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The synthetic approach to ∞-category theory

joint with Dominic Verity and Michael Shulman

Summer School on Higher Topos Theory
and Univalent Foundations

The idea of an ∞-category

An ∞-category — a category weakly enriched over ∞-groupoids — should have:

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

But this definition is tricky to make precise in classical foundations.

Models of ∞-categories

The notion of ∞-category is made precise by several models:

$$
\begin{tikzcd}
& \mathcal{R}\mathrm{ezk} \arrow[r] \arrow[dr, shift left] & \mathcal{S}\mathrm{egal} \arrow[d, shift left] & \\
\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \arrow[ur] \arrow[dr] \arrow[drr] & & & \mathcal{T}\mathrm{op}\text{-}\mathcal{C}\mathrm{at} \arrow[dl] \\
& 1\text{-}\mathcal{C}\mathrm{omp} \arrow[r] & \mathrm{q}\mathcal{C}\mathrm{at} \arrow[ul, shift left] \arrow[u, shift left] &
\end{tikzcd}
$$

- topological categories and relative categories are the simplest to define but do not have enough maps between them
- $\left\{\begin{array}{l} \text{quasi-categories (nee. weak Kan complexes),} \\ \text{Rezk spaces (nee. complete Segal spaces),} \\ \text{Segal categories, and} \\ \text{(saturated 1-trivial weak) 1-complicial sets} \end{array}\right.$

  are cartesian closed, and in fact any of these categories can be enriched over any of the others

The analytic vs synthetic theory of ∞-categories

Q: How might you develop the category theory of ∞-categories?

Strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\mathcal{C}at$; Kazhdan-Varshavsky, Rasekh in $\mathcal{R}ezk$; Simpson in $\mathcal{S}egal$)

- work synthetically to give categorical definitions and prove theorems in all four models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

  (R-Verity: an ∞-cosmos axiomatizes the common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of ∞-categories)

- work synthetically in a simplicial type theory augmenting HoTT to prove theorems in $\mathcal{R}ezk$

  (R-Shulman: an ∞-category is a type with unique binary composites in which isomorphism is equivalent to identity)

Plan

0. The analytic theory of ∞-categories

“∞-category theory for experts”

1. The synthetic theory of ∞-categories (in an ∞-cosmos)

“∞-category theory for graduate students”

2. The synthetic theory of ∞-categories (in homotopy type theory)

“∞-category theory for undergraduates”

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The synthetic theory of ∞-categories (in an ∞-cosmos)

∞-cosmoi of ∞-categories

An ∞-cosmos axiomatizes the structures needed to "develop ∞-category theory."

not-the-defn. An ∞-cosmos is a cartesian closed category $\mathcal{K}$ that has

- certain (flexible weighted enriched) limits
- an adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "\mathrm{ho}"] \arrow[r, phantom, "\perp"] & \mathcal{C}at \arrow[l, bend left] \end{tikzcd}$$

Theorem. $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ define biequivalent ∞-cosmoi.

Henceforth ∞-category and ∞-functor are technical terms that refer to the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
- 1-cells are the ∞-functors $f\colon A \to B$ in the ∞-cosmos
- 2-cells, called ∞-natural transformations $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$, are defined to be the arrows in the homotopy category $\mathrm{ho}(B^A)$

Key fact: equivalences in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{id}_A"] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "\mathrm{id}_B"] \arrow[r, bend right, "fg"'] \arrow[r, phantom, "\Downarrow\cong"] & B \end{tikzcd}$$

coincide with equivalences in the ∞-cosmos.

Thus, non-evil 2-categorical definitions are "homotopically correct."

Adjunctions between ∞-categories

defn. An adjunction between ∞-categories is an adjunction in the homotopy 2-category, consisting of:

- ∞-categories A and B
- ∞-functors $u\colon A \to B$, $f\colon B \to A$
- ∞-natural transformations $\begin{tikzcd} B \arrow[r, bend left, "\mathrm{id}_B"] \arrow[r, bend right, "uf"'] \arrow[r, phantom, "\Downarrow\eta"] & B \end{tikzcd}$ and $\begin{tikzcd} A \arrow[r, bend left, "fu"] \arrow[r, bend right, "\mathrm{id}_A"'] \arrow[r, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$

satisfying the triangle equalities

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & B \\ A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"'] & \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

Write $f \dashv u$ to indicate that f is the left adjoint and u is the right adjoint.

