Quasi-category theory you can use

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Plan

Part I. Introduction to quasi-categories.

- Quasi-categories are models for "abstract homotopy theory"
- Quasi-categories are good for homotopy limits and colimits

Part II. Category theory of quasi-categories (developed by Joyal, Lurie, Nichols-Barrer, Gepner, Haugseng, . . . , R-Verity (v2.0).

- Universal properties in quasi-categories

(case study: initial objects)

- General colimits in quasi-categories

Abstract homotopy theory

Classical **homotopy theory** studies topological spaces up to (weak) homotopy equivalence.

Abstract homotopy theory studies objects up to "weak equivalence": given $X \xrightarrow{\sim} Y$ think of X and Y as "the same".

E.g.,

- chain complexes up to quasi-isomorphism
- spectra up to stable equivalence
- categories up to equivalences
- ...

But it's hard to work in the **homotopy category**. Better to use:

- a Quillen model category,
- a **quasi-category** (aka ∞-**category**).

Some recent work

Precise statements of the following theorems are proven using **quasi-categories**.

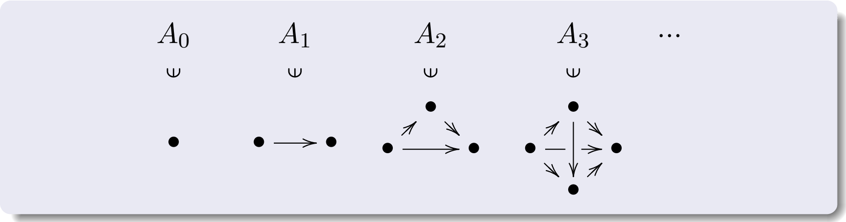
Theorem (Ben-Zvi & Nadler). S^1-equivariant quasi-coherent sheaves on the loop space of a smooth scheme are de Rham modules.

Theorem (Francis). Homology theories for topological n-manifolds are equivalent to n-disk algebras.

Theorem (Barwick & Schommer-Pries). The homotopy theory of (∞, n)-categories is characterized up to equivalence by certain axioms, and its space of automorphisms is equivalent to $(\mathbb{Z}/2)^n$.

Quasi-categories

A **quasi-category** is a simplicial set A with composition.



Composition of 1-simplices: $\begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] & & \bullet \end{tikzcd}$ in A $\rightsquigarrow$ $\begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd}$ in A

Examples of quasi-categories

A **topological space** is a quasi-category: composites exist because any simplex deformation retracts onto each of its horns.

A **category** is a quasi-category: 0-simplices are objects, 1-simplices are arrows, n-simplices are composable strings of n arrows.

Some special quasi-categories:

- $\Delta^n =$ "topological n-simplex" $= (0 \to 1 \to \cdots \to n)$
- $S^\infty = (\bullet \cong \bullet) = \mathrm{colim}_n S^n$

Q: What homotopy theories do these present?

The homotopy category of a quasi-category

The **homotopy category** $\mathrm{ho}(A)$ of a quasi-category A has:

- objects = vertices of A = elements of A_0
- arrows = equivalence classes of 1-simplices

$$f \sim g \colon x \to y \qquad \leftrightsquigarrow \qquad \begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd} \quad \begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, equal] \arrow[rr, "g"'] & & y \end{tikzcd} \quad \begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, equal] \arrow[rr, "f"'] & & y \end{tikzcd}$$

E.g.,

- the homotopy category of a space is its fundamental groupoid
- the homotopy category of a category is the category

$$\mathrm{ho}(\Delta^n) = (0 \to 1 \to \cdots \to n) \qquad \mathrm{ho}(S^\infty) = (\bullet \cong \bullet)$$

Isomorphisms in a quasi-category

An **isomorphism** in a quasi-category is a 1-simplex that represents an isomorphism in its homotopy category.

Theorem (Joyal). Each isomorphism in a quasi-category A extends to a map $S^\infty \to A$.

Data: x, y, $x \xrightarrow{f} y$, $y \xrightarrow{g} x$, $\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & & x \end{tikzcd}$, $\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g"] \arrow[rr, equal] & & y \end{tikzcd}$, ...

