The formal theory of homotopy coherent monads

Emily Riehl

Harvard University
http://www.math.harvard.edu/~eriehl

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Joint with Dominic Verity.

Slogan: “It’s all in the weights!”

Plan

1. Homotopy coherent adjunctions
2. Homotopy coherent monads and the monadic adjunction
3. Codescent in the Eilenberg-Moore quasi-category
4. Monadicity theorem

Quasi-categories

A **quasi-category** is a simplicial set A in which any inner horn

$$\begin{tikzcd} \Lambda^{n,k} \arrow[r] \arrow[d] & A \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad 0 < k < n$$

has a filler.

The **homotopy category** hA has

- objects = vertices
- morphisms = homotopy classes of 1-simplices

Via the adjunction

$$\begin{tikzcd} \underline{\mathrm{Cat}} \arrow[r, hook] \arrow[r, phantom, bend left, "\perp"] & \underline{\mathrm{qCat}} \arrow[l, bend right, "h"'] \end{tikzcd}$$

quasi-category theory extends category theory.

Adjunctions of quasi-categories

$\underline{\mathrm{qCat}}_2$:= the **2-category of quasi-categories**, consisting of

- quasi-categories A, B
- functors (maps of simplicial sets) $g\colon A \to B$
- natural transformations (homotopy classes of 1-simplices)

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Delta^0 \arrow[d, "i_0"'] \arrow[dr, "g"] & \\ \Delta^1 \arrow[r, "\alpha"] & B^A \\ \Delta^0 \arrow[u, "i_1"] \arrow[ur, "k"'] & \end{tikzcd}$$

An **adjunction** of quasi-categories is an adjunction in $\underline{\mathrm{qCat}}_2$.

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd} \qquad \eta\colon \mathrm{id}_B \Rightarrow uf \qquad \epsilon\colon fu \Rightarrow \mathrm{id}_A$$

Some theorems and examples

Theorems.

- $f \dashv u$ induces adjunctions $f^X \dashv u^X$ and $C^u \dashv C^f$ for any simplicial set X and quasi-category C.
- Any equivalence can be promoted to an adjoint equivalence.
- Right adjoints preserve limits.
- $f\colon B \to A$ has a left adjoint iff $f \downarrow a$ has a terminal object for each $a \in A$.

Examples.

- ordinary adjunctions, topological adjunctions
- simplicial Quillen adjunctions
- colim $\dashv$ const $\dashv$ lim
- loops–suspension

A coherence question

$\underline{\mathrm{qCat}}_\infty :=$ the **simplicial category of quasi-categories.**

Given $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ in $\underline{\mathrm{qCat}}_2$, what adjunction data exists in $\underline{\mathrm{qCat}}_\infty$?

- $\mathrm{id}_B \xrightarrow{\eta} uf$ in B^B $\qquad fu \xrightarrow{\epsilon} \mathrm{id}_A$ in A^A
- $\begin{tikzcd} & ufu \arrow[dr, "u\epsilon"] \arrow[d, phantom, "\alpha"] & \\ u \arrow[ur, "\eta u"] \arrow[rr, "\mathrm{id}_u"'] & {} & u \end{tikzcd}$ in B^A $\qquad \begin{tikzcd} & fuf \arrow[dr, "\epsilon f"] \arrow[d, phantom, "\beta"] & \\ f \arrow[ur, "f\eta"] \arrow[rr, "\mathrm{id}_f"'] & {} & f \end{tikzcd}$ in A^B
- $\begin{tikzcd} & uf \arrow[dr, equals] \arrow[dd, "\eta uf"] & \\ \mathrm{id}_B \arrow[ur, "\eta"] \arrow[rr, "\eta"] \arrow[dr, "\eta\cdot\eta"'] & & uf \\ & ufuf \arrow[ur, "u\epsilon f"'] & \end{tikzcd}$ $\quad \begin{tikzcd} & uf \arrow[dr, equals] \arrow[dd, "uf\eta"] & \\ \mathrm{id}_B \arrow[ur, "\eta"] \arrow[rr, "\eta"] \arrow[dr, "\eta\cdot\eta"'] & & uf \\ & ufuf \arrow[ur, "u\epsilon f"'] & \end{tikzcd}$ filling $\Lambda^{3,1} \to B^B$

But do there exist fillers with the same bottom face?