The 2-category theory of adjunctions

Since an adjunction between ∞-categories is just an adjunction in the homotopy 2-category, all 2-categorical theorems about adjunctions become theorems about adjunctions between ∞-categories.

Prop. Adjunctions compose:

$$\begin{tikzcd} C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"] \end{tikzcd}$$

Prop. Adjoints to a given functor $u \colon A \to B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.

Prop. Any equivalence can be promoted to an adjoint equivalence: if $u \colon A \xrightarrow{\sim} B$ then u is left and right adjoint to its equivalence inverse.

Composing adjunctions

Prop. Adjunctions compose:

$$
\begin{tikzcd}
C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"]
\end{tikzcd}
$$

Proof: The composite 2-cells

$$
\begin{tikzcd}
C \arrow[rrrr, equal] \arrow[dr, "f'"'] & & & & C \\
& B \arrow[rr, equal] \arrow[dr, "f"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow\eta'"] & & B \arrow[ur, "u'"'] & \\
& & A \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow\eta"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & C \arrow[dr, "f'"] & & \\
& B \arrow[ur, "u'"] \arrow[rr, equal] \arrow[rr, phantom, shift left=4ex, "\Downarrow\epsilon'"] & & B \arrow[dr, "f"] & \\
A \arrow[ur, "u"] \arrow[rrrr, equal] \arrow[rrrr, phantom, shift left=4ex, "\Downarrow\epsilon"] & & & & A
\end{tikzcd}
$$

define the unit and counit of $ff' \dashv u'u$ satisfying the triangle equalities.

Limits and colimits in an ∞-category

defn. An ∞-category A has

- a terminal element iff $\begin{tikzcd} 1 \arrow[r, phantom, "\perp"] \arrow[r, bend right, dashed, "t"'] & A \arrow[l, bend right, "!"'] \end{tikzcd}$ and
- all limits of shape J iff $\begin{tikzcd} A^J \arrow[r, phantom, "\perp"] \arrow[r, bend right, dashed, "\lim"'] & A \arrow[l, bend right, "\Delta"'] \end{tikzcd}$.

Note: the counit components $\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\lim_d"] \arrow[r, "d"'] \arrow[ur, phantom, shift left=3, "\Downarrow\epsilon"] & A^J \end{tikzcd}$ define the limit cone.

Prop. Right adjoints preserve limits and left adjoints preserve colimits.

Proof: The usual one!

Universal properties of adjunctions and limits

defn. Any ∞-category A has an ∞-category of arrows $A^{\mathbb{2}}$, pulling back

to define the comma ∞-category:

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[r] \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd}$$

This specializes to define the mapping space $\mathrm{Hom}_A(x,y)$ between each pair of elements $x, y \colon 1 \to A$.

Prop. $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ if and only if $\mathrm{Hom}_A(f,A) \simeq_{A\times B} \mathrm{Hom}_B(B,u)$.

Prop. An ∞-functor $d \colon J \to A$ has limit $\ell \colon 1 \to A$ iff
$$\mathrm{Hom}_A(A,\ell) \simeq_A \mathrm{Hom}_{A^J}(\Delta, d).$$

Prop. Mapping spaces are discrete ∞-categories, i.e., ∞-groupoids.

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The synthetic theory of ∞-categories (in homotopy type theory)

The Rosetta Stone for Homotopy Type Theory

type theory	set theory	logic	homotopy theory
A	set	proposition	space
$x : A$	element	proof	point
$\emptyset, 1$	$\emptyset, \{\emptyset\}$	$\bot, \top$	$\emptyset, *$
$A \times B$	set of pairs	A and B	product space
$A + B$	disjoint union	A or B	coproduct
$A \to B$	set of functions	A implies B	function space
$x : A \vdash B(x)$	family of sets	predicate	fibration
$x : A \vdash b : B(x)$	fam. of elements	conditional proof	section
$\prod_{x:A} B(x)$	product	$\forall x.B(x)$	space of sections
$\sum_{x:A} B(x)$	disjoint sum	$\exists x.B(x)$	total space
$p : x =_A y$	$x = y$	proof of equality	path from x to y
$\sum_{x,y:A} x =_A y$	diagonal	equality relation	path space for A

Path induction

The identity type family is freely generated by the terms $\mathsf{refl}_x : x =_A x$.

