c is the slice category, whose objects consists of an object $m \in \mathcal{M}$ together with an arrow $Km \to c$ in $\mathcal{C}$. The universal property of these colimits is used to define $\mathrm{Lan}_K T$ on arrows. The component η_m of the universal map is defined to be the component of the colimiting cone defining $(\mathrm{Lan}_K T)Km$ over the identity arrow at Km in $\mathcal{C}$. Unraveling this definition, one can check that $\mathrm{Lan}_K T$ and η satisfy the required universal property of a left Kan extension.

Two consequences of this explicit construction are the following:

Corollary 2.3. *If $\mathcal{M}$ is small and $\mathcal{A}$ is cocomplete, any functor $T\colon \mathcal{M} \to \mathcal{A}$ has a left Kan extension along any $K\colon \mathcal{M} \to \mathcal{C}$, and $K^*\colon \mathcal{A}^{\mathcal{C}} \to \mathcal{A}^{\mathcal{M}}$ has a left adjoint.*

Corollary 2.4. *If K is full and faithful, then the universal arrow $\eta\colon T \to Lan_K T \circ K$ is a natural isomorphism.*

Proof. For each $m \in \mathcal{M}$, $\mathrm{id}\colon Km \to Km$ is terminal in the comma category $K/(Km)$ because K is full and faithful. So the colimit in (2.2) can be found by evaluating TU on this terminal object. Hence, $(\mathrm{Lan}_K T)Km = Tm$ and $\eta_m = 1$. □

Example 2.5. The usual geometric realization functor $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ is a left Kan extension of the functor $\Delta\colon \mathbf{\Delta} \to \mathbf{Top}$ that takes the object $[n]$ to the standard topological n-simplex Δ_n along the Yoneda embedding $y\colon \mathbf{\Delta} \hookrightarrow \mathbf{sSet}$. As y is full and faithful, we get that $|\Delta^n| \cong \Delta_n$,[1] i.e., the geometric realization of the simplicial set represented by $[n] \in \mathbf{\Delta}$ is the standard topological n-simplex.

2.2. **Coends.** A more elegant formula for left Kan extensions is given using a coend, which is a special type of colimit.

Definition 2.6. A *coend* of a functor $S\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ is a universal dinatural transformation from S to a constant $a \in \mathcal{A}$. Equivalently, a coend is defined to be a coequalizer

$$\coprod_{f\colon c \to d \in \mathrm{mor}\,\mathcal{C}} S(d,c) \underset{S(1,f)}{\overset{S(f,1)}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} S(c,c) \dashrightarrow a$$

Explicitly, the coend consists of an object a and arrows $\omega_c\colon S(c,c) \to a$ for all $c \in \mathcal{C}$ such that for each $f\colon c \to d$ in $\mathcal{C}$, the square

$$\begin{tikzcd} S(d,c) \arrow[r, "S(f,1)"] \arrow[d, "S(1,f)"'] & S(c,c) \arrow[d, "\omega_c"] \\ S(d,d) \arrow[r, "\omega_d"'] & a \end{tikzcd}$$

commutes, and such that the pair (a, ω) is universal with this property.

Notation. The object a in the coned is often denoted by

$$\int^{c \in \mathcal{C}} S(c,c).$$

Example 2.7. Let R be a commutative ring. A right R-module A is an additive functor $A\colon R^{\mathrm{op}} \to \mathbf{Ab}$ and a left R-module B is an additive functor $B\colon R \to \mathbf{Ab}$. Using the usual tensor product $\otimes_{\mathbb{Z}}$ in $\mathbf{Ab}$, A and B form a bifunctor $R \mapsto A \otimes_Z B\colon R^{\mathrm{op}} \times R \to \mathbf{Ab}$. The coend

$$\int^R A \otimes_{\mathbb{Z}} B = A \otimes_R B$$

is the usual tensor product over R of a right and left R-module.

[1] As this example illustrates, the symbol "Δ" will be severely overloaded in this paper. The author hopes that each meaning is clear from context, and the fact that all the notations used here are reasonably standard.

Example 2.8. The above example extends to the functor tensor product. Given a monoidal category $\mathcal{A}$ and functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ and $G\colon \mathcal{C} \to \mathcal{A}$, the external tensor product defines a bifunctor $F \otimes G\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$. Again, the coend

$$\int^{\mathcal{C}} F \otimes G = F \otimes_{\mathcal{C}} G$$

gives the usual functor tensor product.

For example, the geometric realization of a simplicial set $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set} \hookrightarrow \mathbf{Top}$ considered as a simplicial space with the discrete topology, is the functor tensor product

$$|X| \colon = X \otimes_{\mathbf{\Delta}} \Delta,$$

where $\Delta\colon \mathbf{\Delta} \to \mathbf{Top}$ is as in Example 2.5.

