TWO-SIDED DISCRETE FIBRATIONS
IN 2-CATEGORIES AND BICATEGORIES

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Abstract. Fibrations over a category B, introduced to category theory by Grothendieck, determine pseudo-functors $B^{\mathrm{op}} \to \mathbf{Cat}$. A two-sided discrete variation models functors $B^{\mathrm{op}} \times A \to \mathbf{Set}$. By work of Street, both notions can be defined internally to an arbitrary 2-category or bicategory. While the two-sided discrete fibrations model profunctors internally to $\mathbf{Cat}$, unexpectedly, the dual two-sided codiscrete cofibrations are necessary to model $\mathcal{V}$-profunctors internally to $\mathcal{V}$-$\mathbf{Cat}$. There are many categorical prerequisites, particularly in the later sections, but we believe they are strictly easier than the topics below that take advantage of them. These notes were written to accompany a talk given in the Algebraic Topology and Category Theory Proseminar in the fall of 2010 at the University of Chicago.

1. Introduction

Fibrations were introduced to category theory in [Gro61, Gro95] and developed in [Gra66]. Ross Street gave definitions of fibrations internal to an arbitrary 2-category [Str74] and later bicategory [Str80]. For the case $\mathcal{K} = \mathbf{Cat}$, the 2-categorical definitions agree with the classical ones, while the bicategorical definitions are more general.

While these notes present fibrations in these contexts, the real goal is to define two-sided discrete fibrations, which provide a model profunctors $B^{\mathrm{op}} \times A \to \mathbf{Set}$ in $\mathbf{Cat}$ and the dual two-sided codiscrete cofibrations, which model $\mathcal{V}$-profunctors $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ in $\mathcal{V}$-$\mathbf{Cat}$.

In our attempt to cover a lot of material as expediently as possible, we give only a few proofs but do provide thorough citations.

This theory has been extended to quasi-categories, a model for $(\infty, 1)$-categories, by André Joyal and Jacob Lurie. The Grothendieck construction, which is called *straightening* in [Lur09], plays a particularly important role. A sequel to these notes, which may or may not be written, would address this extension.

Comma categories. One categorical prerequisite is so important to merit a brief review. Given a pair of functors $B \xrightarrow{f} C \xleftarrow{g} A$ (an *opspan* in $\mathbf{Cat}$), the *comma category* f/g has triples $(b \in B, fb \to ga, a \in A)$ as objects and morphisms $(b, fb \to ga, a) \to (b', fb' \to ga', a')$ given by a pair of arrows $a \to a' \in A$, $b \to b' \in B$ such that the obvious triangle commutes. This category is equipped with canonical

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projects to A and B as well as a 2-cell

$$\begin{tikzcd} f/g \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\Leftarrow"] & B \arrow[d, "f"] \\ A \arrow[r, "g"'] & C \end{tikzcd}$$

and is universal among such data. Equivalently, it is the limit of the opspan $B \xrightarrow{f} C \xleftarrow{g} A$ weighted by $\mathbb{1} \xrightarrow{d} \mathbb{2} \xleftarrow{c} \mathbb{1}$, the inclusions of the terminal category as the domain and codomain of the walking arrow.

Often, we are interested in comma categories in which either f or g is an identity (in which case it is denoted by the name of the category) or in which either A or B is terminal (in which case the functor is denoted by the object it identifies). Such categories are sometimes called *slice categories*.

A few notes on terminology. What we call *fibrations* in **Cat** are sometimes called *categorical*, *Grothendieck*, *Cartesian*, or *right* (and unfortunately also *left*) *fibrations*. The left-handed version, now *opfibrations*, was originally called *cofibrations*, though this name was rejected to avoid confusing topologists. Somewhat unfortunately, as we shall see below, once fibrations have been defined internally to a 2-category $\mathcal{K}$, the *opfibrations* are precisely the fibrations in $\mathcal{K}^{\mathrm{co}}$ (formed by reversing the 2-cells only), while the *cofibrations* are precisely the fibrations in $\mathcal{K}^{\mathrm{op}}$ (formed by reversing the 1-cells only).

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2. Fibrations in 1-category theory

Loosely, a *fibration* is a functor $p\colon E \to B$ such that the fibers E_b depend contravariantly pseudo-functorially on the objects $b \in B$. Many categories are naturally *fibered* in this way.

2.1. Discrete fibrations.

We start with an easier variant.

Definition 2.1.1. A functor $p\colon E \to B$ is a *discrete fibration* if for each object $e \in E$ and arrow $f\colon b' \to pe \in B$, there exists a unique lift $g\colon e' \to e$.

Let $\mathbf{DFib}(B)$ denote the category of discrete fibrations over B, defined to be a full subcategory of the comma category $\mathbf{Cat}/B$. Two facts about discrete fibrations are particularly important:

Theorem 2.1.2. *There is an isomorphism of categories*

$$\mathbf{DFib}(B) \cong [B^{op}, \mathbf{Set}].$$

Proof. Given a discrete fibration $E \to B$, define $B^{\mathrm{op}} \to \mathbf{Set}$ by $b \mapsto E_b$, the category whose objects sit over $b \in B$ and whose arrows map to the identity at b. Because 1_b has a unique lift for each $e \in E_b$, this category is discrete. For each morphism $f\colon b' \to b$, define $f^*\colon E_b \to E_{b'}$ by mapping $e \in E_b$ to the domain of the unique lift of f with codomain e. Functoriality follows from uniqueness of lifts.

