

Class 19 : 3/24/14 Section 4.7 I

Last class we showed: For $f(x) = a^x$

$$\frac{d}{dx} [f(x)] = a^x f'(0), \text{ where } f'(0) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

and that $e \approx 2.71828$ was defined as the value for a so that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.

$$\Rightarrow \frac{d}{dx} [e^x] = e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^x (1) = e^x$$

Notes ① How many other functions can you think of where the derivative function equals the function itself

$$\textcircled{2} \frac{d}{dx} [e^{f(x)}] \stackrel{\text{chain rule}}{=} e^{f(x)} \cdot f'(x)$$

$$\text{ex: } \frac{d}{dx} [e^{\sin x}] = e^{\sin x} \cdot \cos x$$

$$\frac{d}{dx} [e^{2x^2+3x}] = e^{2x^2+3x} \cdot (4x+3)$$

Notes cont'd

$$\textcircled{3} \quad \frac{d}{dx}[a^x] = ? \quad \text{Rewrite } a^x = e^{\ln a^x}$$

(Remember, $\ln a^x$ only works when $a^x > 0$.

But a^x is always > 0 !!!)

$$a^x = e^{\ln a^x} = e^{x \ln a}, \text{ where } \ln a \text{ is a constant.}$$

$$\text{Let } f(x) = x \ln a. \text{ Then } f'(x) = \ln a.$$

$$\begin{aligned} \text{And } \frac{d}{dx}[a^x] &= \frac{d}{dx}[e^{x \ln a}] = \frac{d}{dx}[e^{f(x)}] \\ &= e^{f(x)} \cdot f'(x) = e^{x \ln a} \cdot \ln a \end{aligned}$$

$$\boxed{\frac{d}{dx}[a^x] = a^x \ln a}$$

$$\textcircled{4} \quad \text{Recall that } \frac{d}{dx}[a^x] = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}. \text{ Then}$$

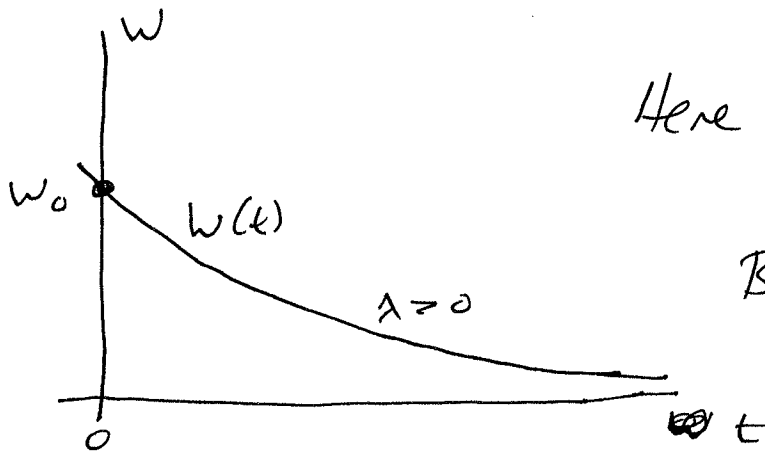
$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln a.$$

Hence for $a = e \approx 2.71828 \dots$,

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = \ln e = 1.$$

Here is an application

Let $w(t) = w_0 e^{-\lambda t}$, an exponential function representing radioactive decay of some substance (note the exponent is a linear func of t)



$$\text{Here } \frac{dw}{dt} = w_0 (-\lambda) e^{-\lambda t}$$

But $w_0 e^{-\lambda t} = w(t)$. Hence

$$\frac{dw}{dt} = (-\lambda) w(t)$$

- This means for this type of function, the rate of decay is proportional to the amount of material present.
- The derivative of an exponential function is always a constant times the original function.

- New question: Find a function whose derivative satisfies the equation

$$\frac{dy}{dt} = ry$$

Since this equation states: The derivative of some unknown function $y(t)$ is a constant times the original function.

The answer is: $y(t) = C_0 e^{rt}$. Does this work?

$$\begin{aligned} y'(t) &= \frac{dy}{dt} = \frac{d}{dt} [C_0 e^{rt}] = C_0 \frac{d}{dt} [e^{rt}] \\ &= C_0 e^{rt} (r) \\ &= r(C_0 e^{rt}) = ry. \quad \blacksquare \end{aligned}$$

Back to inverse functions

Recall, if a function $f(x)$ has an inverse, then there exists a function $f^{-1}(x)$, where

$$\text{domain of } f^{-1}(x) = \text{range of } f(x)$$

$$\text{range of } f^{-1}(x) = \text{domain of } f(x)$$

Also, if $f^{-1}(x)$ exists, then

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, \text{ and}$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x.$$

ex. Let $f(x) = x^2$ on the domain $[0, \infty)$.

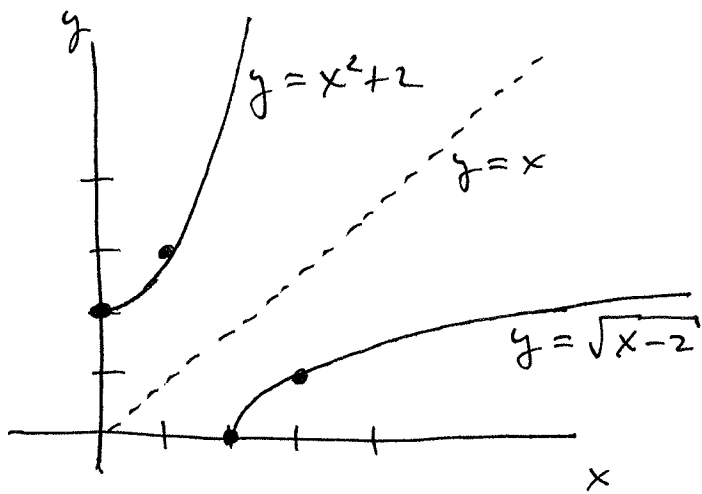
Its inverse is $f^{-1}(x) = \sqrt{x}$ on $[0, \infty)$.

