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Categorifying cardinal arithmetic

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Goal: prove $a \times (b+c) = (a \times b) + (a \times c)$ for any natural numbers a, b, and c.



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- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof



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- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?





Step I: categorification

The idea of categorification



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Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

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Q: What is true of A and B if a = b?

A: a=b if and only if A and B are isomorphic, which means there exist functions $f\colon A\to B$ and $g\colon B\to A$ that are inverses in the sense that $g\circ f=\operatorname{id}$ and $f\circ g=\operatorname{id}$. In this case, we write $A\cong B$.

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Categorification: the truth behind a = b is $A \cong B$.





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Q: What is the deeper meaning of the symbols "+" and " \times "?

Categorifying +



Q: If b := |B| and c := |C| what set has b + c elements?

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Q: If b := |B| and c := |C| what set has b + c elements?

A: The disjoint union B+C is a set with b+c elements.

$$B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\} , \qquad C = \left\{ \begin{array}{c} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{array} \right\} , \qquad B + C = \left\{ \begin{array}{c} \sharp & \flat & \spadesuit & \heartsuit \\ \natural & \diamondsuit & \clubsuit \end{array} \right\}$$

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$$b + c \coloneqq |B + C|$$

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$$A = \left\{ \begin{array}{cc} * & \star \end{array} \right\} , \qquad B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\} , \qquad A \times B = \left\{ \begin{array}{cc} (*, \sharp) & (\star, \sharp) \\ (*, \flat) & (\star, \flat) \\ (*, \natural) & (\star, \natural) \end{array} \right\}$$

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In summary:

• Natural numbers define cardinalities: there are sets A, B, and C so that a := |A|, b := |B|, and c := |C|.



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Q: What is the deeper meaning of the equation

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A: It means that the sets $A \times (B+C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B+C) \cong (A \times B) + (A \times C)$$

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(*,,\flat) & (\star,,\flat) \\
(*,,) & (\star,,) \\
(*,\lozenge) & (\star,\diamondsuit) \\
(*,\diamondsuit) & (\star,\diamondsuit) \\
(*,\clubsuit) & (\star,\clubsuit)
\end{array}
\right\}
\cong
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Step 2: the Yoneda lemma



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• for all sets X, the sets of functions

$$\operatorname{Fun}(A,X) := \{h \colon A \to X\} \quad \text{and} \quad \operatorname{Fun}(B,X) := \{k \colon B \to X\}$$
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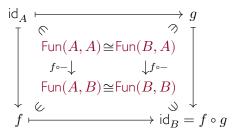
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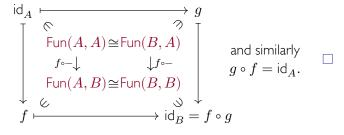
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Step 3: representability

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A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f \colon B + C \to X$ is determined by two functions $f_B \colon B \to X$ and $f_C \colon C \to X$.

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By "pairing"
$$\begin{array}{ccc} \operatorname{Fun}(B+C,X) &\cong & \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X) \\ & & & & & \\ f & & & & \\ f & & & & \\ \end{array}$$

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A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a,b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-,b) \colon A \to X$ sending a to f(a,b).

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By "currying"
$$\begin{array}{cccc} \operatorname{Fun}(A\times B,X) &\cong & \operatorname{Fun}(B,\operatorname{Fun}(A,X)) \\ & & & & & \\ & & & & \\ f\colon A\times B\to X & \Leftrightarrow & f\colon B\to \operatorname{Fun}(A,X) \\ \end{array}$$

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 $\quad \operatorname{Fun}(B+C,X) \cong \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X) \text{ by "pairing"}$

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Step 3 summary:

- $\operatorname{Fun}(B+C,X) \cong \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X)$ by "pairing" and
- $\operatorname{Fun}(A \times B, X) \cong \operatorname{Fun}(B, \operatorname{Fun}(A, X))$ by "currying."



Step 4: the proof



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5

Epilogue: what was the point of that?



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Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma}$$
 and $(\alpha^{\beta})^{\gamma} = \alpha^{\beta \times \gamma} = (\alpha^{\gamma})^{\beta}$.

Generalization to other mathematical contexts

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$$U\otimes (V\oplus W)\cong (U\otimes V)\oplus (U\otimes W).$$

For nice topological spaces X, Y, Z,

$$X\times (Y\sqcup Z)=(X\times Y)\sqcup (X\times Z).$$

• For abelian groups A, B, C,

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Thank you!