Conversely, given a functor $F\colon B^{\mathrm{op}} \to \mathbf{Set}$, the canonical functor from its category of elements $*/F$ to B is a discrete fibration. The comma category $*/F$, whose objects are elements of Fb for some $b \in B$, is sometimes called the *Grothendieck construction* on the presheaf F. □

A functor $f\colon C \to D$ is *final* if any diagram of shape D can be restricted along f to a diagram of shape C without changing its colimit or, equivalently, if for all $d \in D$, the comma category d/f is non-empty and connected.

Theorem 2.1.3. *There is an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on* **Cat** *with $\mathcal{E}$ the final functors and $\mathcal{M}$ the discrete fibrations.*

Proof. Exercise. (Indeed, this was an exam problem for the Part III course I took from Peter Johnstone.) □

This is called the *comprehensive factorization system* for reasons I've never understood. Note that there is an "internal" rephrasing of this definition. Write B_0 for the set of objects and B_1 for the set of arrows of a small category B.

Definition 2.1.4. A functor $p : E \to B$ between small categories is a *discrete fibration* iff

$$\begin{tikzcd} E_1 \arrow[r, "\mathrm{cod}"] \arrow[d, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E_0 \arrow[d, "p_0"] \\ B_1 \arrow[r, "\mathrm{cod}"'] & B_0 \end{tikzcd}$$

is a pullback in **Set**.

Finally, we mention *discrete opfibrations*, which are discrete fibrations $p\colon E^{\mathrm{op}} \to B^{\mathrm{op}}$ which have unique lifts of morphisms with specified domain. These correspond bijectively to functors $B \to \mathbf{Set}$ and form an orthogonal factorization system with the class of *initial* functors which are those such that restriction preserves limits.

2.2. **Fibrations.** Now we're ready for the real thing.

Definition 2.2.1. Given a functor $p\colon E \to B$, an arrow $g\colon e' \to e$ in E is *p-cartesian* if for any $g'\colon e'' \to e$ such that $pg' = pg \cdot h$ in B, there is a unique lift k of h such that $g' = g \cdot h$. A functor $p : E \to B$ is a *fibration* if each $f : b \to pe$ in B has a p-cartesian lift with codomain e.

Some sources differentiate between fibrations and those with chosen cartesian lifts which may satisfy additional properties. See [Gra66] or the nLab.

Theorem 2.2.2. *A functor $p\colon E \to B$ is a fibration if and only if either*

(i) *for each $e \in E$, the functor $p\colon E/e \to B/p(e)$ has a right adjoint right inverse*
(ii) *the canonical functor $E^2 \to B/p$ has a right adjoint right inverse*

Proof. In each case, the right adjoint picks out p-cartesian lifts for each morphism. See [Gra66, Prop 3.11]. □

Let $\mathbf{Fib}(B)$ denote the sub 2-category of $\mathbf{Cat}/B$ of fibrations, functors that preserve cartesian arrows, and all 2-cells.

Theorem 2.2.3. *There is a 2-equivalence of 2-categories*

$$\mathbf{Fib}(B) \cong [B^{op}, \mathbf{Cat}]_{ps},$$

where the latter is the 2-category of pseudo-functors, pseudo-natural transformations, and modifications.

Proof. The lax colimit of a pseudo-functor $B^{\mathrm{op}} \to \mathbf{Cat}$ is canonically a fibration over B. Conversely, a fibration $E \to B$ gives rise to a pseudo-functor $b \mapsto E_b$, $f\colon b' \to b \mapsto f^*\colon E_b \to E'_b$. By the universal property of the cartesian lifts, this assignment is functorial up to natural isomorphism. $\square$

Fibrations enjoy similar stability properties to their topology analogs.

Theorem 2.2.4. *Fibrations are closed under composition and pullback along arbitrary functors.*

Proof. See [Gra66, 3.1]. $\square$

Before giving examples, we mention the dual notion. A functor $p\colon E \to B$ is an *opfibration* if $p\colon E^{\mathrm{op}} \to B^{\mathrm{op}}$. A functor $p\colon E \to B$ that is both a fibration and an opfibration is called a *bifibration*. The proof of the following lemma is left as an exercise.

Lemma 2.2.5. *A fibration $p\colon E \to B$ is also an opfibration if and only if each functor $f^*\colon E_b \to E_{b'}$ has a left adjoint $f_!$.*

Finally, some examples:

Example 2.2.6.
- (i) The codomain functor $C^{\mathbf{2}} \to C$ is an opfibration that is a fibration iff C has pullbacks. (Hence the name "cartesian".)
- (ii) The domain functor $C^{\mathbf{2}} \to C$ is a fibration this is an opfibration iff C has pushouts.
- (iii) The forgetful functor $\mathbf{Mod} \to \mathbf{Ring}$ is a bifibration. For each ring homomorphism f, f^* is restriction of scalars, $f_!$ is extension of scalars.
- (iv) For any category C, the category of set-indexed families of objects of C is a fibration over $\mathbf{Set}$ with the forgetful functor taking a family to its indexing set. The functors f^* are given by reindexing and have left adjoints iff C has small coproducts, and right adjoints iff C has small products.