2.3. **Left Kan extensions as Coends.** A *tensor* or *copower* in a category $\mathcal{A}$ of an object $a \in \mathcal{A}$ with a set S, denoted $S \cdot a$ or $S \odot a$ is simply the coproduct, indexed by S, of a with itself, i.e., $\coprod_S a$.

Theorem 2.9. *Given functors $K\colon \mathcal{M} \to \mathcal{C}$ and $T\colon \mathcal{M} \to \mathcal{A}$ such that the following tensors and coends exist, T has a left Kan extension along K defined on objects by*

$$(Lan_K T)c = \int^{m \in \mathcal{M}} \mathcal{C}(Km, c) \cdot Tm.$$

Proof. See [6, §X.4]. □

As above, we use ω for the colimiting wedge of the tensor. We may then define η_n for $n \in \mathcal{M}$ to be the composite

$$Tn \xrightarrow{\mathrm{incl}_{\mathrm{id}_{Kn}}} \mathcal{C}(Kn, Kn) \cdot Tn \xrightarrow{\omega_n} \int^{m \in \mathcal{M}} C(Km, Kn) \cdot Tm = \mathrm{Lan}_K T(Kn).$$

3. Local Homotopy Colimits

For the idea of a "homotopical universal property" to be meaningful, we want $\mathcal{M}$ to be in some sense topological. So for this section, let $\mathcal{M}$ be a cocomplete category enriched in simplicial sets. We will define the homotopy colimit of an ordinary functor $F\colon \mathcal{C} \to \mathcal{M}$, where $\mathcal{C}$ is an arbitrary small category.

First, we need a slight generalization of the functor tensor product introduced in Example 2.8. If $\mathcal{U}$ is a category enriched in **sSet** such that $\mathcal{U}$ is tensored over **sSet**, we can define the functor tensor product of $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$ and $G\colon \mathcal{C} \to \mathcal{U}$ as follows

$$F \otimes_{\mathcal{C}} G \colon = \int^{c \in \mathcal{C}} Fc \odot Gc,$$

whenever the desired coends exist. So, for example, the functor tensor product of a simplicial set $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and $\Delta\colon \mathbf{\Delta} \to \mathbf{Top}$ makes sense without regarding the X as a discrete simplicial space.

Now we are prepared for the following definition.

Definition 3.1. Given an ordinary functor $F\colon \mathcal{C} \to \mathcal{M}$, with $\mathcal{M}$ enriched in simplicial sets, define

$$\mathrm{hocolim} F = N(-/\mathcal{C}) \otimes_{\mathcal{C}} F = \int^{\mathcal{C}} N(d/\mathcal{C}) \odot Fd.$$

$$\mathrm{holim} F = \hom_{\mathcal{C}}(N(\mathcal{C}/-), F) = \int_{\mathcal{C}} F d^{N(\mathcal{C}/d)}.$$

Assuming these limits and colimits exist, these define functors

$$\mathrm{hocolim}, \mathrm{holim}\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}.$$

Example 3.2. Familiar examples of homotopy colimits in **Top** include the mapping cylinder (the colimit of an arrow $\bullet \longrightarrow \bullet$), the double mapping cylinder (the colimit of $\bullet \longleftarrow \bullet \longrightarrow \bullet$), and the mapping telescrope (the colimit of $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$). This will be more readily seen after we redefine the homotopy colimit in terms of the familiar bar construction.

3.1. **The Bar Construction.**

In practice, it is easier to compute these colimits by means of the bar construction. In this section, we will introduce the bar construction. In the next, we will connect it to the notion of homotopy colimit introduced above.

The bar construction can be done in a great deal of generality (e.g., see [8]). Here we'll let $\mathcal{V}$ be a symmetric monoidal category and let $\mathcal{U}$ be enriched in $\mathcal{V}$. Let $Z\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{U}$ be a functor. We will consider two cases simultaneously — where $\mathcal{C}$ is an ordinary category and where $\mathcal{C}$ is a $\mathcal{V}$-category. For the latter, we want Z to be a $\mathcal{V}$-functor as well. We also want some form of geometric realization, so we fix an ordinary functor $\Delta\colon \mathbf{\Delta} \to \mathcal{U}$. Geometric realization will then be defined for a simplicial object X in $\mathcal{U}$ as the functor tensor product

$$|X|\colon = X \otimes_{\mathbf{\Delta}} \Delta.$$

If enriched categories are confusing, just take $\mathcal{U} = \mathbf{Top}$ and $\mathcal{C}$ an ordinary category and forget about this added generality.

Before defining the bar construction, we must define the simplicial bar construction.

