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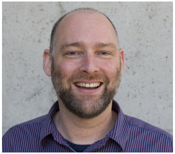
A proof of the model-independence of $(\infty, 1)$-category theory

joint with Dominic Verity



CT2018, Universidade dos Açores

Plan

Goal: build model-independent foundations of $(\infty, 1)$-category theory

1. What are model-independent foundations?
2. ∞-cosmoi of $(\infty, 1)$-categories
3. A taste of the formal category theory of $(\infty, 1)$-categories
4. The proof of model-independence of $(\infty, 1)$-category theory

I

What are model-independent foundations?

Models of $(\infty, 1)$-categories

Schematically, an $(\infty, 1)$-category is a category "weakly enriched" over ∞-groupoids/homotopy types ... but this is tricky to make precise.

$$\begin{tikzcd}
& \mathcal{R}\text{ezk} \arrow[r] \arrow[dr, shift left] & \mathcal{S}\text{egal} \arrow[d, shift left] & \\
\mathcal{R}\text{el}\mathcal{C}\text{at} \arrow[ur] \arrow[dr] \arrow[drr] & & & \mathcal{T}\text{op-}\mathcal{C}\text{at} \arrow[dl] \\
& 1\text{-}\mathcal{C}\text{omp} \arrow[r, shift left] & \text{q}\mathcal{C}\text{at} \arrow[l, dashed, shift left] \arrow[u, shift left] \arrow[ul, shift left] &
\end{tikzcd}$$

- topological categories and relative categories are the simplest to define but do not have enough maps between them
- $\left\{\begin{array}{l}\text{quasi-categories (nee. weak Kan complexes),} \\ \text{Rezk spaces (nee. complete Segal spaces),} \\ \text{Segal categories, and} \\ \text{(saturated 1-trivial weak) 1-complicial sets}\end{array}\right.$

  each have a homotopically meaningful internal hom.

The analytic vs synthetic theory of $(\infty, 1)$-categories

Q: How might you develop the category theory of $(\infty, 1)$-categories?

Two strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\mathcal{C}at$; Kazhdan-Varshavsky, Rasekh in $\mathcal{R}ezk$; Simpson in $\mathcal{S}egal$)

- work synthetically to give categorical definitions and prove theorems in all four models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

Our method: introduce an ∞-cosmos to axiomatize common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of $(\infty, 1)$-categories.

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∞-cosmoi of $(\infty, 1)$-categories

∞-cosmoi of ∞-categories

Idea: An ∞-cosmos is an "$(\infty, 2)$-category with $(\infty, 2)$-categorical limits" whose objects we call ∞-categories.

An ∞-cosmos is a category that

- is enriched over quasi-categories, i.e., functors $f\colon A \to B$ between ∞-categories define the points of a quasi-category $\mathrm{Fun}(A, B)$,
- has a class of isofibrations $E \twoheadrightarrow B$ with familiar closure properties,
- and has flexibly-weighted simplicially-enriched limits, constructed as limits of diagrams of ∞-categories and isofibrations.

Theorem. $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, and $1\text{-}\mathcal{C}omp$ define ∞-cosmoi, and so do certain models of (∞, n)-categories for $0 \leq n \leq \infty$, fibered versions of all of the above, and many more things besides.

Henceforth ∞-category and ∞-functor are technical terms that mean the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
- 1-cells are the ∞-functors $f\colon A \to B$ in the ∞-cosmos
- 2-cells we call ∞-natural transformations $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ which are defined to be homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$

Prop. Equivalences in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "1_B"] \arrow[r, bend right, "fg"'] \arrow[r, phantom, "\Downarrow\cong"] & B \end{tikzcd}$$

coincide with equivalences in the ∞-cosmos.

Thus, non-evil 2-categorical definitions are "homotopically correct."