We mention one final result which will motivate the definitions in Section 3.

Theorem 2.2.7. *A functor $p\colon E \to B$ is a fibration if and only if the functor $[X, p]\colon [X, E] \to [X, B]$ is a fibration for every category X.*

Proof. See [Gra66, 3.6]. $\square$

2.3. **Two-sided discrete fibrations.** Finally, we reach the variant of interest.

Definition 2.3.1. A *two-sided discrete fibration* is a span $A \xleftarrow{q} E \xrightarrow{p} B$ such that
- (i) each $qe \to a'$ in A has a unique lift in E that has domain e and lies in the fiber over pe
- (ii) each $b' \to pe$ in B has a unique lift in E that has codomain e and lies in the fiber over qe

(iii) for each $f\colon e \to e'$ in E the codomain of the lift of qf equals the domain of the lift of pf and their composite is f.

Let $\mathbf{DFib}(A, B)$ denote the full subcategory of $\mathbf{Span}(A, B)$ on the two-sided discrete fibrations.

Theorem 2.3.2. *There exist equivalences of categories*

$$\mathbf{DFib}(A, B) \simeq [B^{op} \times A, \mathbf{Set}],$$

pseudo-natural in A and B.

Proof. Given a two-sided discrete fibration $A \xleftarrow{q} E \xrightarrow{p} B$, define $B^{\mathrm{op}} \times A \to \mathbf{Set}$ by $(b, a) \mapsto E_{a,b}$, the objects in the fiber over a and b. Given $g\colon b' \to b$, the corresponding function $g^*\colon E_{a,b} \to E_{a,b'}$ sends $e \in E_{a,b}$ to the domain of the unique lift of g in the fiber over a with codomain e; likewise, given $f\colon a \to a'$ the corresponding $f_* E_{a,b} \to E_{a',b}$ sends e to the codomain of the unique lift of f in the fiber over b with domain a'.

Conversely, $P\colon B^{\mathrm{op}} \times A \to \mathbf{Set}$, let the objects of E be triples $(b \in B, e \in P(b, a), a \in A,)$ and morphisms $(b, e, a) \to (b', e', a')$ be pairs of arrows $f\colon a \to a'$ in A and $g\colon b \to b'$ in B such that $f_*(e) = g^*(e')$. Note this isn't the *collage* of P, a category living over the walking arrow $\mathbb{2}$, defined below. Rather it's the category of sections of this functor, with morphisms the natural transformations. See the nLab discussion of two-sided fibrations. □

Comma categories provide an important class of examples of two-sided discrete fibrations. In fact, in **Cat**, they tell the whole story.

Theorem 2.3.3. *For any opspan $B \xrightarrow{f} C \xleftarrow{g} A$, its comma category*

$$\begin{tikzcd} f/g \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\Leftarrow"] & B \arrow[d, "f"] \\ A \arrow[r, "g"'] & C \end{tikzcd}$$

is a two-sided discrete fibration $A \xleftarrow{c} f/g \xrightarrow{d} B$. Furthermore, all two-sided discrete fibrations in **Cat** *arise this way.*

Proof. See [Str74, 14]. □

Finally, for completeness, we give the definition of two-sided fibrations, which aren't required to be discrete.

Definition 2.3.4. A span $A \xleftarrow{q} E \xrightarrow{p} B$ is a *two-sided fibration* if

(i) any $g\colon qe \to a \in A$ has an opcartesian lift with domain e that lies in the fiber over the identity at pe

(ii) any $f\colon b \to pe$ has a cartesian lift with codomain e that lies in the fiber over the identity at qe

(iii) given a cartesian lift $f^*e \to e$ of f and an opcartesian lift $e \to g_!e$ of g, as above, the composite

$$f^*e \to e \to g_!e$$

lies over both f and g. Write $f^*e \to g_!f^*e$ and $f^*g_!e \to g_!e$ for its opcartesian and cartesian lifts. The canonical comparison $g_!f^*e \to f^*g_!e$ induced by the universal property of either of these must be an isomorphism.

Two-sided fibrations determine pseudo-functors $\mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Cat}$.

3. Fibrations in 2-categories

The notions of fibration and two-sided discrete fibration internal to a 2-category are due to [Str74]; a good summary of the main results can be found in [Web07, §2]. In order to perform desired constructions, we work in *finitely complete* 2-categories $\mathcal{K}$, i.e., a 2-category that admits finite conical limits and cotensors with the "walking arrow" category $\mathbb{2}$. In particular:

Lemma 3.0.5. *A finitely complete 2-category $\mathcal{K}$ has all comma objects.*