Interlude: classical homotopy colimits

hocolimit cone :

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \\ Z \arrow[r] & P \end{tikzcd}$$

$$P := Z \cup_X X \times I \cup_X X \times I \cup_X Y$$

hocolimit cone :

$$\begin{tikzcd} Z \arrow[r] \arrow[drr, bend right] & Y \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d] & W \arrow[r] \arrow[dl] & \cdots \\ & & P & & \end{tikzcd}$$

$$P := Z \times I \cup_Z Y \times I \cup_Y X \times I \cup_X W \times I \cup_W \cdots$$

…and **initial** with this data.

Initial objects, $\varnothing$

defn $\varnothing$. A vertex a in a quasi-category A is **initial** iff any sphere in A with a as its starting vertex can be filled to a simplex.

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "0\mapsto a"] \arrow[d] & A \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

$$a \longrightarrow x \qquad \begin{tikzcd} & x \arrow[dr] & \\ a \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\sim"] & & y \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dd] \arrow[dr] & \\ a \arrow[ur] \arrow[dr] \arrow[r, "\sim"] & \arrow[r, "\sim"] & z \\ & x \arrow[ur] & \end{tikzcd} \qquad \ldots$$

Initial objects, I & II

defn I. A vertex $a \in A$ is **initial** iff $\Delta^0 \xrightarrow{a} A$ is left adjoint to the constant functor $A \xrightarrow{!} \Delta^0$.

defn II. A vertex $a \in A$ is **initial** iff $a \downarrow A \xrightarrow{\sim} A$.

$$(a \downarrow A)_0 \ni \quad a \longrightarrow x$$

$$(a \downarrow A)_1 \ni \quad \begin{tikzcd} a \arrow[r] \arrow[d, equal] \arrow[dr] & x \arrow[d] \\ a \arrow[r] & y \end{tikzcd}$$

$$(a \downarrow A)_2 \ni \quad \begin{tikzcd} a \arrow[rr] \arrow[dd, equal] \arrow[dr, equal] & & x \arrow[dr] \arrow[dd] & \\ & a \arrow[rr] & & y \arrow[dl] \\ a \arrow[rr] \arrow[ur, equal] & & z & \end{tikzcd}$$

$$\cdots$$

Initial objects, III

defn III. A vertex $a \in A$ is **initial** iff there exists a map $\epsilon\colon \mathrm{const}_a \rightsquigarrow \mathrm{id}_A$ in A^A.

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, "\mathrm{const}_a"] & \\
A \times \Delta^1 \arrow[r, "\epsilon"] & A \\
A \arrow[u] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "\epsilon_x"] & x \end{tikzcd} \qquad x \in A_0$$

$$\begin{tikzcd}
a \arrow[r] \arrow[d, equal] \arrow[dr, "\epsilon_f" description] & x \arrow[d, "f"] \\
a \arrow[r] & y
\end{tikzcd} \qquad f \in A_1$$

$$\begin{tikzcd}
a \arrow[rr] \arrow[dd, equal] \arrow[dr, equal] & & x \arrow[dr] \arrow[dd] & \\
& a \arrow[rr, "\epsilon_\alpha" description] \arrow[dl, equal] & & y \arrow[dl] \\
a \arrow[rr] & & z &
\end{tikzcd} \qquad \alpha \in A_2$$

$\cdots \qquad \cdots$

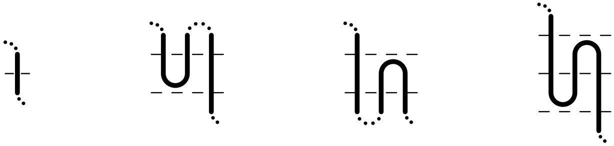
Initial objects, IV

defn IV. A vertex $a \in A$ is **initial** iff $\Delta^0 \xrightarrow{a} A$ and $A \xrightarrow{!} \Delta^0$ extend to a homotopy coherent adjunction.

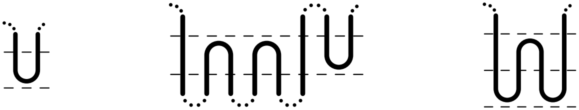
adjunction data: $a \in A_0$ and $\epsilon\colon \mathrm{const}_a \rightsquigarrow \mathrm{id}_A \in (A^A)_1$

homotopy coherent adjunction data:

- higher simplices in A spanning the vertex a



- higher simplices in A^A spanning the vertices const_a and id_A



Initial objects, V

defn V. A vertex $a \in A$ is **initial** iff for any $f: X \to A$ there exists a map $\mathrm{const}_a \rightsquigarrow f$ unique up to equivalence in A^X

Thus $\epsilon: \mathrm{const}_a \rightsquigarrow \mathrm{id}_A$ represents a unique equivalence class in A^A.