Path induction. If $B(x, y, p)$ is a type family dependent on $x, y : A$ and $p : x =_A y$, then to prove $B(x, y, p)$ it suffices to assume y is x and p is refl_x. I.e., there is a function

$$\mathsf{path\text{-}ind} : \left(\prod_{x:A} B(x, x, \mathsf{refl}_x) \right) \to \left(\prod_{x,y:A} \prod_{p:x=_A y} B(x, y, p) \right).$$

A model for the type theory for synthetic ∞-categories

$$
\begin{array}{ccccccc}
\mathcal{S}et^{\mathbf{\Delta}^{op}\times\mathbf{\Delta}^{op}} & \supset & \mathcal{R}eedy & \supset & \mathcal{S}egal & \supset & \mathcal{R}ezk \\
\| & & \| & & \| & & \| \\
\text{bisimplicial sets} & & \text{types} & & \text{types with composition} & & \text{types with composition \& univalence}
\end{array}
$$

Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

Theorem (Rezk). ∞-categories are modeled by Rezk spaces aka complete Segal spaces.

Shapes in the theory of the directed interval

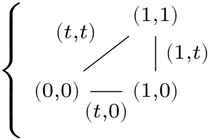
Our types may depend on other types and also on shapes $\Phi \subset \mathbb{2}^n$, polytopes embedded in a directed cube, defined in a language

$$\top, \bot, \wedge, \vee, \equiv \qquad \text{and} \qquad 0, 1, \leq$$

satisfying intuitionistic logic and strict interval axioms.

$$\Delta^n := \{(t_1, \dots, t_n) : \mathbb{2}^n \mid t_n \leq \cdots \leq t_1\} \qquad \text{e.g.} \qquad \Delta^1 := \mathbb{2}$$

$\Delta^2 :=$



$$\partial\Delta^2 := \{(t_1, t_2) : \mathbb{2}^2 \mid (t_2 \leq t_1) \wedge ((0 = t_2) \vee (t_2 = t_1) \vee (t_1 = 1))\}$$

$$\Lambda^2_1 := \{(t_1, t_2) : \mathbb{2}^2 \mid (t_2 \leq t_1) \wedge ((0 = t_2) \vee (t_1 = 1))\}$$

Extension types

Formation rule for extension types

$$\frac{\Phi \subset \Psi \text{ shape} \qquad A \text{ type} \qquad a : \Phi \to A}{\left\langle \begin{tikzcd} \Phi \arrow[r, "a"] \arrow[d, tail] & A \\ \Psi \arrow[ur, dashed] & \end{tikzcd} \right\rangle \text{ type}}$$

A term $f : \left\langle \begin{tikzcd} \Phi \arrow[r, "a"] \arrow[d, tail] & A \\ \Psi \arrow[ur, dashed] & \end{tikzcd} \right\rangle$ defines

$$f : \Psi \to A \text{ so that } f(t) \equiv a(t) \text{ for } t : \Phi.$$

The simplicial type theory allows us to *prove* equivalences between extension types along composites or products of shape inclusions.

Hom types

The hom type for A depends on two terms in A:

$$x, y : A \vdash \mathrm{Hom}_A(x, y)$$

$$\mathrm{Hom}_A(x,y) := \left\langle \begin{tikzcd} \partial\Delta^1 \arrow[r, "{[x,y]}"] \arrow[d, hook] & A \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd} \right\rangle \text{ type}$$

A term $f : \mathrm{Hom}_A(x, y)$ defines an arrow in A from x to y.

In the ∞-cosmos $\mathcal{R}$ezk:

- $\mathrm{Hom}_A(x, y)$ recovers the mapping space from x to y and
- $\sum_{x,y:A} \mathrm{Hom}_A(x, y)$ recovers the ∞-category of arrows $A^{\mathbb{2}}$.

Segal types $\equiv$ types with binary composition

A type A is Segal iff every composable pair of arrows has a unique composite., i.e., for every $f : \mathrm{Hom}_A(x,y)$ and $g : \mathrm{Hom}_A(y,z)$ the type

$$\left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[f,g]}"] \arrow[d, tail] & A \\ \Delta^2 \arrow[ur, dashed] \end{tikzcd} \right\rangle \qquad \text{is contractible.}$$

Semantically, a Reedy fibrant bisimplicial set A is *Segal* if and only if $A^{\Delta^2} \twoheadrightarrow A^{\Lambda^2_1}$ has contractible fibers.