Proof. First note that

$$\begin{tikzcd} A^{\mathbb{2}} \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\Leftarrow"] & A \arrow[d, "1_A"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$$

is a comma object, where d and c are induced by the domain and codomain inclusions $\mathbb{1} \rightrightarrows \mathbb{2}$. To see this, recall that comma objects, like all weighted limits, are defined *representably*, meaning in this case that the comma object A/A of the depicted opspan must induce isomorphisms of categories

$$\mathcal{K}(X, A/A) \cong \mathcal{K}(X, A)/\mathcal{K}(X, A)$$

for all $X \in \mathcal{K}$, where the right hand side denotes the comma category for the pair of identity functors on $\mathcal{K}(X, A)$. But we know that in **Cat**, this comma category is $\mathcal{K}(X, A)^{\mathbb{2}}$. Hence, A/A must induce isomorphisms of categories

$$\mathcal{K}(X, A/A) \cong \mathcal{K}(X, A)^{\mathbb{2}}$$

which is the defining universal property of the cotensor of $A \in \mathcal{K}$ by $\mathbb{2}$.

Given an opspan $A \xrightarrow{g} C \xleftarrow{f} B$ in $\mathcal{K}$, its comma object is the (composite) 2-pullback

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "1_A"'] \arrow[dr, "g"] & & C^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & B \arrow[dl, "f"'] \arrow[dr, "1_B"] & \\ A & & C & & C & & B \end{tikzcd}$$

with 2-cell defined by whiskering the 2-cell of the comma object $C^{\mathbb{2}}$. This can be proven directly in **Cat**, implying the result for a generic 2-category $\mathcal{K}$ by the representability of weighted limits. □

Another proof of the previous lemma uses the pasting lemma for comma squares.

Lemma 3.0.6. *Given a diagram in a 2-category $\mathcal{K}$ such that the right-hand square is a comma square*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Leftarrow"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

the whole diagram is a comma square if and only if the left-hand square is a 2-pullback.

Proof. Analogous to the pasting lemma for ordinary pullbacks. □

3.1. **Fibrations.** Fibrations in a 2-category are defined representably.

Definition 3.1.1. A 1-cell $p\colon E \to B$ in a 2-category $\mathcal{K}$ is a *fibration* iff $\mathcal{K}(X,p)$ is a fibration for all $X \in \mathcal{K}$ and if

$$\begin{tikzcd}
\mathcal{K}(X,E) \arrow[r, "\mathcal{K}(x,E)"] \arrow[d, "\mathcal{K}(X,p)"'] & \mathcal{K}(Y,E) \arrow[d, "\mathcal{K}(Y,p)"] \\
\mathcal{K}(X,B) \arrow[r, "\mathcal{K}(x,B)"'] & \mathcal{K}(Y,B)
\end{tikzcd}$$

is a map of fibrations for all $x\colon Y \to X$ in $\mathcal{K}$.

Unpacking this definition, $p\colon E \to B$ is a *fibration* if every 2-cell

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "b"'] \arrow[dr, phantom, shift left=3, "\Rightarrow\beta"] & E \arrow[d, "p"] \\
 & B
\end{tikzcd}$$

has a p-cartesian lift $\alpha\colon e' \Rightarrow e$ so that $p\alpha = \beta$. A 2-cell

$$\begin{tikzcd}
X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\alpha"] & E
\end{tikzcd}$$

is *p-cartesian* when for all $x\colon Y \to X$, αx is a $\mathcal{K}(Y,p)$-cartesian arrow in $\mathcal{K}(Y,E)$. This means that for all 2-cells

$$\begin{tikzcd}
Y \arrow[rr, "e''"] \arrow[dr, "x"'] & \arrow[d, phantom, "\Downarrow\xi"] & E \\
 & X \arrow[ur, "e"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, "e''"] \arrow[d, "x"'] & \arrow[d, phantom, "\Downarrow\gamma"] & E \arrow[d, "p"] \\
X \arrow[r, "e'"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

such that $p\xi = p\alpha \cdot \gamma$, then there is a unique 2-cell $\zeta\colon e'' \Rightarrow e'x$ such that $\xi = \alpha\zeta$ and $p\zeta = \gamma$.

Note this definition did not require any hypotheses on the 2-category $\mathcal{K}$; we do make use of finite completeness going forward.

Theorem 3.1.2. *In any finitely complete 2-category $\mathcal{K}$*

(i) *the composite of fibrations is a fibration*
(ii) *the pullback of a fibration is a fibration*

Proof. Follows from Theorem 2.2.4. □

Theorem 3.1.3. *Let $\mathcal{K}$ be a finitely complete 2-category, $p\colon E \to B$ a 1-cell. The following are equivalent:*

(i) *p is a fibration*

(ii) *for all* $b\colon X \to B$*, the map* $i\colon X \times_B E \to b/p$ *has a right adjoint in* $\mathcal{K}/X$

$$\begin{tikzcd}
X \times_B E \arrow[dr, dashed, "i"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& b/p \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & E \arrow[d, "p"] \\
& X \arrow[r, "b"'] & B
\end{tikzcd}$$

(iii) *the map* $E \to B/p$ *has a right adjoint in* $\mathcal{K}/B$.