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A taste of the formal category theory of $(\infty, 1)$-categories

Absolute lifting diagrams

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

is an absolute right lifting diagram if it and any restriction are right liftings:

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\forall\Downarrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, shift left=3, "\exists!\Downarrow\zeta"] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

, in which case:

- $$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$ is absolute right lifting

- $$\begin{tikzcd} & E \arrow[d, "k"] \\ C \arrow[ur, "s"] \arrow[r, "r"'] \arrow[ur, phantom, shift right=3, "\Downarrow\sigma"] & B \end{tikzcd}$$ is absolute right lifting iff $$\begin{tikzcd} & E \arrow[d, "k"] \\ & B \arrow[d, "f"] \\ C \arrow[uur, bend left, "s"] \arrow[ur, "r"] \arrow[r, "g"'] \arrow[uur, phantom, shift right=2, "\Downarrow\sigma"] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$ is

Adjunctions and limits

An adjunction between ∞-categories is an adjunction $(A, B, f, u, \eta, \epsilon)$ in the homotopy 2-category.

$\rightsquigarrow$ Hence all 2-categorical theorems about adjunctions become theorems about adjunctions between ∞-categories! In particular:

A right adjoint $\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, dashed, "u"] \end{tikzcd}$ is an absolute right lifting $\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, dashed, "u"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon", shift right=3] & A \end{tikzcd}$

Hence, a limit functor or limit of $d \colon 1 \to A^J$ is an absolute right lifting

$$\begin{tikzcd} A \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend left, dashed, "\lim"] \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^J \arrow[ur, dashed, "\lim"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\lambda", shift right=3] & A^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\lambda d", shift right=3] & A^J \end{tikzcd}$$

Right adjoints preserve limits

Prop (right adjoints preserve limits). If $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ and $\lambda \colon \Delta\ell \Rightarrow d$ is a limit cone then

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd}$$

is absolute right lifting.

Proof: It suffices to show the transposed cone is absolute right lifting

$$\begin{tikzcd} & & B \arrow[d, "\Delta"] \\ & A \arrow[ur, "u"] \arrow[d, "\Delta"] & B^J \arrow[d, "f^J"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \arrow[ur, "u^J"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon^J"] & A^J \end{tikzcd} = \begin{tikzcd} & & B \arrow[d, "f"] \\ & A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \arrow[r, equal] & A^J \end{tikzcd}$$

which is the case by 2-naturality and composition of absolute right liftings.

Universal properties of adjunctions and limits

Any cospan has a comma ∞-category
$$\begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl, two heads, "\mathrm{cod}"'] \arrow[dr, two heads, "\mathrm{dom}"] & \\ C \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$
with

comma span a two-sided discrete fibration aka a module $C \overset{\mathrm{Hom}_A(f,g)}{\longrightarrow\!\!\!\!\!\!\!+\!\!\!\!\longrightarrow} B$.

Thm.
$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\rho"] & A \end{tikzcd}$$
absolute lifting iff $\mathrm{Hom}_B(B,r) \underset{C\times B}{\simeq} \mathrm{Hom}_A(f,g)$.

Cor. $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ iff $\mathrm{Hom}_A(f,A) \simeq_{A\times B} \mathrm{Hom}_B(B,u)$.

Cor. $d\colon 1 \to A^J$ has a limit ℓ iff $\mathrm{Hom}_A(A,\ell) \simeq_A \mathrm{Hom}_{A^J}(\Delta,d)$.

The calculus of modules

Thm. Any ∞-cosmos has a virtual equipment of ∞-categories, ∞-functors, modules, and "multilinear" module maps:

$$\begin{tikzcd} A_0 \arrow[r, "E_1", "\mid" marking] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\mid" marking] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n", "\mid" marking] & A_n \arrow[d, "g"] \\ B_0 \arrow[rrr, "F"', "\mid" marking] & & & B_n \end{tikzcd}$$

with units
$$\begin{tikzcd} A \arrow[r, equal, "\mid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[d, equal] \\ A \arrow[r, dashed, "\mathrm{Hom}_A"', "\mid" marking] & A \end{tikzcd}$$
and restriction of scalars
$$\begin{tikzcd} X \arrow[r, dashed, "E(b,a)", "\mid" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & Y \arrow[d, "b"] \\ A \arrow[r, "E"', "\mid" marking] & B \end{tikzcd}$$

$\leadsto$ The homotopy 2-category embeds covariantly and contravariantly.