The free adjunction

$\underline{\mathrm{Adj}}$:= the **free adjunction**, a 2-category with

- objects $+$ and $-$
- $\underline{\mathrm{Adj}}(+,+) = \underline{\mathrm{Adj}}(-,-)^{\mathrm{op}} := \mathbb{\Delta}_+$
- $\underline{\mathrm{Adj}}(-,+) = \underline{\mathrm{Adj}}(+,-)^{\mathrm{op}} := \mathbb{\Delta}_\infty$

Theorem (Schanuel-Street)**.** 2-functors $\underline{\mathrm{Adj}} \to \underline{\mathrm{qCat}_2}$ correspond to adjunctions in $\underline{\mathrm{qCat}_2}$.

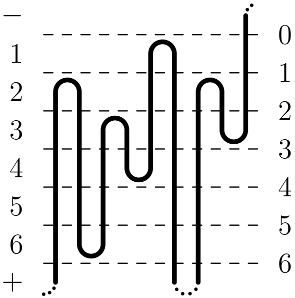
$$\begin{tikzcd}[column sep=large]
\mathrm{id} \arrow[r, "\eta"] & uf \arrow[r, shift left=4, "\eta"] \arrow[r, leftarrow, "u\epsilon" description] \arrow[r, shift right=4, "uf\eta"'] & ufuf \arrow[r, shift left=8, "\eta"] \arrow[r, leftarrow, shift left=4, "u\epsilon" description] \arrow[r, "uf\eta" description] \arrow[r, leftarrow, shift right=4, "ufu\epsilon" description] \arrow[r, shift right=8, "ufuf\eta"'] & ufufuf \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
u \arrow[r, shift left=2, "\eta"] \arrow[r, leftarrow, shift right=2, "u\epsilon"'] & ufu \arrow[r, shift left=6, "\eta"] \arrow[r, leftarrow, shift left=2, "u\epsilon" description] \arrow[r, shift right=2, "uf\eta" description] \arrow[r, leftarrow, shift right=6, "ufu\epsilon"'] & ufufu \arrow[r, shift left=10, "\eta"] \arrow[r, leftarrow, shift left=6, "u\epsilon" description] \arrow[r, shift left=2, "uf\eta" description] \arrow[r, leftarrow, shift right=2, "ufu\epsilon" description] \arrow[r, shift right=6, "ufuf\eta" description] \arrow[r, leftarrow, shift right=10, "ufufu\epsilon"'] & ufufufu \cdots
\end{tikzcd}$$

The free homotopy coherent adjunction

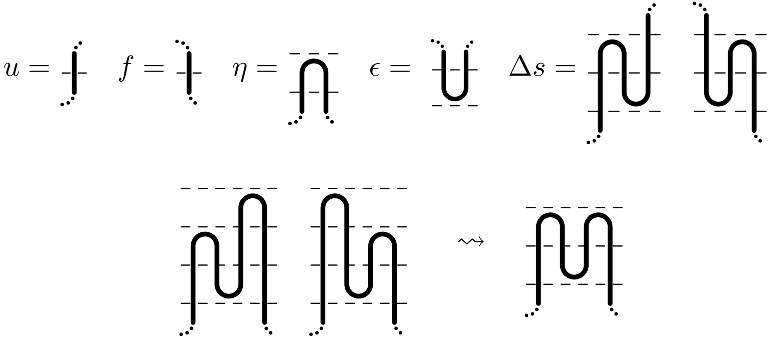
Conjecture. The **free homotopy coherent adjunction** is $\underline{\mathrm{Adj}}$, regarded as a simplicial category under $2\text{-}\underline{\mathrm{Cat}} \hookrightarrow \underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$.

n-arrows are **strictly undulating squiggles** on $n+1$ lines



Proposition. $\underline{\mathrm{Adj}}$ is a simplicial computad (i.e., cofibrant).

Homotopy coherent adjunctions



Theorem. Any adjunction $\underline{\mathrm{Adj}} \to \underline{\mathrm{qCat}_2}$ lifts to a homotopy coherent adjunction $\underline{\mathrm{Adj}} \to \underline{\mathrm{qCat}_\infty}$.

Theorem. Such extensions are homotopically unique: the spaces of extensions are contractible Kan complexes.

Homotopy coherent monads

$\underline{\mathrm{Mnd}}$:= full subcategory of $\underline{\mathrm{Adj}}$ on $+$.