Here $(f \circ f^{-1})(x) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ on $[0, \infty)$

and $(f^{-1} \circ f)(x) = f^{-1}(x^2) = \sqrt{x^2} = x$ on $[0, \infty)$.

Sometimes we can find $f^{-1}(x)$ by solving for x as a function of y and then switching the variables.

ex. For $y = x^2 + 2$ on $[0, \infty)$, solve for x as a function of y : $x = \pm \sqrt{y-2}$. This is not a function, but restricted to ~~the positive part~~ $x \in [0, \infty)$, only the positive part of the parabola is represented in the original function. Hence $x = \sqrt{y-2}$. To graph, switch x, y . $\boxed{y = \sqrt{x-2}}$.



• Notice the symmetry with respect to the $y=x$ line of $f(x)$ and $f^{-1}(x)$, when it exists.

Q: How are the derivatives of $f(x)$ and $f^{-1}(x)$ related?

Note: This is good to know in the case where the derivative of $f^{-1}(x)$ is needed but one cannot calculate by solving for x as a func of y .

Suppose $f(x)$ has an inverse. Find $\frac{d}{dx} [f^{-1}(x)]$.

Use the Chain Rule: Since $(f \circ f^{-1})(x) = x$, then

$$\frac{d}{dx} [(f \circ f^{-1})(x)] = \frac{d}{dx} [x] = 1.$$

$$\frac{d}{dx} [f(f^{-1}(x))] = f'(f^{-1}(x)) \cdot \frac{d}{dx} [f^{-1}(x)] = 1.$$

Hence

$$\boxed{\frac{d}{dx} [f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}}$$

What does this mean?

Notes ① This is a form of implicit differentiation.

One can use this formula without actually knowing $f^{-1}(x)$.

② For $x = f^{-1}(y)$, the inverse to $y = f(x)$, then in Leibniz notation $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

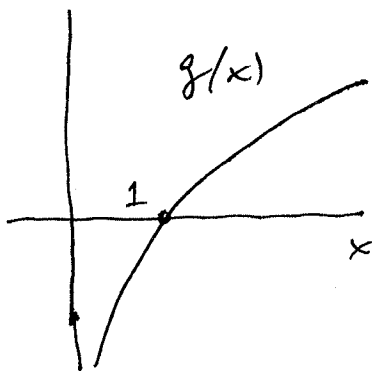
③ Suppose we did not know $\frac{d}{dx}[\sqrt{x}]$.

But we knew $y = f^{-1}(x) = \sqrt{x}$ is the inverse to $y = f(x) = x^2$. Then

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

since $f'(x) = 2x$.

④ So what is the derivative of $g(x) = \ln x$?



Here $g(x)$ is the inverse to $y = e^x$. Hence $g^{-1}(x) = e^x$

Notes cont'd.

④ could So use the previous equation:

$$\begin{aligned}
 x &= (g^{-1} \circ g)(x) = (g \circ g^{-1})(x) \\
 &= e^{\ln x} = \ln e^x
 \end{aligned}$$

Since $x = e^{\ln x}$, $\frac{d}{dx} [x] = \frac{d}{dx} [e^{\ln x}]$

$$\begin{aligned}
 1 &= e^{\ln x} \cdot \frac{d}{dx} [\ln x] \\
 &= x \cdot \frac{d}{dx} [\ln x].
 \end{aligned}$$

Solve for the derivative:

$$\boxed{\frac{1}{x} = \frac{d}{dx} [\ln x]}$$

⑤ Notice that now we can also do the same thing for other base logs ~~the~~

$\frac{d}{dx} [\log_a x]$ Here $x = a^{\log_a x}$, so

$$\begin{aligned}
 \frac{d}{dx} [x] &= \frac{d}{dx} [a^{\log_a x}] = \cancel{a^{\log_a x}} \cdot \frac{d}{dx} [\log_a x] \\
 &= [a^{\log_a x} (\ln a)] \frac{d}{dx} [\log_a x] \\
 &= x \ln a \frac{d}{dx} [\log_a x].
 \end{aligned}$$

Hence $\boxed{\frac{d}{dx} [\log_a x] = \frac{1}{x \ln a}}$

Notes cont'd

⑥ How about $\frac{d}{dx} [\ln A(x)]$, when $A(x)$ is a differentiable function?

$$\frac{d}{dx} [\ln A(x)] = \frac{1}{A(x)} \cdot A'(x) = \frac{A'(x)}{A(x)}$$

derivative of \ln
fnc evaluated
at $A(x)$

ex. $\frac{d}{dx} [\ln (x^2 + 3x)] = \frac{1}{x^2 + 3x} \cdot (2x + 3) = \frac{2x + 3}{x^2 + 3x}$

ex. $\frac{d}{dx} [\ln \cos x] = \frac{1}{\cos x} (-\sin x) = -\tan x$

ex. $\frac{d}{dx} [\log_2 \sqrt{x^2 + 2x}] = \frac{1}{\sqrt{x^2 + 2x} \ln 2} \cdot \frac{1}{2\sqrt{x^2 + 2x}} \cdot (2x + 2)$

$= \frac{x + 1}{(x^2 + 2x) \ln 2}$ Can you see that?