(iv) *the canonical arrow* $E^{\mathbb{2}} \to B/p$ *has a right adjoint with counit an isomorphism.*

Proof. (iii) is (ii) with $b = 1_B$. (iii) implies (ii) by the pasting lemma 3.0.6. Equivalence with (i) requires some cleverness. See [Web07, 2.7]. (i) $\Leftrightarrow$ (iv) is analogous to Theorem 2.2.2; see [Str74, 9]. $\square$

An *opfibration* in $\mathcal{K}$ is a fibration in $\mathcal{K}^{\mathrm{co}}$. It follows from characterization (iii) above that any 2-functor between finitely complete 2-categories that preserves comma objects preserves fibrations and opfibrations. We briefly mention the very simplest examples.

Example 3.1.4.

(i) The fibrations internal to the 2-category **Cat** are exactly the fibrations defined above.

(ii) A fibration internal to the 2-category **Cat**$/A$ is a functor $p\colon E \to B$ such that arrows $b \to pe$ in the fiber over an identity in A have p-cartesian lifts. If the functors $E \to A$ and $B \to A$ are fibrations in **Cat** and p preserves cartesian arrows, then p is a fibration in **Cat** if and only if it is a fibration in **Cat**$/A$. In general, the notion of fibration in **Cat**$/A$ is weaker.

Discrete fibrations in a 2-category $\mathcal{K}$ with cotensors by $\mathbb{2}$ can either be defined representably or in analogy with Definition 2.1.4 and these definitions are equivalent.

3.2. **Two-sided discrete fibrations.** In a bicategory $\mathcal{K}$, we write **Span**$(\mathcal{K})$ for the bicategory of spans in $\mathcal{K}_0$, the 1-category underlying $\mathcal{K}$. If $\mathcal{K}$ has binary products, the hom-categories **Span**$(\mathcal{K})(A, B)$ are isomorphic to the comma categories $\mathcal{K}/A \times B$; hence, they are actually 2-categories. Furthermore, the composition is 2-functorial.

Definition 3.2.1. A span $A \xleftarrow{q} E \xrightarrow{p} B$ is a *two-sided discrete fibration* if and only if it is representably so, i.e., if for all $X \in \mathcal{K}$,

$$\mathcal{K}(X, A) \xleftarrow{\mathcal{K}(X,q)} \mathcal{K}(X, E) \xrightarrow{\mathcal{K}(X,p)} \mathcal{K}(X, B)$$

is a two-sided discrete fibration.

As in **Cat**, comma objects provide examples of two-sided discrete fibrations.

Theorem 3.2.2. *Given* $f\colon A \to C$ *and* $g\colon B \to C$*, the span* $A \leftarrow f/g \to B$ *is a two-sided discrete fibration.*

Proof. Because weighted limits are also defined representably, it suffices to prove when $\mathcal{K} = \mathbf{Cat}$. See Theorem 2.3.3. □

Theorem 3.2.3. *If* $A \xleftarrow{q} E \xrightarrow{p} B$ *is a* two-sided discrete fibration*, then* p *is a fibration and* q *is an opfibration.*

Proof. Technical, but suffices to prove for p because the second part follows by interpreting this in $\mathcal{K}^{\mathrm{co}}$. □

In his original paper, Street defines fibrations, opfibrations, and two-sided discrete fibrations to be pseudo-algebras for certain 2-monads on the appropriate hom-2-category of $\mathbf{Span}(K)$. For instance, the 2-monad on $\mathbf{Span}(K)(A, B)$ for two-sided discrete fibrations sends a span $A \xleftarrow{q} E \xrightarrow{p} B$ to the 2-pullback of

$$\begin{tikzcd} & A^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & & E \arrow[dl, "q"'] \arrow[dr, "p"] & & B^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ A & & A & & B & & B \end{tikzcd}$$

See [Str74] for details.

3.3. **Yoneda lemma.** Part of the motivation for defining two-sided discrete fibrations internally to a 2-category was to state and prove a Yoneda lemma in this context. While this is peripheral to our discussion, we nonetheless take a brief detour to give the statement.

Theorem 3.3.1. *Let* $\mathcal{K}$ *be finitely complete 2-category,* $A \xleftarrow{q} E \xrightarrow{p} B$ *a two-sided discrete fibration and* $f\colon B \to A$ *a 1-cell. There is a unique arrow* $i\colon B \to f/A$ *from the 2-pullback of* f *along the identity at* A *to the comma object. Precomposition with* i *induces a bijection between arrows of spans* $f/A \to E$ *and arrows of spans* $B \to E$*.*

Proof. See [Str74, 16] or [Web07, 2.12]. □

4. Fibrations in bicategories

The notions of fibration and two-sided discrete fibration internal to a 2-category are due to [Str80]; a good summary of the main results can be found in [CJSV94]. The first two sections are somewhat abbreviated; we excuse this laxity by mentioning that it enables us to quickly get to the main point in the final two sections. The reader who wishes to see statements analogous to those of Section 3 is encouraged to prove them, replacing any 2-limits that appear with the appropriate bilimits.

Section 4.3 relies heavily on the "codiscrete cofibration" entry at the nLab.

