Class 16: 3/10/14 Section 4.4

Another Rule: So what about compositions of functions?

$$\frac{d}{dx}\left[(f \circ g)(x)\right] = \frac{d}{dx}\left[f(g(x))\right] = ?$$

Here, if $g(x)$ is diff at $x=c$, and $f(x)$ is diff at $x = g(c)$, then $f(g(x))$ is differentiable at $x=c$.

We can "see" the pattern simply by understanding the definition:

$$\left.\frac{d}{dx}\left[f(g(x))\right]\right|_{x=c} = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x-c}$$

(This is the other definition of a derivative at a pt $x=c$, from the middle of pg 134.)

$$\frac{d}{dx}\left[f(g(x))\right]\Big|_{x=c} = \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x-c}$$

$$= \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x-c} \underbrace{\left(\frac{g(x)-g(c)}{g(x)-g(c)}\right)}_{\substack{\text{clever form of 1}\\ \text{to better understand}\\ \text{the structure}}}$$

$$= \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{g(x)-g(c)} \cdot \frac{g(x)-g(c)}{x-c}$$

$$\overset{\substack{\text{prod}\\ \text{rule}}}{\underset{\substack{\text{for}\\ \text{limits}}}{=}} \underbrace{\left(\lim_{x\to c} \frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\right)}_{\substack{\text{think of this as}\\ \text{a change of variables}\\ \text{from } x \text{ to } g(x)}} \cdot \underbrace{\left(\lim_{x\to c} \frac{g(x)-g(c)}{x-c}\right)}_{g'(c)}$$

$$\underbrace{\text{where as } x\to c,\ g(x)\to g(c).}_{f'(g(c))}$$

Hence $\frac{d}{dx}\left[f(g(x))\right]\Big|_{x=c} = f'(g(c)) \cdot g'(c)$

This is called the <u>Chain Rule</u> for derivatives of <u>compositions</u>.

Notes ① In essence, the derivative of a composition of functions is the product of the derivatives, but with a "twist": the outside function derivative is evaluated at the inside function.

② As a function:

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Can you see the pattern? One works from the outside in.

ex Calculate $h'(x)$, where $h(x) = \sqrt{x^2+1}$

Strategy: Use the Chain Rule, with $f(x) = \sqrt{x}$, $g(x) = x^2+1$, so that

$$h(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}.$$

Solution: $h'(x) = f'(g(x)) \cdot g'(x)$.

Here $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(x^2+1) = \frac{1}{2\sqrt{x^2+1}}$

and $g'(x) = \frac{d}{dx}[x^2+1] = 2x$.

Hence $h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x$ $\blacksquare$

(No need to simplify here).

~~ex.~~

Note: Notice a pattern:

$$\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{1}{2\sqrt{f(x)}} \cdot f'(x).$$

ex. Calculate $\frac{d}{dx}\left[\frac{1}{3x^2+6}\right]$.

Solution: By Quotient Rule, we know

$$\frac{d}{dx}\left[\frac{1}{3x^2+6}\right] = \frac{\frac{d}{dx}[1](3x^2+6) - 1\,\frac{d}{dx}[3x^2+6]}{(3x^2+6)^2} = \frac{0 - 6x}{(3x^2+6)^2}$$

$$= \frac{-6x}{(3x^2+6)^2}$$

Solution cont'd.:

By the Chain Rule, $\frac{1}{3x^2+6} = f(g(x))$, where

$f(x) = \frac{1}{x}$, and $g(x) = 3x^2+6$.

Thus $\frac{d}{dx}\left[\frac{1}{3x^2+6}\right] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$= \frac{-1}{(3x^2+6)^2} \cdot 6x = \frac{-6x}{(3x^2+6)^2}$ ∎

Notice the pattern here: For any function $f(x)$ which is differentiable,

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{[f(x)]^2}$$

There are other patterns like this on page 162. It is not necessary to know them, but they are useful.

Example 10 pg 167 is a good example of a nested function.

ex Calculate $\frac{d}{dx}\left[(\sqrt{x^2+1}+1)^2\right]$

Solution: First, we calculate the composition. Let $f(x)=x^2$, and $g(x)=\sqrt{x^2+1}+1$.

Then the Chain Rule stipulates that

$$\frac{d}{dx}\left[(\sqrt{x^2+1}+1)^2\right] = 2\left(\sqrt{x^2+1}+1\right)\cdot \frac{d}{dx}\left[\sqrt{x^2+1}+1\right]$$

$$= 2\left(\sqrt{x^2+1}+1\right)\cdot\Big(\underbrace{\frac{d}{dx}\left[\sqrt{x^2+1}\right] + \cancelto{0}{\frac{d}{dx}[1]}}_{\text{Sum Rule}}\Big)$$

We now need to apply the Chain Rule again to the part we still need to differentiate.