Modules $A \overset{E}{\nrightarrow} B$ and $A \overset{F}{\nrightarrow} B$ are isomorphic iff $E \simeq_{A\times B} F$ so the virtual equipment captures the formal category theory of ∞-categories.

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The proof of model-independence of $(\infty, 1)$-category theory

Cosmological biequivalences and change-of-model

A cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ between ∞-cosmoi is

- a cosmological functor: a simplicial functor that preserves isofibrations and the simplicial limits
- surjective on objects up to equivalence: if $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ with $FA \simeq C \in \mathcal{L}$
- a local equivalence: $\mathrm{Fun}(A, B) \xrightarrow{\sim} \mathrm{Fun}(FA, FB) \in \mathrm{q}\mathcal{C}\mathrm{at}$

Prop. A cosmological biequivalence induces a biequivalence of homotopy 2-categories, defining (local) bijections on:

- equivalence classes of ∞-categories
- isomorphism classes of parallel ∞-functors
- 2-cells with corresponding boundary

and fibered equivalence classes of modules, respecting representability.

Idea: $FA \simeq A', FB \simeq B' \rightsquigarrow \mathcal{K}_{/A\times B} \xrightarrow{\sim} \mathcal{L}_{/FA\times FB} \xrightarrow{\sim} \mathcal{L}_{/A'\times B'}$

Model-independence

$$\begin{tikzcd} & \mathcal{R}\mathrm{ezk} \arrow[r] \arrow[dr, shift left] & \mathcal{S}\mathrm{egal} \arrow[d, shift left] \\ 1\text{-}\mathcal{C}\mathrm{omp} \arrow[rr, shift left] & & \mathrm{q}\mathcal{C}\mathrm{at} \arrow[u, shift left] \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd} \quad \rightsquigarrow \quad \text{cosmological biequivalences between models of } (\infty,1)\text{-categories}$$

Model-Independence Theorem. A cosmological biequivalence induces a biequivalence of virtual equipments of modules and thus preserves, reflects, and creates all ∞-categorical properties and structures.

- The existence of an adjoint to a given functor.
- The existence of a limit for a given diagram.
- The property of a given functor defining a cartesian fibration.
- The existence of a pointwise Kan extension.

Analytically-proven theorems also transfer along biequivalences:

- Universal properties in an $(\infty, 1)$-category A are determined elementwise, by each $a\colon 1 \to A$.

Summary

- In the past, the theory of $(\infty, 1)$-categories has been developed analytically, in a particular model.
- A large part of that theory can be developed simultaneously in many models by working synthetically with $(\infty, 1)$-categories as objects in an ∞-cosmos.
- The axioms of an ∞-cosmos are chosen to simplify proofs by allowing us to work strictly up to isomorphism insofar as possible.
- Much of this development in fact takes place in a strict 2-category of $(\infty, 1)$-categories, $(\infty, 1)$-functors, and $(\infty, 1)$-natural transformations using the methods of formal category theory.
- Both analytically- and synthetically-proven results about $(\infty, 1)$-categories transfer across "change-of-model" functors called biequivalences.
- Open problems: many $(\infty, 1)$-categorical notions are yet to be incorporated into ∞-cosmology.

References

For more on the model-independent theory of $(\infty, 1)$-categories see:

Emily Riehl and Dominic Verity

- mini-course lecture notes:

  ∞-Category Theory from Scratch
  arXiv:1608.05314

- draft book in progress:

  Elements of ∞-Category Theory
  www.math.jhu.edu/~eriehl/elements.pdf

Obrigada!