Definition 3.3. The *simplicial bar construction* $B_*(\mathcal{C}, Z)$ is a simplicial object in $\mathcal{U}$. The n-simplices are

$$B_n(\mathcal{C}, Z) = \coprod_{(\mathrm{ob}\,\mathcal{C})^n} (\mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_0, c_1)) \odot Z(c_n, c_0).$$

The tensors here are from the monoidal structure on $\mathcal{V}$ if $\mathcal{C}$ is simplicially enriched or from the cartesian monoidal structure on **Set** if $\mathcal{C}$ is ordinary. When $\mathcal{C}$ is unenriched, it is more convenient describe the n-simplices as the following coproduct

$$B_n(\mathcal{C}, Z) = \coprod_{\gamma\colon [n] \to \mathcal{C}} Z(\gamma(n), \gamma(0)), \tag{3.4}$$

where we're using the fact that tensors involving sets are just coproducts indexed by that set.

It remains to define the maps that make $B_*(\mathcal{C}, Z)$ a simplicial object in $\mathcal{U}$. In the case where $\mathcal{C}$ is ordinary, we regard the B_n as an object of the form (3.4) and note that the coproduct is over the set $N\mathcal{C}_n$ of n-simplices of the nerve of $\mathcal{C}$. The simplicial maps d_i and s_i are simply induced by the corresponding maps of the nerve $N\mathcal{C}$.

The enriched case is slightly trickier to describe. For $0 < i < n$, the map $d_i\colon B_n(\mathcal{C}, Z) \to B_{n-1}(\mathcal{C}, Z)$ is induced by the composition map

$$\circ\colon \mathcal{C}(c_i, c_{i+1}) \otimes \mathcal{C}(c_{i-1}, c_i) \to \mathcal{C}(c_{i-1}, c_{i+1}),$$

an arrow in $\mathcal{V}$. For all i, the map $s_i\colon B_n(\mathcal{C},Z)\to B_{n+1}(\mathcal{C},Z)$ is induced by the identity arrow $\mathbf{1}\to\mathcal{C}(c_i,c_i)$, where $\mathbf{1}$ is the unit of the monoidal structure on $\mathcal{V}$. It remains only to define d_0 and d_n, and these definitions are analogous. Explicitly, $d_0\colon B_n(\mathcal{C},Z)\to B_{n-1}(\mathcal{C},Z)$ is the map induced by

$$\begin{tikzcd}
(\mathcal{C}(c_{n-1},c_n)\otimes\cdots\otimes\mathcal{C}(c_0,c_1))\odot Z(c_n,c_0) \arrow[r, "\mathrm{incl}"] \arrow[d, "\cong"'] & B_n(\mathcal{C},Z) \arrow[dd, dashed, "d_0"] \\
(\mathcal{C}(c_{n-1},c_n)\otimes\cdots\otimes\mathcal{C}(c_1,c_2))\odot(\mathcal{C}(c_0,c_1)\odot Z(c_n,c_0)) \arrow[d, "\mathrm{id}\odot\phi_0"'] & \\
(\mathcal{C}(c_{n-1},c_n)\otimes\cdots\otimes\mathcal{C}(c_1,c_2))\odot Z(c_n,c_1) \arrow[r, "\mathrm{incl}"] & B_{n-1}(\mathcal{C},Z)
\end{tikzcd}$$

where $\phi_0\colon\mathcal{C}(c_0,c_1)\odot Z(c_n,c_0)\to Z(c_n,c_1)$ is adjunct, in the adjunction defining the tensor, to the arrow $\mathcal{C}(c_0,c_1)\to\mathcal{U}(Z(c_n,c_0),Z(c_n,c_1))$ which is specified as part of the data that makes $Z(c_n,-)$ is a $\mathcal{V}$-functor. It is straightforward to check that these d_i and s_i satisfy the desired relations to make $B_*(\mathcal{C},Z)$ a simplicial object in $\mathcal{U}$.

For an easy example, when $\mathcal{U}=\mathbf{Set}$ and Z is the constant functor that sends everything to the terminal object, $B_*(\mathcal{C},Z)$ is the familiar nerve $N\mathcal{C}$ of $\mathcal{C}$.

Definition 3.5. The *bar construction* is the geometric realization of the simplicial bar construction, i.e.,

$$B(\mathcal{C},Z)=|B_*(\mathcal{C},Z)|=B_*(\mathcal{C},Z)\otimes_{\mathbf{\Delta}}\Delta.$$

An important special case occurs when $\mathcal{U}$ also has some sort of monoidal structure. In this case, the functor Z is often defined instead as the "external tensor product" of two functors $G\colon\mathcal{C}^{\mathrm{op}}\to\mathcal{U}$ and $F\colon\mathcal{C}\to\mathcal{U}$; explicitly

$$Z=G\overline{\otimes}F\colon(a,b)\mapsto Ga\otimes Fb.$$

When Z has this form, we write $B_*(G,\mathcal{C},F)$ for the simplicial bar construction and $B(G,\mathcal{C},F)$ for the bar construction. This notation is consistent with [7].