Let $f(x)=\sqrt{x}$, $g(x)=x^2+1$, as in the last example above. We get

$$\frac{d}{dx}\left[(\sqrt{x^2+1}+1)^2\right] = 2\left(\sqrt{x^2+1}+1\right)\cdot\left(\frac{1}{2\sqrt{x^2+1}}\right)\cdot\frac{d}{dx}(x^2+1)$$

$$= 2\left(\sqrt{x^2+1}+1\right)\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x$$

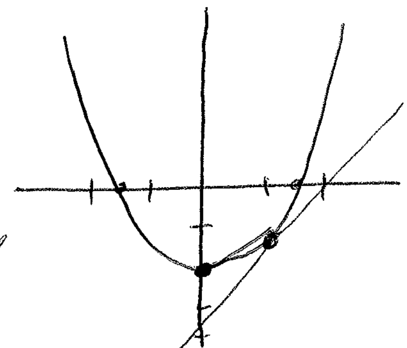
Simplify if you like, but it is not necessary.

In general, if $i(x) = f(g(h(x)))$, then

$$i'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Consider the equation relating x and y

$$y = x^2 - 2$$



Instead of thinking of y as a fnc of x, think instead of each side of equation as an expression involving x, where y is some unknown fnc of x:

Since LHS = RHS, as functions of x, their derivatives are also equal, and

Thinking of y as some unknown fnc of x, use chain rule $\frac{d}{dx}[y] = \frac{dy}{dx} \cdot \frac{d}{dx}[x]$

$$\frac{dy}{dx} = \frac{d}{dx}[x^2 - 2] = 2x, \Rightarrow y' = 2x.$$

And at $x = 1$, we find $y' = 2$, so $y + 1 = 2(x - 2)$

$$y = 2x - 5$$

We can now generalize:

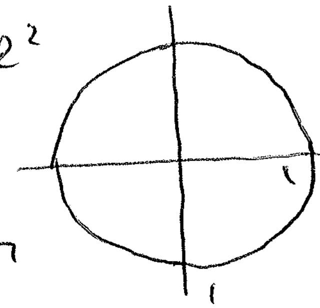
① Rewrite as $0 = x^2 - 2 - y$

Then $\frac{d}{dx}[0] = \frac{d}{dx}[x^2 - 2 - y] = 2x - 0 - \frac{dy}{dx}$

so again $\frac{dy}{dx} = 2x$.

② But now, we can also do more complicated expressions:

ex. The set of all solutions to $x^2+y^2=1$ is the unit circle in $\mathbb{R}^2$



Here, y cannot be made a fnc of x. $y = \pm\sqrt{1-x^2}$

(the northern semicircle is $y=\sqrt{1-x^2}$)

"Implicitly", it is a function of x in the sense that its values will change as those of x change.

We can still analyze how y changes as we change x by taking derivatives of the equation (deriving implicitly):

$$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[1]$$

$$\underbrace{2x + 2y\frac{dy}{dx}}_{\text{chain Rule}} = 0$$

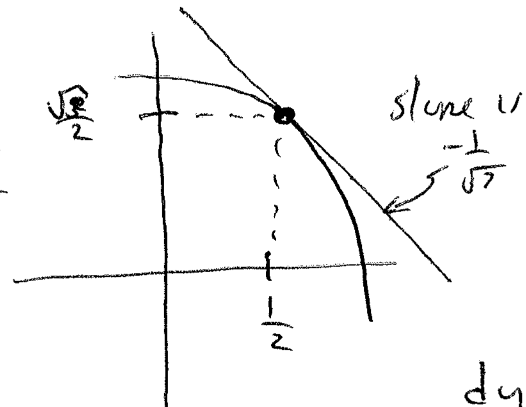
$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}.$$

<u>Notes</u> ① This works everywhere as long as expression makes sense.

② For a choice of pt (x,y), interpretation is still that $\frac{dy}{dx}$ is the slope of the line tangent to curve at (x,y).

$\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=1$

$\frac{1}{4}+\frac{3}{4}=1$



$$\left.\frac{dy}{dx}\right|_{\substack{x=\frac{1}{2}\\ y=\sqrt{3}/2}} = \frac{-(1/2)}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

③ This particular expression for $\frac{dy}{dx}$ works as long as $y\neq 0$. But what happens here to the tangent line?

④ Whenever an expression involving x, y cannot be written as y is a func of x, then $\frac{dy}{dx}$ can always be found but <u>will</u> involve both x and y. Why?

(only way to denote a pt on the curve without ambiguity).