Colimits, $\varnothing$

defn $\varnothing$. A **colimit** of a diagram $d\colon \ulcorner \to A$ in a quasi-category A is an initial object in the quasi-category of cones under d.

data:

$$\text{diagram}: \begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd} \qquad \text{colimit cone}: \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d] \\ z \arrow[r] & p \end{tikzcd}$$

The universal property of the colimit is encoded by the universal property of an initial object.

Colimits, I & II

defn I. A quasi-category A **has $\ulcorner$-shaped colimits** iff the constant diagram functor $A \xrightarrow{!} A^{\ulcorner}$ has a left adjoint $A^{\ulcorner} \xrightarrow{\text{colim}} A$.

defn II. The map $A^{\ulcorner} \xrightarrow{\text{colim}} A$ defines a **$\ulcorner$-colimit functor** in A iff $\text{colim} \downarrow A$ is equivalent to the quasi-category of cones.

Colimits, III

defn III. A quasi-category A **has $\ulcorner$-shaped colimits** iff there exist maps $\epsilon\colon \mathrm{colim}\,! \rightsquigarrow \mathrm{id}_A$ in A^A and $\eta\colon \mathrm{id}_{A^\ulcorner} \rightsquigarrow !\,\mathrm{colim}$ in $(A^\ulcorner)^{A^\ulcorner}$ satisfying the triangle identities.

colimit cone:

$$\begin{tikzcd} A^\ulcorner \arrow[d] \arrow[dr, "\mathrm{id}_{A^\ulcorner}"] & \\ A^\ulcorner \times \Delta^1 \arrow[r, "\eta"] & A^\ulcorner \\ A^\ulcorner \arrow[u] \arrow[ur, "!\,\mathrm{colim}"'] & \end{tikzcd}$$

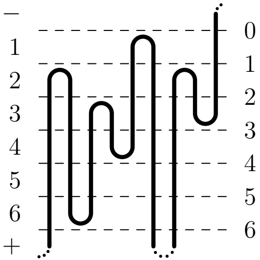
universal property:

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "\mathrm{colim}\,!"] & \\ A \times \Delta^1 \arrow[r, "\epsilon"] & A \\ A \arrow[u] \arrow[ur, "\mathrm{id}_A"'] & \end{tikzcd}$$

Colimits, IV

defn IV. A quasi-category A **has** $\ulcorner$**-shaped colimits** iff the constant diagram functor $A \xrightarrow{!} A^\ulcorner$ and its left adjoint $A^\ulcorner \xrightarrow{\text{colim}} A$ extend to a homotopy coherent adjunction.

homotopy coherent adjunction data: higher simplices in each of the four hom-quasi-categories between A and $A^\ulcorner$ corresponding to strictly undulating squiggles:



Colimits, V

defn V. A vertex $p \in A$ and cone under $d : \Gamma \to A$ define a **colimit cone** iff for any $f : X \to A$ and f-indexed cone under d, that cone factors uniquely (up to equivalence in A^X) through the colimit cone, inducing a map $\mathrm{const}_p \rightsquigarrow f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow"] & A \arrow[d, "!"] \\ \Delta^0 \arrow[r, "d"'] & A^\Gamma \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\exists!\Uparrow p" near start] & A \arrow[d, "!"] \\ \Delta^0 \arrow[ur, "p"' near end] \arrow[r, "d"'] \arrow[ur, phantom, "\Uparrow" near end, shift right=4ex] & A^\Gamma \end{tikzcd}$$

The universal property defines a unique 2-cell in the 2-category of quasi-categories.

References

"The 2-category theory of quasi-categories" `arXiv:1306.5144`

"Homotopy coherent adjunctions and the formal theory of monads" `arXiv:1310.8279`

"Completeness results for quasi-categories of algebras, homotopy limits, and related general constructions" `arXiv:1401.6247`

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