3.2. **Relation to Homotopy Colimits.** The relationship between the bar construction and homotopy colimits is made apparent by the following theorem.

Theorem 3.6. *Let $F\colon\mathcal{C}\to\mathcal{M}$ with $\mathcal{M}$ a simplicially enriched category with a monoidal structure. Let $*$ denote the constant functor from $\mathcal{C}$ to the unit of the monoidal structure. Suppose also, for convenience, that the simplicial enrichment is given by a functor S that is left adjoint to geometric realization. Then*

$$\mathit{hocolim}\,F\cong B(*,\mathcal{C},F).$$

When $\mathcal{M}$ is a simplicially enriched category with a monoidal structure $\otimes$ such that the simplicial enrichment is given by a functor that is left adjoint to geometric realization, $|X|\otimes m$ satisfies the defining universal property of the tensor $X\odot m$, where $m\in\mathcal{M}$ and X is a simplicial set. We will need this fact below.

Proof. Let $y\colon\mathcal{C}^{\mathrm{op}}\to[\mathcal{C},\mathbf{Set}]$ denote the functor $c\mapsto\mathcal{C}(c,-)$, which sends an object of c to its covariant represented functor. Then

$$N(-/\mathcal{C})=B_*(*,\mathcal{C},y)\colon\mathcal{C}^{\mathrm{op}}\to\mathbf{sSet},$$

so from the definition

$$\begin{aligned}
\operatorname{hocolim} F &= N(-/\mathcal{C}) \otimes_{\mathcal{C}} F \\
&= B_*(*, \mathcal{C}, y) \otimes_{\mathcal{C}} F \\
&= \int^{c \in \mathcal{C}} B_*(*, \mathcal{C}, \mathcal{C}(c, -)) \odot Fc \\
&= \int^{c \in \mathcal{C}} |B_*(*, \mathcal{C}, \mathcal{C}(c, -))| \otimes Fc \\
&= \int^{c \in \mathcal{C}} \left(\int^{n \in \mathbf{\Delta}} B_n(*, \mathcal{C}, \mathcal{C}(c, -)) \odot \Delta(n) \right) \otimes Fc \\
&= \int^{n \in \mathbf{\Delta}} \left(\int^{c \in C} B_n(*, \mathcal{C}, \mathcal{C}(c, -)) \odot Fc \right) \otimes \Delta(n) \qquad (3.7)
\end{aligned}$$

by Fubini's theorem for iterated coends. Similarly,

$$\begin{aligned}
B(*, \mathcal{C}, F) &= |B_*(*, \mathcal{C}, F)| \\
&= \int^{n \in \mathbf{\Delta}} B_n(*, \mathcal{C}, F) \otimes \Delta(n) \\
&= \int^{n \in \mathbf{\Delta}} \left(\coprod_{\gamma\colon [n] \to \mathcal{C}} F\gamma(0) \right) \otimes \Delta(n) \qquad (3.8)
\end{aligned}$$

So if we can show that

$$\int^{c \in \mathcal{C}} B_n(*, \mathcal{C}, \mathcal{C}(c, -)) \odot Fc = \coprod_{\gamma\colon [n] \to \mathcal{C}} F\gamma(0)$$

then we may conclude that (3.7)=(3.8). The tensor on the left is with a set, so we may rewrite the left hand side as

$$\int^{c \in C} \coprod_{N(c/\mathcal{C})_n} Fc,$$

bearing in mind that $B_*(*, \mathcal{C}, \mathcal{C}(c, -)) = N(c/\mathcal{C})$. Elements of $N(c/\mathcal{C})_n$ are strings $\gamma\colon [n] \to \mathcal{C}$ of n composable arrows in $\mathcal{C}$ together with an arrow $c \to \gamma(0)$ in $\mathcal{C}$. As coproducts and coends commute,

$$\int^{c \in C} \coprod_{N(c/\mathcal{C})_n} Fc = \int^{c \in C} \coprod_{\gamma\colon [n] \to \mathcal{C}} \coprod_{c \to \gamma(0)} Fc = \coprod_{\gamma\colon [n] \to \mathcal{C}} \int^{c \in \mathcal{C}} \coprod_{c \to \gamma(0)} Fc,$$

and by inspection the coend on the right is what we want, completing the proof. □

3.3. **An Example.** One of the most familiar homotopy colimits is the topological mapping cylinder, which is the homotopy colimit of a single arrow $X \xrightarrow{f} Y$ in **Top**. Let us compute it using the bar construction.