By contractibility, $\left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[f,g]}"] \arrow[d, tail] & A \\ \Delta^2 \arrow[ur, dashed] \end{tikzcd} \right\rangle$ has a unique inhabitant. Write $g \circ f : \mathrm{Hom}_A(x,z)$ for its inner face, *the* composite of f and g.

Identity arrows

For any $x : A$, the constant function defines a term

$$\mathsf{id}_x := \lambda t.x : \mathsf{Hom}_A(x,x) := \left\langle \begin{tikzcd} \partial\Delta^1 \arrow[r, "{[x,x]}"] \arrow[d, hook] & A \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd} \right\rangle,$$

which we denote by id_x and call the identity arrow.

For any $f : \mathsf{Hom}_A(x,y)$ in a Segal type A, the term

$$\lambda(s,t).f(t) : \left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[\mathsf{id}_x, f]}"] \arrow[d, hook] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \right\rangle$$

witnesses the unit axiom $f = f \circ \mathsf{id}_x$.

Associativity of composition

Let A be a Segal type with arrows

$$f : \mathrm{Hom}_A(x, y), \quad g : \mathrm{Hom}_A(y, z), \quad h : \mathrm{Hom}_A(z, w).$$

Prop. $$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof: Consider the composable arrows in the Segal type $\Delta^1 \to A$:

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[d, "g"'] \arrow[dddr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[rr, dashed] \arrow[dr, "g\circ f"'] \arrow[ddrr, dashed, bend left=15, "\ell"] \arrow[dd, "f"'] & & z \arrow[dd, "h"] \\
 & z \arrow[dr, "h"] & \\
y \arrow[ur, "g"] \arrow[rr, dashed] & & w
\end{tikzcd}$$

Composing defines a term in the type $\Delta^2 \to (\Delta^1 \to A)$ which yields a term ℓ: $\mathrm{Hom}_A(x, w)$ so that $\ell = h \circ (g \circ f)$ and $\ell = (h \circ g) \circ f$.

Isomorphisms

An arrow $f\colon \mathrm{Hom}_A(x,y)$ in a Segal type is an isomorphism if it has a two-sided inverse $g\colon \mathrm{Hom}_A(y,x)$. However, the type

$$\sum_{g\colon \mathrm{Hom}_A(y,x)} (g \circ f = \mathrm{id}_x) \times (f \circ g = \mathrm{id}_y)$$

has higher-dimensional structure and is *not* a proposition. Instead define

$$\mathrm{isiso}(f) := \left(\sum_{g\colon \mathrm{Hom}_A(y,x)} g \circ f = \mathrm{id}_x \right) \times \left(\sum_{h\colon \mathrm{Hom}_A(y,x)} f \circ h = \mathrm{id}_y \right).$$

For $x, y : A$, the type of isomorphisms from x to y is:

$$x \cong_A y := \sum_{f : \mathrm{Hom}_A(x,y)} \mathrm{isiso}(f).$$

Rezk types $\equiv$ ∞-categories

By path induction, to define a map

$$\text{path-to-iso}\colon (x =_A y) \to (x \cong_A y)$$

for all $x, y : A$ it suffices to define

$$\text{path-to-iso}(\text{refl}_x) :\equiv \text{id}_x.$$

A Segal type A is Rezk iff every isomorphism is an identity, i.e., iff the map

$$\text{path-to-iso}\colon \prod_{x,y:A} (x =_A y) \to (x \cong_A y)$$

is an equivalence.

Discrete types $\equiv$ ∞-groupoids

Similarly by path induction define

$$\text{path-to-arr} \colon (x =_A y) \to \text{Hom}_A(x, y)$$

for all $x, y : A$ by $\text{path-to-arr}(\text{refl}_x) := \text{id}_x$.

A type A is discrete iff every arrow is an identity, i.e., iff path-to-arr is an equivalence.

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms.

Proof:

$$\begin{tikzcd}
x =_A y \arrow[rr, "\text{path-to-arr}"] \arrow[dr, "\text{path-to-iso}"'] & & \text{Hom}_A(x, y) \\
& x \cong_A y \arrow[ur, hook] &
\end{tikzcd}$$

∞-categories for undergraduates

defn. An ∞-groupoid is a type in which arrows are equivalent to identities:

$$\text{path-to-arr}\colon (x =_A y) \to \mathrm{Hom}_A(x, y) \text{ is an equivalence.}$$

defn. An ∞-category is a type

- which has unique binary composites of arrows:

$$\left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[f,g]}"] \arrow[d, tail] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \right\rangle \quad \text{is contractible}$$

- and in which isomorphisms are equivalent to identities:

$$\text{path-to-iso}\colon (x =_A y) \to (x \cong_A y) \text{ is an equivalence.}$$

Covariant type families $\equiv$ categorical fibrations

A type family $x : A \vdash B(x)$ over a Segal type A is covariant if for every $f : \mathrm{Hom}_A(x, y)$ and $u : B(x)$ there is a unique lift of f with domain u.

