Class 34: 5/2/14 Section 7.2. I

Integration by Parts

Replacing an integral of a product with an expression involving another integral (possibly easier to solve) is given by the Product Rule in differentiation

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)\,dx}_{du}$$

$$\int u\,dv = uv - \int v\,du$$

One can refer to this as an Anti Product Rule.

ex. Find the anti derivative of xe^{2x}.

Here, $\int xe^{2x}dx$ is not obvious and a u-substitution will not work. Try it!

Via IbP, try the double substitution
$$u = x \quad \text{and} \quad dv = e^{2x}dx.$$

Fill in the missing data:
$$du = dx \quad \text{and} \quad v = \frac{1}{2}e^{2x}$$

Then
$$\int xe^{2x}dx = \int u\,dv = uv - \int v\,du$$
$$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$$
$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

Is it correct? Take the derivative to see.
$$\frac{d}{dx}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right] = \frac{1}{2}e^{2x} + \frac{1}{2}x(2e^{2x}) - \frac{1}{4}(2e^{2x}) = xe^{2x}.$$

ex. Find $\int x \cos x \, dx$. Here try

$u = x$, $dv = \cos x \, dx$. Then

$du = dx$, $v = \sin x$, and

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C.$$

Does this work? Check!

Notes ① Sometimes, IBP is needed repeatedly:

ex. $$\int x^2 \sin x \, dx \;\underset{\substack{du = 2x\,dx \\ v = -\cos x}}{\overset{\substack{u = x^2 \\ dv = \sin x\,dx}}{=\!=}}\; x^2(-\cos x) - \int (-\cos x)\, 2x \, dx$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx.$$

(like the previous problem, now do IBP again on last term)

(2) With practice, patterns emerge as to which substitution to make: For example, if one factor in the integrand is a polynomial, try making that u. Then the du on the right will be a polynomial of less degree.

<u>But just try stuff!</u>

(3) Sometimes one of the substitutions for u or v is simply the function 1.

ex. $\int \ln x \, dx$. (What is the anti derivative of $\ln x$?)

We solve this by IBP: Let $u = \ln x$, $dv = dx$ so $du = \frac{1}{x} dx$ and $v = x$. Then

$$\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = \underbrace{x}_{v} \underbrace{\ln x}_{u} - \int \underbrace{x}_{v} \underbrace{\left(\tfrac{1}{x}\right) dx}_{du} = x \ln x - \int dx$$
$$= x \ln x - x + C.$$

Does it work?

(4) <u>Last example</u>: Sometimes one can go in circles by IbP. But we can still solve algebraically:

ex. Find $\int e^x \sin x \, dx$.

Here, try IbP with $u = e^x$, $dv = \sin x \, dx$, so $du = e^x dx$ and $v = -\cos x$. We get

$$\int e^x \sin x \, dx = -e^x \cos x + \underbrace{\int e^x \cos x \, dx}$$

IbP with $u = e^x$, $dv = \cos x \, dx$
$du = e^x dx$, $v = \sin x$.

$$= -e^x \cos x + \left(e^x \sin x - \underbrace{\int e^x \sin x \, dx}\right)$$

bring to other side

back where we started? Not really! Solve for this unknown

$$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x, \text{ or}$$

$$\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C.$$

Check by differentiation whether this is the antiderivative.