Let $\mathcal{C} = \mathbf{2} = (0 \to 1)$ be the category with objects 0 and 1 and one non-identity arrow. Let $F\colon \mathcal{C} \to \mathbf{Top}$ be the ordinary functor with image $X \xrightarrow{f} Y$. The simplical bar construction $B_*(*, \mathcal{C}, F)$ yields

$$B_0 = X \sqcup Y \quad \text{and} \quad B_1 = X \sqcup X \sqcup Y,$$

where the first X in B_0 corresponds to the domain of the image of the identity at 0, the second X corresponds to the domain of f, and the Y corresponds to the domain of the image of the identity at Y. We may write $B_1 = X_0 \sqcup X_f \sqcup Y_1$ to keep track of which object arises from which arrow.

The homotopy colimit of f is the geometric realization of B_*, which is a quotient of a coproduct indexed only on the non-degenerate simplices of B_*. Since the nerve of $\mathcal{C}$ is degenerate above level one (one might say 1-skeletal), all the n-simplices of $B_*(*, \mathcal{C}, F)$ are degenerate when $n > 1$. So it suffices to stop at level zero in the computation that follows.

By definition

$$\operatorname{hocolim} F = |B_*(*, \mathcal{C}, F)| = \int^{n \in \mathbf{\Delta}} B_n(*, \mathcal{C}, F) \times \Delta_n,$$

where Δ_n is the standard topological n-simplex. Expanding this coend, we see that

$$\operatorname{hocolim} F = \operatorname{colim} \left(\begin{tikzcd} B_0 \times \Delta_1 \arrow[r, "s_0"] \arrow[dr, "s^0"', "d_1"] & B_1 \times \Delta_1 \\ B_1 \times \Delta_0 \arrow[ur, shift left, "d_0"] \arrow[ur, shift right] \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & B_0 \times \Delta_0 \end{tikzcd} \right)$$

$$= \operatorname{colim} \left(\begin{tikzcd} X \times I \sqcup Y \times I \arrow[r, "s_0"] \arrow[dr, "s^0"', "d_1"] & (X \times I)_0 \sqcup (X \times I)_f \sqcup (Y \times I)_1 \\ X_0 \sqcup X_f \sqcup Y_1 \arrow[ur, shift left, "d_0"] \arrow[ur, shift right] \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & X \sqcup Y \end{tikzcd} \right)$$

The colimit is computed by first taking the disjoint union

$$(X \sqcup Y) \sqcup ((X \times I) \sqcup (X \times I) \sqcup (Y \times I))$$

of the two objects on the right and then form the quotient that identifies any two points that appear in the images of any pair of corresponding maps (s_0 and s^0 or d_i and d^i). In particular s_0 includes $X \times I$ into the 0 component of the top coproduct and s^0 projects onto X in the bottom coproduct. So $(X \times I)_0$ gets squashed down onto and identified with X. Similarly, $(Y \times I)_1$ gets squashed down onto and identified with Y. The result so far is

$$X \sqcup (X \times I)_f \sqcup Y, \tag{3.9}$$

but we haven't quotiented by the d_i and d^i yet!

The images of X_0 and Y_1 under d_0 and d^0 and d_1 and d^1 have already been identified when we quotiented using the degeneracies. The image of X_f under d^1 is X and the map restricts to the identity. The image of X_f under d_0 is $X \times \{0\} \subset (X \times I)_f$, so the 0-th face of this cylinder gets glued to the X in (3.9). The image of X_f under d^0 is $f(X) \subset Y$, while the image of X_f under d_0 is $X \times \{1\} \subset X \times I$. So the 1-th face of this cylinder gets identified with Y by gluing $x \times 1$ to $f(x)$. The result is

$$Mf = X \times I \sqcup Y / \sim$$

where $\sim$ denotes this gluing. This is the usual mapping cylinder.

3.4. **The Cobar Construction.** There is a dual to Theorem 3.6 that constructs homotopy limits by means of the *cobar construction*. We give a few details about this construction because it is slightly trickier than its dual version and also is explicated less frequently.

There are two ways we could dualize the bar construction. Replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$ yields the same construction defined previously, but when we replace $\mathcal{U}$ with $\mathcal{U}^{\mathrm{op}}$, the result looks substantially different. The resulting dual construction is called the *cobar construction*.

Let $\mathcal{V}$, $\mathcal{C}$, $\mathcal{U}$, and Z be as in §3.1. As before, we start by defining the cosimplicial cobar construction.