4.1. **Fibrations.** In a bicategory, it is generally considered unreasonable to ask for an equality of 1-cells, but there is no moral objection to asking 2-cells to be equal. Thus, when defining fibrations internally to a generic bicategory $\mathcal{K}$, we can use the definition of p-cartesian 2-cells that was "unpacked" above, enabling the definition:

Definition 4.1.1. A 1-cell $p\colon E \to B$ in a bicategory $\mathcal{K}$ is a *fibration* if for all 1-cells $e\colon X \to E$ and 2-cells $\alpha\colon b \Rightarrow pe\colon X \to B$, there exists a p-cartesian $\chi\colon e' \Rightarrow e$ for which there is an isomorphism $b\colon pe'$ whose composite with $p\chi$ is α.

Example 4.1.2. The fibrations internal to **Cat** as a bicategory are sometimes called *Street fibrations*. Explicitly, a functor $p\colon E \to B$ is a Street fibration if for every $f\colon b \to pe$ in B, there is a p-cartesian arrow $g\colon e' \to e$ and an isomorphism $h\colon b \to pe'$ such that $f = pg \cdot h$.

This notion of fibration is invariant under equivalence of categories. In particular, equivalences of categories are Street fibrations, though they are not necessarily fibrations in the classical sense.

Lemma 4.1.3. *A 1-cell $p\colon E \to B$ in a bicategory $\mathcal{K}$ is a fibration if and only if*

(i) *for all $X \in \mathcal{K}$, $\mathcal{K}(X,p)\colon \mathcal{K}(X,E) \to \mathcal{K}(X,B)$ is a Street fibration*
(ii) *for all 1-cells $x\colon Y \to X$ in $\mathcal{K}$, precomposition with x induces a map of fibrations $\mathcal{K}(X,p) \to \mathcal{K}(Y,p)$.*

4.2. **Two-sided fibrations and two-sided discrete fibrations.** First, we should say a few words about the tricategory $\mathbf{Span}(K)$. When $\mathcal{K}$ is a bicategory, not a 2-category, we define the 1-cells and 2-cells of the bicategory $\mathbf{Span}(K)(A,B)$ slightly differently. A morphism of spans from A to B is given by a 1-cell f in $\mathcal{K}$ and isomorphic 2-cells as depicted

$$\begin{tikzcd} & E \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\ A \arrow[r, phantom, "\mu\cong"] & & B \arrow[l, phantom, "\cong\nu"] \\ & E' \arrow[ul, "q'"] \arrow[ur, "p'"'] & \end{tikzcd}$$

A 2-cell in $\mathbf{Span}(K)(A,B)$ is a 2-cell $\theta\colon f \Rightarrow f'$ that pastes together with one of each pair of 2-cells isomorphisms to give the other.

Definition 4.2.1. A span $A \xleftarrow{q} E \xrightarrow{p} B$ in $\mathcal{K}$ is a two-sided discrete fibration if

(i) for every $e\colon X \to E$ and 2-cell $\alpha\colon qe \Rightarrow a\colon X \to A$, there exists an opcartesian 2-cell $\chi\colon e \Rightarrow e'$ and isomorphism $qe' \Rightarrow a$ whose composite with $q\chi$ is α and such that $p\chi$ is an isomorphism.
(ii) for every $e\colon X \to E$ and 2-cell $\beta\colon b \Rightarrow pe\colon X \to B$, there exists a cartesian 2-cell $\zeta\colon e' \Rightarrow e$ and isomorphism $b \Rightarrow pe'$ whose composite with $p\zeta$ is β and such that $q\zeta$ is an isomorphism.
(iii) for all $\eta, \eta'\colon e \Rightarrow e'\colon X \to E$, if $p\eta = p\eta'$, $q\eta = q\eta'$, and $p\eta$ and $q\eta$ are invertible, then $\eta = \eta'$ and is invertible.

Condition (iii) is equivalent to saying that the span is representably essentially discrete, i.e., for all spans E' from A to B, the hom-category

$$\mathbf{Span}(K)(A,B)(E',E)$$

is equivalent to a discrete category.

Proof of the following alternate characterization, which is due to [CJSV94] and should be compared with Definition 2.3.1, is left as an exercise.

Lemma 4.2.2. *A span $A \xleftarrow{q} E \xrightarrow{p} B$ is a two-sided discrete fibration if and only if the following conditions hold.*

(i) *for all arrows* $e\colon X \to E$ *and 2-cells* $\alpha\colon qe \Rightarrow a$*, the category whose objects are pairs* $(\chi\colon e \Rightarrow e', \nu\colon qe' \cong a)$ *with* $\alpha = \nu \cdot q\chi$ *and* $p\chi$ *invertible is essentially discrete and non-empty;*

(ii) *for all arrows* $e\colon X \to E$ *and 2-cells* $\beta\colon b \Rightarrow pe$*, the category whose objects are pairs* $(\zeta\colon e' \Rightarrow e, \mu\colon b \cong pe')$ *with* $\beta = p\zeta \cdot \mu$ *and* $q\zeta$ *invertible is essentially discrete and non-empty;*

(iii) *each 2-cell* $\eta\colon e \Rightarrow e'\colon X \to E$ *is a composite* $\zeta\chi$ *where* $p\chi$ *and* $q\zeta$ *are invertible.*

By a comma object in a bicategory, we mean the bilimit with the shape described above, relaxing the defining isomorphism $\mathcal{K}(X, f/g) \simeq \mathcal{K}(X, f)/\mathcal{K}(X, g)$ of categories to an equivalence.