Definition. A **homotopy coherent monad** is a simplicial functor $T\colon \underline{\mathrm{Mnd}} \to \underline{\mathrm{qCat}}_\infty$., i.e.,

- $+ \mapsto B \in \underline{\mathrm{qCat}}_\infty$
- $\mathbb{\Delta}_+ \xrightarrow{t} B^B$ =: the **monad resolution**

$$\begin{tikzcd} \mathrm{id}_B \arrow[r, "\eta"] & t \arrow[r, shift left=2, "\eta"] \arrow[r, shift right=2, "t\eta"'] & t^2 \arrow[l, "\mu"'] \arrow[r, shift left=4, "\eta"] \arrow[r, "t\eta"] \arrow[r, shift right=4, "tt\eta"'] & t^3 \cdots \arrow[l, shift right=2, "\mu"'] \arrow[l, shift left=2, "t\mu"] \end{tikzcd}$$

and higher data, e.g.,

$$\begin{tikzcd} & t^2 \arrow[dr, "\mu"] & \\ t \arrow[ur, "\eta t"] \arrow[rr, equal] \arrow[rr, phantom, bend left=20, "\sim"] & & t \end{tikzcd}$$

Weighted limits

Fix a simplicial functor T, a diagram of shape $\mathbf{A}$.

A **weight** is a simplicial functor $W \colon \mathbf{A} \to \underline{\mathrm{sSet}}$.

The **weighted limit** $\{W, T\}$ represents the simplicial set of cones of shape W over T.

Key facts:

- The limit weighted by $\hom_a$ evaluates at a.
- The weighted limit bifunctor is cocontinuous in the weights.

Upshot: Weights built by gluing representables will define cones of the expected shape.

Weighted limits in the quasi-categorical context

Proposition. $\underline{\mathrm{qCat}_\infty}$ has all limits weighted by projective cofibrant simplicial functors.

$$\begin{tikzcd} \underline{\mathrm{Mnd}} \arrow[d] \arrow[dr, "W_+"] & \\ \underline{\mathrm{Adj}} \arrow[r, shift left, "\mathrm{hom}_+"] \arrow[r, shift right, "\mathrm{hom}_-"'] & \underline{\mathrm{sSet}} \\ \underline{\mathrm{Mnd}} \arrow[u] \arrow[ur, "W_-"'] & \end{tikzcd}$$

$\underline{\mathrm{Adj}}$ a simplicial computad $\Rightarrow W_+$ and W_- projective cofibrant.

The Eilenberg-Moore quasi-category

Fix a homotopy coherent monad $T\colon \underline{\mathrm{Mnd}} \to \underline{\mathrm{qCat}}_\infty$

- $\{W_+, T\} = B$
- $\{W_-, T\} =: B[t]$, the **Eilenberg-Moore quasi-category**

By definition

$$B[t] = \mathrm{eq}\left(B^{\mathbb{\Delta}_\infty} \rightrightarrows B^{\mathbb{\Delta}_+ \times \mathbb{\Delta}_\infty}\right)$$

so a vertex is a map $\mathbb{\Delta}_\infty \to B$ of the form:

$$\begin{tikzcd}
b \arrow[r, shift left, "\eta"] & tb \arrow[l, shift left, "\beta"] \arrow[r, shift left=4.5, "\eta"] \arrow[r, shift left=1.5, "t\eta"] & t^2b \arrow[l, shift left=1.5, "\mu"] \arrow[l, shift left=4.5, "t\beta"] \arrow[r, shift left=7.5, "\eta"] \arrow[r, shift left=1.5, "t\eta"] \arrow[r, shift right=4.5, "tt\eta"] & t^3b \cdots \arrow[l, shift right=4.5, "\mu"] \arrow[l, shift left=1.5, "t\mu"] \arrow[l, shift left=7.5, "tt\beta"]
\end{tikzcd}$$

and higher data, e.g.,

$$\begin{tikzcd}
 & tb \arrow[dr, "\beta"] & \\
b \arrow[ur, "\eta"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3, "\sim"] & & b
\end{tikzcd}$$

The monadic homotopy coherent adjunction

…is all in the weights!