Definition 3.10. The *cosimplicial cobar construction* $C^*(\mathcal{C}, Z)$ is a cosimplicial object in $\mathcal{U}$, defined dually to the simplicial bar construction. The n-simplices are

$$C^n(\mathcal{C}, Z) = \prod_{(\mathrm{ob}\,\mathcal{C})^n} (\mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_0, c_1)) \pitchfork Z(c_0, c_n),$$

where $v \pitchfork u$ denotes the *cotensor* (aka *power*) of an object $u \in \mathcal{U}$ by an object $v \in \mathcal{V}$ (see [10]).[2] When $\mathcal{C}$ is an ordinary category, the repeated tensor product is just a product of sets and the cotensor is just the product of the object $Z(c_0, c_n)$ of $\mathcal{U}$ with itself indexed by this set. When $\mathcal{C}$ is unenriched, it is more convenient describe the n-simplices as the following product

$$C^n(\mathcal{C}, Z) = \prod_{\gamma\colon [n] \to \mathcal{C}} Z(\gamma(0), \gamma(n)), \tag{3.11}$$

using the fact mentioned above that cotensors involving sets are just products indexed by that set.

It remains to define the maps that make $C^*(\mathcal{C}, Z)$ a cosimplicial object in $\mathcal{U}$. Before, we do so, it is important to note that an arrow $v \to v'$ in $\mathcal{V}$ induces an arrow $v' \pitchfork u \to v \pitchfork u$ in $\mathcal{U}$ by the defining universal property. A slogan to help remember this is that cotensors are similar to homs; indeed if $\mathcal{V} = \mathcal{U} = \mathbf{Set}$, then $v \pitchfork u = \prod_v u = \hom_{\mathbf{Set}}(v, u)$. This is the essential reason why $C^*(\mathcal{C}, Z)$ is cosimplicial, rather than simplicial.

In the case where $\mathcal{C}$ is ordinary, we regard the C^n as an object of the form (3.11) and note that the product (or cotensor, if you prefer) is over (with) the set $N\mathcal{C}_n$ of n-simplices of the nerve of $\mathcal{C}$. The cosimplicial maps d^i and s^i are simply induced by the corresponding maps d_i and s_i of the nerve $N\mathcal{C}$.

For the enriched case, we define one of the "harder" maps and leave the remaining definitions as an exercise to the reader. The map $d^0\colon C^{n-1}(\mathcal{C}, Z) \to C^n(\mathcal{C}, Z)$ is

[2]We dislike the notation $v \pitchfork u$. In other contexts, we prefer $\{v, u\}$ but here we've decided it looks more confusing.

the map induced by the universal property of the product as shown below

$$\begin{tikzcd} C^{n-1}(\mathcal{C}, Z) \arrow[r, "\mathrm{proj}"] \arrow[dd, dashed, "d^0"'] & (\mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_1, c_2)) \pitchfork Z(c_1, c_n) \arrow[d, "\mathrm{id} \pitchfork \phi^0"] \\ & (\mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_1, c_2)) \pitchfork [\mathcal{C}(c_0, c_1) \pitchfork Z(c_0, c_n)] \arrow[d, "\cong"] \\ C^n(\mathcal{C}, Z) \arrow[r, "\mathrm{proj}"'] & (\mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_0, c_1)) \pitchfork Z(c_0, c_n) \end{tikzcd}$$

where $\phi^0 \colon Z(c_1, c_n) \to \mathcal{C}(c_0, c_1) \pitchfork Z(c_0, c_n)$ is adjoint in the defining adjunction of the cotensor to the arrow $\mathcal{C}(c_0, c_1) \to \mathcal{U}(Z(c_1, c_n), Z(c_0, c_n))$ which is specified as part of the data that makes $Z(-, c_n)$ is a $\mathcal{V}$-functor.

The definitions of the other d^i and s^i that make $C^*(\mathcal{C}, Z)$ a cosimplicial object in $\mathcal{U}$ are similar.

Definition 3.12. The *cobar construction* $C(\mathcal{C}, Z)$ is defined by taking the hom of the cosimplicial cobar construction, i.e.,

$$C(\mathcal{C}, Z) \colon = \hom_{\mathbf{\Delta}}(\Delta, C^*(\mathcal{C}, Z))$$

where $\Delta \colon \mathbf{\Delta} \to \mathcal{U}$ is the functor we used above to define geometric realization.

4. Homotopy Colimit Functors

We now shift perspectives to consider categories $\mathcal{M}$ where the "homotopy theory" comes in the form of a Quillen model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on $\mathcal{M}$. (See [4] for an introduction to model categories, including much of the material in this section.) Rather than define an object with a homotopical universal property, as in the previous section, we will seek to define a global homotopy colimit functor that is "homotopically well-behaved."

4.1. **A concrete example.** Let's start with a few observations that may be familiar to the topologists: homotopy equivalences are not (in general) preserved by pushouts. For example, consider a pushout in $\mathbf{Top}^{\mathbf{2}}$, the category of arrows and commutative squares, of the following diagram

$$\begin{tikzcd} D^n \arrow[d, "\sim"] & S^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, equal] & D^n \arrow[d, "\sim"] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

All of the vertical maps are homotopy equivalences. But the pushout of the top row is S^n and the pushout of the bottom row is $*$ and the induced map $S^n \to *$ is certainly not a homotopy equivalence!