The codomain of the unique lift defines a term $f_* u : B(y)$.

Prop. For $u : B(x)$, $f : \mathrm{Hom}_A(x, y)$, and $g : \mathrm{Hom}_A(y, z)$,

$$g_*(f_* u) = (g \circ f)_* u \qquad \text{and} \qquad (\mathrm{id}_x)_* u = u.$$

Prop. If $x : A \vdash B(x)$ is covariant then for each $x : A$ the fiber $B(x)$ is discrete. Thus covariant type families are fibered in ∞-groupoids.

Prop. Fix $a : A$. The type family $x : A \vdash \mathrm{Hom}_A(a, x)$ is covariant.

The Yoneda lemma

Let $x : A \vdash B(x)$ be a covariant family over a Segal type and fix $a : A$.

Yoneda lemma. The maps

$$\text{ev-id} := \lambda\phi.\phi(a, \mathrm{id}_a) : \left(\prod_{x:A} \mathrm{Hom}_A(a,x) \to B(x)\right) \to B(a)$$

and

$$\text{yon} := \lambda u.\lambda x.\lambda f.f_* u : B(a) \to \left(\prod_{x:A} \mathrm{Hom}_A(a,x) \to B(x)\right)$$

are inverse equivalences.

Corollary. A natural *isomorphism* $\phi : \prod_{x:A} \mathrm{Hom}_A(a,x) \cong \mathrm{Hom}_A(b,x)$ induces an *identity* $\text{ev-id}(\phi) : b =_A a$ if the type A is Rezk.

The dependent Yoneda lemma

Yoneda lemma. If A is a Segal type and $B(x)$ is a covariant family dependent on $x : A$, then evaluation at (a, id_a) defines an equivalence

$$\text{ev-id} : \left(\prod_{x:A} \mathrm{Hom}_A(a, x) \to B(x) \right) \to B(a)$$

The Yoneda lemma is a "directed" version of the "transport" operation for identity types, suggesting a dependently-typed generalization analogous to the full induction principle for identity types.

Dependent Yoneda lemma. If A is a Segal type and $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \mathrm{Hom}_A(x, y)$, then evaluation at (x, x, id_x) defines an equivalence

$$\text{ev-id} : \left(\prod_{x,y:A} \prod_{f:\mathrm{Hom}_A(x,y)} B(x, y, f) \right) \to \prod_{x:A} B(x, x, \mathrm{id}_x)$$

Dependent Yoneda is directed path induction

Slogan: the dependent Yoneda lemma is directed path induction.

Path induction. If $B(x, y, p)$ is a type family dependent on $x, y : A$ and $p : x =_A y$, then to prove $B(x, y, p)$ it suffices to assume y is x and p is refl_x. I.e., there is a function

$$\mathsf{path\text{-}ind} : \left(\prod_{x:A} B(x, x, \mathsf{refl}_x) \right) \to \left(\prod_{x,y:A} \prod_{p:x=_A y} B(x, y, p) \right).$$

Arrow induction. If $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \mathsf{Hom}_A(x, y)$ and A is Segal, then to prove $B(x, y, f)$ it suffices to assume y is x and f is id_x. I.e., there is a function

$$\mathsf{id\text{-}ind} : \left(\prod_{x:A} B(x, x, \mathsf{id}_x) \right) \to \left(\prod_{x,y:A} \prod_{f:\mathsf{Hom}_A(x,y)} B(x, y, f) \right).$$

References

For more on the synthetic theories of ∞-categories, see:

Emily Riehl and Dominic Verity

- draft book in progress:

  Elements of ∞-Category Theory
  www.math.jhu.edu/~eriehl/elements.pdf

- mini-course lecture notes:

  ∞-Category Theory from Scratch
  arXiv:1608.05314

Emily Riehl and Michael Shulman

- A type theory for synthetic ∞-categories, Higher Structures 1(1):116–193, 2017; arXiv:1705.07442

Thank you!