$$\underline{\mathrm{Adj}}^{\mathrm{op}} \xrightarrow{\hom} \underline{\mathrm{sSet}}^{\underline{\mathrm{Adj}}} \xrightarrow{\mathrm{res}} \underline{\mathrm{sSet}}^{\underline{\mathrm{Mnd}}} \xrightarrow{\{-,T\}} \underline{\mathrm{qCat}}_{\infty}^{\mathrm{op}}$$

$$\begin{tikzcd}
- \arrow[r, phantom, "\mapsto"] \arrow[d, bend left, "u"] \arrow[d, phantom, "\dashv"] & \hom_- \arrow[r, phantom, "\mapsto"] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & W_- \arrow[r, phantom, "\mapsto"] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & B[t] \arrow[d, bend left, "u^t"] \arrow[d, phantom, "\dashv"] \\
+ \arrow[r, phantom, "\mapsto"] \arrow[u, bend left, "f"] & \hom_+ \arrow[r, phantom, "\mapsto"] \arrow[u, bend right] & W_+ \arrow[r, phantom, "\mapsto"] \arrow[u, bend right] & B \arrow[u, bend left, "f^t"]
\end{tikzcd}$$

Proposition. If $V \hookrightarrow W$ is identity-on-0-cells, then $\{W,T\} \to \{V,T\}$ is conservative. E.g., $W_+ \to W_-$.

Corollary. The monadic forgetful functor $u^t \colon B[t] \to B$ is conservative.

Codescent in the Eilenberg-Moore category

Suppose (b, β) is an algebra for a monad t on a category B.

Fact. There is a canonical colimit diagram in $B[t]$

$$\begin{tikzcd}
\cdots\ t^3b \arrow[r, shift left=8, "tt\beta"] \arrow[r, shift left=4, leftarrow, "tt\eta"] \arrow[r, "t\mu"] \arrow[r, shift right=4, leftarrow, "t\eta"] \arrow[r, shift right=8, "\mu"] & t^2b \arrow[l, bend left=60, "\eta"] \arrow[r, shift left=4, "t\beta"] \arrow[r, leftarrow, "t\eta"] \arrow[r, shift right=4, "\mu"] & tb \arrow[l, bend left=60, "\eta"] \arrow[r, "\beta"] & b \arrow[l, bend left=40, "\eta"]
\end{tikzcd}$$

which is a u^t-split reflexive coequalizer diagram, and preserved by u^t.

$(b, \beta) \rightsquigarrow$ a u^t-split (augmented) simplicial object

$$\begin{tikzcd}
\Delta_+^{\mathrm{op}} \arrow[r] \arrow[d] & B[t] \arrow[d, "u^t"] \\
\Delta_\infty \arrow[r] & B
\end{tikzcd}$$

Codescent in the Eilenberg-Moore quasi-category

Theorem. Every vertex in $B[t]$ is the colimit of a canonical u^t-split simplicial object that is preserved by u^t.

Proof. By cocontinuity,

$$\begin{tikzcd} \mathbb{\Delta}_+^{\mathrm{op}} \times W_+ \arrow[r] \arrow[d] & \mathbb{\Delta}_\infty \times W_+ \arrow[d] \\ \mathbb{\Delta}_+^{\mathrm{op}} \times W_- \arrow[r] & W \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \{W,T\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B[t]^{\mathbb{\Delta}_+^{\mathrm{op}}} \arrow[d, "u^t"] \\ B^{\mathbb{\Delta}_\infty} \arrow[r, "\mathrm{res}"'] & B^{\mathbb{\Delta}_+^{\mathrm{op}}} \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} & & B[t] \arrow[d, "\mathrm{const}"] \\ \{W,T\} \arrow[r] & B[t]^{\mathbb{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[r, phantom, bend left, "\Uparrow"] & B[t]^{\mathbb{\Delta}^{\mathrm{op}}} \end{tikzcd} \quad \text{in } \underline{\mathrm{qCat}}_2.$$

Codescent in the Eilenberg-Moore quasi-category

Theorem.

$$\begin{tikzcd} & & B[t] \arrow[d, "\mathrm{const}"] \\ \{W,T\} \arrow[r] & B[t]^{\Delta_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift left=3, "\Uparrow"] & B[t]^{\Delta^{\mathrm{op}}} \end{tikzcd}$$

defines an absolute left lifting diagram in $\underline{\mathrm{qCat}_2}$ that u^t preserves.

Proof. Similar to:

Theorem.

$$\begin{tikzcd} & B \arrow[d, "\mathrm{const}"] \\ B^{\Delta_\infty} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] & B^{\Delta^{\mathrm{op}}} \end{tikzcd}$$

defines an absolute left lifting diagram in $\underline{\mathrm{qCat}_2}$ that is preserved by any functor.