A slogan here is that we need to replace the maps we are pushing out along by cofibrations to get a homotopically meaningful pushout. We formalize this below.

4.2. **Derived functors from Quillen adjunctions.** If $\mathcal{M}$ has all colimits of shape $\mathcal{D}$, these define an adjoint pair of functors

$$\operatorname{colim}\colon \mathcal{M}^{\mathcal{D}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M}\colon \Delta \tag{4.1}$$

Now suppose $\mathcal{M}$ is a model category. If $\mathcal{M}^{\mathcal{D}}$ has a model structure such that (4.1) is a Quillen adjunction then a general theorem implies the existence of a derived adjunction

$$\mathbb{L}\operatorname{colim}\colon \operatorname{Ho}\mathcal{M}^{\mathcal{D}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \operatorname{Ho}\mathcal{M}\colon \mathbb{R}\Delta$$

between the homotopy categories. Here is the general theorem:

Theorem 4.2. *Let $\mathcal{M}$ and $\mathcal{K}$ be model category and $F\colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}\colon G$ be an adjoint pair. If $F \dashv G$ is a Quillen adjunction, then the derived functors $\mathbb{L}F$ and $\mathbb{R}G$ exist and form an adjoint pair*

$$\mathbb{L}F\colon Ho\,\mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ho\,\mathcal{K}\colon \mathbb{R}G.$$

Proof. Ken Brown's lemma and a few technicalities. □

Furthermore, we can define $\mathbb{L}F$ and $\mathbb{R}G$ in a particularly simple manner in good settings. Supposing that $\mathcal{M}$ has a functorial cofibrant replacement $Q\colon \mathcal{M} \to \mathcal{M}$ and $\mathcal{K}$ has a functorial fibrant replacement $R\colon \mathcal{K} \to \mathcal{K}$ then we may define $\mathbb{L}F = FQ$ and $\mathbb{R}G = GR$.

We call the left derived functor $\mathbb{L}\operatorname{colim}\colon \operatorname{Ho}(\mathcal{M}^{\mathcal{D}}) \to \operatorname{Ho}\mathcal{M}$ of the colimit functor the *homotopy colimit functor*. It is important to note that

Remark 4.3. The canonical map $\operatorname{Ho}(\mathcal{M}^{\mathcal{D}}) \to \operatorname{Ho}(\mathcal{M})^{\mathcal{D}}$ induced by the universal property of localization is not typically a categorical equivalence. Hence, $\mathbb{L}\operatorname{colim}$ is not usually left adjoint to $\Delta_{\operatorname{Ho}\mathcal{M}}\colon \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}(\mathcal{M})^{\mathcal{D}}$. Thus, "homotopy pushouts" are not "pushouts in the homotopy category," a potential source of confusion.

Let's consider a specific example. Let $\mathcal{D} = \{\, a \xleftarrow{f} b \xrightarrow{g} c \,\}$. Then given any functor $X\colon \mathcal{D} \to \mathcal{M}$, the colimit of X is the pushout of Xf and Xg. We wish to define a model structure on $\mathcal{M}^{\mathcal{D}}$ compatible with the adjunction (4.1). Weak equivalences in the category $\mathcal{M}^{\mathcal{D}}$ should be the pointwise weak equivalences. As we've seen in the example above, colim will not preserve these and so does not induce a functor $\operatorname{Ho}(\mathcal{M}^{\mathcal{D}}) \to \operatorname{Ho}(\mathcal{M})$ directly.

Instead, we hope to apply the theorem above. Keeping in mind that our right adjoint is the diagonal functor, we want a model structure on $\mathcal{M}^{\mathcal{D}}$ that has pointwise fibrations as well as pointwise weak equivalences. It then follows immediately that Δ preserves both fibrations and trivial fibrations, so $\operatorname{colim} \dashv \Delta$ is a Quillen adjunction. Since Δ is furthermore a *homotopical* functor (i.e., preserves weak equivalences), the universal property of localization tells us that $\operatorname{Ho}\Delta = \mathbb{R}\Delta$. So $\mathbb{L}\operatorname{colim} \dashv \operatorname{Ho}\Delta$ is the desired "homotopy colimit functor."