Theorem 4.2.3. *Any comma object in a bicategory gives a two-sided discrete fibration.*

Proof. See [Str80, 3.44]. □

4.3. **Two-sided codiscrete cofibrations.** For this section, the motivating example is the 2-category $\mathcal{K} = \mathcal{V}$-**Cat** of categories enriched in some closed symmetric monoidal category $(\mathcal{V}, \otimes, I)$. We'll see in the next section what is special about the case $\mathcal{V} = \mathbf{Set}$, $\mathcal{K} = \mathbf{Cat}$.

In enriched category theory, $\mathcal{V}$-profunctors play an important role; if $\mathcal{A}, \mathcal{B} \in \mathcal{V}$-**Cat**, a $\mathcal{V}$*-profunctor* from $\mathcal{A}$ to $\mathcal{B}$ is a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$. A warning: unless $\mathcal{V}$ is cartesian monoidal, the tensor product of $\mathcal{V}$-categories is distinct from their cartesian product. The tensor product of $\mathcal{V}$-categories gives the morally correct notion of $\mathcal{V}$-profunctors and it is necessary for the construction of collages below.

We would like to be able to model $\mathcal{V}$-profunctors internally to the 2-category of $\mathcal{V}$-categories because this will make it easier to understand which pseudo-functors $\mathcal{V}$-**Cat** $\to \mathcal{K}$ "preserve" profunctors. One way to describe the data of a $\mathcal{V}$-profunctor in $\mathcal{V}$-**Cat** is through its collage.

Definition 4.3.1. The *collage* of $F\colon \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ is a cospan $\mathcal{A} \to \mathcal{E} \leftarrow \mathcal{B}$, where $\mathcal{E}$ is the $\mathcal{V}$-category with objects $\mathrm{ob}\mathcal{A} \sqcup \mathrm{ob}\mathcal{B}$ and hom-objects

$$\mathcal{E}(b', b) = \mathcal{B}(b', b), \quad \mathcal{E}(b, a) = F(b, a), \quad \mathcal{E}(a, a') = \mathcal{A}(a, a'), \quad \mathcal{E}(a, b) = \emptyset,$$

for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. The $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{E}$, $\mathcal{B} \to \mathcal{E}$ are the inclusions.

The main result is the following theorem of [Str80]:

Theorem 4.3.2. *The collages for* $\mathcal{V}$*-profunctors are exactly the two-sided codiscrete cofibrations in* $\mathcal{V}$-**Cat***, regarded as a bicategory.*

The reader may have already guessed the following definitions.

Definition 4.3.3. A cospan $A \to E \leftarrow B$ in a bicategory $\mathcal{K}$ is a *two-sided cofibration* if and only if it is a two-sided fibration in $\mathcal{K}^{\mathrm{op}}$, the bicategory with 1-cells reversed. The span is *codiscrete* if it is representably discrete in $\mathbf{Cospan}(\mathcal{K})(A, B) \cong A \sqcup B/\mathcal{K}$, that is, if the hom-category $\mathbf{Cospan}(\mathcal{K})(A, B)(E, E')$ is equivalent to a discrete category for all cospans $A \to E' \leftarrow B$.

In order for the model of profunctors in $\mathcal{K}$ to be complete, we need to be able to compose a two-sided codiscrete cofibration from A to B and from B to C and obtain a two-sided codiscrete cofibration from A to C. If we removed the word "codiscrete," this would be a piece of cake. So long as $\mathcal{K}$ has finite colimits, cofibrations are stable under pushout and composition. Hence, the pushout-composite of a cospan from A to B and a cospan from B to C is a cospan from A to C that is a two-sided cofibration if the original cospans were. This composition law is associative up to isomorphism, which is good enough.

However, the resulting two-sided cofibration is unlikely to be codiscrete, whether or not the original two-sided cofibrations were. For instance, given $\mathcal{V}$-profunctors $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ and $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{B} \to \mathcal{V}$ and considering their collages, the pushout $\mathcal{A} \to \mathcal{E} \leftarrow \mathcal{C}$ is a $\mathcal{V}$-category with objects $\mathrm{ob}\mathcal{A} \sqcup \mathrm{ob}\mathcal{B} \sqcup \mathrm{ob}\mathcal{C}$ called a *gamut*; because of the presence of objects of $\mathcal{B}$, this is too fat to be a collage for a $\mathcal{V}$-profunctor $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$.

This problem can be solved provided there is a method for coreflecting from two-sided cofibrations into two-sided codiscrete cofibrations; a subcategory is *coreflective* if the inclusion has a right adjoint. In some examples, there may be a limit construction that achieves this. This is the approach that Street takes originally, but see [Str87].