Proof. See "The 2-category theory of quasi-categories."

The classical monadicity theorem

Let t be the monad induced by an adjunction $f \dashv u$.

Theorem (Beck).

- There is a comparison functor commuting with the adjunctions.

$$\begin{tikzcd} A \arrow[rr, dashed, "R"'] \arrow[dr, "u"] & \arrow[d, phantom, "\perp"] & B[t] \arrow[ll, dashed, bend right, "L"'] \arrow[dl, "u^t"] \\ & B \arrow[ul, "f"] \arrow[ur, "f^t"] & \end{tikzcd}$$

- If A has u-split coequalizers, then R has a left adjoint.
- If u preserves them, then L is fully faithful.
- If u is conservative, then $L \dashv R$ is an adjoint equivalence.

Goal. Prove the analogous theorem for the homotopy coherent monad of a homotopy coherent adjunction.

Defining the comparison map

$$\begin{tikzcd} & \underline{\mathrm{Adj}} \arrow[r, "H"] & \underline{\mathrm{qCat}_\infty} \\ \underline{\mathrm{Mnd}} \arrow[ur] \arrow[urr, bend right, "T"'] \end{tikzcd} \quad \rightsquigarrow \quad B[t] \cong \{W_-, \mathrm{res}\, H\} \cong \{\mathrm{lan}\, W_-, H\}$$

Weights for the monadic adjunction, revisited.

- weight for the Eilenberg-Moore quasi-category: $\mathrm{lan}\,\mathrm{res}\,\mathrm{hom}_-$
- weight for the monadic adjunction: $\mathrm{lan}\,\mathrm{res}\,\mathrm{hom}$

The counit of $\underline{\mathrm{sSet}^{\mathrm{Adj}}} \underset{\mathrm{res}}{\overset{\mathrm{lan}}{\underset{\perp}{\rightleftarrows}}} \underline{\mathrm{sSet}^{\mathrm{Mnd}}}$ defines a map of weights $\mathrm{lan}\,\mathrm{res}\,\mathrm{hom} \to \mathrm{hom}$ and hence a natural transformation

$$\begin{tikzcd} A \arrow[rr, dashed, "R"] \arrow[dr, shift left, "u"] & & B[t] \arrow[dl, shift left, "u^t"] \\ & B \arrow[ul, shift left, "f"] \arrow[ur, shift left, "f^t"] & \end{tikzcd}$$

between homotopy coherent adjunctions.

The weight for u-split simplicial objects

Define a weight

$$\begin{tikzcd} \mathbb{\Delta}^{\mathrm{op}} \times \hom_+ \arrow[r] \arrow[d] & \mathbb{\Delta}_\infty \times \hom_+ \arrow[d] \\ \mathbb{\Delta}^{\mathrm{op}} \times \hom_- \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & W' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \{W', H\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{\Delta}^{\mathrm{op}}} \arrow[d, "u"] \\ B^{\mathbb{\Delta}_\infty} \arrow[r, "\mathrm{res}"'] & B^{\mathbb{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

Definition. The quasi-category A **admits colimits of u-split simplicial objects** if there is an absolute left lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\mathrm{const}"] \\ \{W', H\} \arrow[r] \arrow[ur, "\mathrm{colim}"] & A^{\mathbb{\Delta}^{\mathrm{op}}} \arrow[ul, phantom, "\Uparrow" near start] \end{tikzcd} \quad \text{in } \underline{\mathrm{qCat}}_2.$$

The proof of the monadicity theorem

Proof.

- The obvious map $W' \to \operatorname{lan} W_-$ induces $B[t] \to \{W', H\}$.
- If A has colimits of u-split simplicial objects, define $L := B[t] \to \{W', H\} \xrightarrow{\text{colim}} A$.
- From the universal property of absolute left liftings, $L \dashv R$.
- If u preserves these colimits, then u^t carries the unit of $L \dashv R$ to an isomorphism.
- As u^t is conservative, the unit is an isomorphism.
- If u is conservative, it follows that the counit is also an isomorphism, and $A \simeq B[t]$ is an adjoint equivalence of quasi-categories.

Further reading

- "The 2-category theory of quasi-categories" arXiv:1306.5144.
- "A weighted limits proof of monadicity" on the n-category café.
- "Homotopy coherent adjunctions and the formal theory of monads" — coming soon!