For this particular $\mathcal{D}$, pointwise weak equivalences and pointwise fibrations do indeed determine a model structure on $\mathcal{M}^{\mathcal{D}}$ with no additional hypotheses on $\mathcal{M}$ called the *Reedy model structure*. We will define the cofibrations briefly and refer the reader to [4, §10] for proof. Given a morphism $\alpha\colon X \Rightarrow Y$ in $\mathcal{M}^{\mathcal{D}}$, we define arrows by the pushouts indicated below

$$\begin{tikzcd}
 & Xa \arrow[d] \arrow[ddl, bend right, "\alpha_a"'] & Xb \arrow[l, "Xf"'] \arrow[r, "Xg"] \arrow[d, "\alpha_b"] & Xc \arrow[d] \arrow[ddr, bend left, "\alpha_c"] & \\
 & \cdot \arrow[dl, dashed, "i_a"'] & Yb \arrow[l] \arrow[r] \arrow[dll, bend left, "Yf"] \arrow[drr, bend right, "Yg"'] & \cdot \arrow[dr, dashed, "i_c"] & \\
Ya & & & & Yc
\end{tikzcd}$$

We declare α to be a cofibration if and only if i_a, α_b, and i_c are cofibrations.

Unraveling this definition, the cofibrant replacement of $X \in \mathcal{M}^{\mathcal{D}}$ substitutes the cofibrant replacement of Xb for this object and replaces the maps Xf and Xg by cofibrations. This accords with the topologist's intuition for the example at the beginning of this section.

4.3. **Other Homotopy Limit and Colimit Functors.** A dual model structure can be used to define a homotopy pullback functor for any model category $\mathcal{M}$. Unsurprisingly, this model structure has pointwise cofibrations and weak equivalences, with fibrations defined using a pullback condition.

Under very restrictive hypotheses on the category $\mathcal{D}$, this construction can be generalized. However, we can make much more progress by placing a few restrictions on $\mathcal{M}$.

Theorem 4.4. *If $\mathcal{M}$ is cofibrantly generated, then $\mathcal{M}^{\mathcal{D}}$ has the* projective model structure*, in which fibrations and weak equivalences are defined pointwise.*

As we saw above, in this case the colimit functor is left Quillen in the projective model structure, so $\mathbb{L}\mathrm{colim}$ exists by Theorem 4.2.

Many model structures are cofibrantly generated, so we can define the projective model structure on the corresponding diagram category and obtain the homotopy colimit functor $\mathbb{L}\mathrm{colim}$. However, the hypotheses for the dual injective model structure are much more restrictive (for definitions, see [1]).

Theorem 4.5. *When $\mathcal{M}$ is sheafifiable, then $\mathcal{M}^{\mathcal{D}}$ has the* injective model structure*, in which cofibrations and weak equivalences are defined pointwise.*

Again, in this case the limit functor is right Quillen, so $\mathbb{R}\mathrm{colim}$ exists.

5. Comparison

It remains to compare the two approaches to defining homotopy colimits in the case where both may be applied; namely, when $\mathcal{M}$ is a simplicial model category. To get the local and global approaches to agree, we have to modify the local homotopy colimits as follows:

$$\begin{aligned} \operatorname{hocolim} F &:= \operatorname{hocolim} QF \\ \operatorname{holim} F &:= \operatorname{holim} RF \end{aligned}$$

where Q and R are cofibrant and fibrant replacement applied pointwise to F.[3]

The comparison theorem now states

[3] The functors defined in 3.1 are often called the *uncorrected* homotopy colimits and limits.

Theorem 5.1. *As defined above,* hocolim *is a left derived functor of* colim *and* holim *is a right derived functor of* lim.

The proof is too long for these notes. Fortunately, an excellent reference exists: Mike Shulman's [9], which was a source for much of this material.

References

[1] T. Beke, Sheafifiable homotopy model categories, in *Math. Proc. Camb. Phil. Soc.*, 2000.
[2] A.K. Bousfield and D.M. Kan, *Homotopy Limits, Completions and Localizations*, SLNM 304, Springer, Berlin 1972.
[3] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, J.H. Smith, *Homotopy Limit Functors on Model Categories and Homotopical Categories*, 2004.
[4] W.G. Dwyer and J. Spalinski, Homotopy theories and model categories, in *Handbook of Algebraic Topology*, 1995.
[5] P. Hirschhorn, *Model Categories and Their Localizations*, MSM 99, 2002.
[6] S. Mac Lane, *Categories for the Working Mathematician, Second Edition, Graduate Texts in Mathematics 5*, Springer-Verlag, 1997.
[7] J.P. May, Classifying spaces and fibrations, in *Mem. Amer. Math. Soc. 155*, 1975.
[8] J.-P. Meyer, Bar and Cobar Constructions, I, in *J. Pure and Applied Algebra 33*, 163-207, 1984.
[9] M. Shulman, Homotopy limits and colimits and enriched homotopy theory, arXiv:math/0610194v3, 2006.
[10] E. Riehl, Weighted Limits and Colimits, available at math.harvard.edu/∼eriehl.

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