A simpler approach is to ask that $\mathcal{K}$ have an orthogonal factorization system whose left class is generated by the two-sided codiscrete cofibrations $A \sqcup B \to E$. An *orthogonal factorization system* in a bicategory consists of two classes $(\mathcal{L},\mathcal{R})$ of 1-cells such that

(i) every 1-cell in $\mathcal{K}$ is isomorphic to the composite of a 1-cell in $\mathcal{L}$ followed by a 1-cell in $\mathcal{R}$
(ii) for all $l\colon X \to Y \in \mathcal{L}$, $r\colon Z \to W \in \mathcal{R}$, the square

$$\begin{tikzcd}
\mathcal{K}(Y,Z) \arrow[r, "\mathcal{K}(Y,r)"] \arrow[d, "\mathcal{K}(l,Z)"'] \arrow[dr, phantom, "\cong"] & \mathcal{K}(Y,W) \arrow[d, "\mathcal{K}(l,W)"] \\
\mathcal{K}(X,Z) \arrow[r, "\mathcal{K}(X,r)"'] & \mathcal{K}(X,W)
\end{tikzcd}$$

is a bipullback in **Cat**.

An orthogonal factorization system $(\mathcal{L},\mathcal{R})$ is *generated* by a collection of 1-cells if the right class consists of precisely those 1-cells that satisfy axiom (ii) for all l in this collection. When the generators are taken to be the codiscrete cofibrations, arrows in the right class are necessarily representably fully faithful. If the right class is stable under pushout and cotensor with $\mathbb{2}$, then the composite of a pair of two-sided codiscrete cofibrations can be defined by factoring the cospan $A \sqcup C \to E$ formed by taking their pushout. This is the approach of [CJSV94] and the nLab.

We record this fact in the following theorem.

Theorem 4.3.4. *Suppose $\mathcal{K}$ is a bicategory with finite limits and colimits. If the two-sided codiscrete cofibrations $A \sqcup B \to E$ generate an orthogonal factorization system whose right class is closed under pushout and cotensor with $\mathbb{2}$, then there is a bicategory $\mathbf{DCof}(\mathcal{K})$ whose objects are the objects of $\mathcal{K}$, whose 1-cells $A \to B$ are the two-sided codiscrete cofibrations from A to B, and whose 2-cells are isomorphism classes of morphisms of cospans.*

Proof. See [CJSV94, 4.20]. ∎

Here is how this works in our main example.

Lemma 4.3.5. $\mathcal{V}$-**Cat** *has an orthogonal factorization system whose left class consists of the essentially surjective $\mathcal{V}$-functors and whose right class consists of the $\mathcal{V}$-fully faithful functors that is generated by the two-sided codiscrete cofibrations.*

Proof. We leave it to the reader to prove that this orthogonal factorization exists; we show that it is generated by the two-sided codiscrete cofibrations. The collages are surjective on objects, so $\mathcal{V}$-fully faithful functors are necessarily right orthogonal to them. It remains to show that any $\mathcal{V}$-functor $F\colon \mathcal{C} \to \Delta$ right orthogonal to the collages $\mathcal{A} \sqcup \mathcal{B} \to \mathcal{E}$ is necessarily $\mathcal{V}$-fully faithful. Let $\mathcal{I}$ denote the $\mathcal{V}$-category with one-object and the unit as its hom-object. A $\mathcal{V}$-profunctor from $\mathcal{I}$ to itself is specified by a single object in $\mathcal{V}$. Given $c, c' \in \mathcal{C}$, form the collage of the $\mathcal{V}$-profunctor $\mathcal{I}^{\mathrm{op}} \otimes \mathcal{I} \to \mathcal{V}$ determined by $\Delta(Fc, Fc')$. This collage has the form $\mathcal{I} \sqcup \mathcal{I} \to \mathcal{E}$, where $\mathcal{E}$ has two objects 0,1 and one non-trivial hom $\mathcal{E}(0,1) = \Delta(Fc, Fc')$. The obvious lifting problem

$$\begin{tikzcd}
\mathcal{I} \sqcup \mathcal{I} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{E} \arrow[r, "1"'] \arrow[ur, dashed] & \Delta
\end{tikzcd}$$

must have a unique solution, which shows that F is $\mathcal{V}$-fully faithful. $\square$

It remains to check that $\mathcal{V}$-fully faithful functors are stable under pushout and cotensors with $\mathbb{2}$; we leave this to the reader.

4.4. **A final note on modeling profunctors in Cat.** We now have two models for profunctors in **Cat**, the two-sided discrete fibrations and the two-sided codiscrete fibrations. It turns out there is a formal reason that these are the same.

In any 2-category $\mathcal{K}$ with comma and cocomma objects, there is an adjunction

$$\text{cocomma}\colon \mathbf{Span}(\mathcal{K})(A,B) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cospan}(\mathcal{K})(A,B) \colon \text{comma}$$

We've seen above that comma objects are always two-sided discrete fibrations; dually, cocomma objects are always two-sided codiscrete cofibrations. In **Cat**, this adjunction is *idempotent* the comma object of the cocomma object of a comma object is isomorphic to the original comma object; this is equivalent to the dual statement. Any such adjunction restricts to an adjoint equivalence between the full subcategories in the image of each functor, which are consequently reflective and coreflective subcategories of the originals. So this adjunction restricts to an equivalence between the reflective subcategory of two-sided discrete fibrations and the coreflective subcategory of two-sided codiscrete cofibrations. Hence, both of these are equivalent to the 2-category of profunctors from A to B.

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