

Class 15: 3/5/14 Section 4.3 I

Patterns found in the formal definition of a derivative lead to Rules:

Cont'd.

We have already seen:

Constant Multiple Rule: $\frac{d}{dx}[c f(x)] = c f'(x)$
for $c \in \mathbb{R}$ any constant

Sum/Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
whenever f, g differentiable

Here is another:

Thm Let $n \in \mathbb{N}$. Then $\frac{d}{dx}[x^n] = n x^{n-1}$

Called the Power Rule, the derivative of a power function is another power function of one less degree.

pt. Calculate directly from the formal definition:

$$\frac{d}{dx}[x^n] = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

To play with this, understand how to expand powers of binomials: for $n \in \mathbb{N}$,

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \dots + nx^{n-1}y + y^n.$$

where the powers of x are descending, the powers of y are ascending, and the coefficients follow patterns of combinations: $\binom{n}{m}x^{n-m}y^m$, and also follow Pascal's Triangle.

1	---	---	---	---	$(x+y)^0$
1	1	---	---	---	$(x+y)^1 = 1x + 1y$
1	2	1	---	---	$(x+y)^2 = 1x^2 + 2xy + 1y^2$
1	3	3	1	---	⋮
1	4	6	4	1	
1	5	10	10	5	1

proof cont'd

Back to the derivative calculation: Let $n \in \mathbb{N}$,

$$\begin{aligned} \frac{d}{dx}[x^n] &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^n + n x^{n-1} h + \overbrace{\frac{n(n-1)}{2} x^{n-2} h^2 + \dots + n x h^{n-1} + h^n - x^n}^{\text{all have } h^2 \text{ or higher power.}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{n x^{n-1} + \overbrace{\frac{n(n-1)}{2} x^{n-2} h + \dots + n x h^{n-2} + h^{n-1}}^{\text{all still have } h}}{1} \end{aligned}$$

every term with a h in it now goes to 0.
what is left is only

$= n x^{n-1}$

Notes ① Actually, the Power Rule works for any real number power:

$$\frac{d}{dx}[x^r] = r x^{r-1}, \quad r \in \mathbb{R}.$$

However proving this is harder.

(keep in mind, $r=0$ is a strange one. Try it: it still works).

Notes cont'd.

② Now all polynomials are easily differentiable:

$$\text{ex: } \frac{d}{dx} [3x^6 - 4x^2 + 6x - 2] =$$

$$\text{Sum Rule} \rightarrow \frac{d}{dx} [3x^6] + \frac{d}{dx} [-4x^2] + \frac{d}{dx} [6x] + \frac{d}{dx} [2]$$

$$\text{Constant Mult Rule} \rightarrow = 3 \frac{d}{dx} [x^6] + -4 \frac{d}{dx} [x^2] + 6 \frac{d}{dx} [x] + 0$$

$$\text{Power Rule} \rightarrow = 3(6x^5) - 4(2x^1) + 6(1x^0) + 0$$

$$= 18x^5 - 8x + 6$$

Look for the ability to do many of these steps without writing down each step.

③ Works with negative powers:

$$\frac{d}{dx} \left[\frac{1}{x} \right] = \frac{d}{dx} [x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

And with fractional powers:

$$\frac{d}{dx} [\sqrt{x}] = \frac{d}{dx} [x^{1/2}] = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Notes cont'd.

④ Exponential functions are not power functions:

$$\frac{d}{dx}[e^x] \neq x e^{x-1} \quad !!!$$

Q: Here is a funny problem:

Q: Is it possible to find 2 functions

$f(x), g(x)$ that satisfy the equation

$$x = f(x)g(x)$$

where $f(x), g(x)$ are differentiable and

$$f(0) = g(0) = 0?$$

A: The answer is no, it is not possible,

but the reason involves the derivatives of the product $f(x)g(x)$.

Yet another Rule:

Thm Let $f(x), g(x)$ be differentiable at x .

$$\text{Then } \frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Note: The derivative of a product of functions is NOT the product of the derivatives. This is important.

pt. Calculate:

$$\begin{aligned} \frac{d}{dx} [f(x)g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \quad \text{clever form of zero.} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h} \\ &= \underbrace{\left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right)}_{f \text{ is diff.}} \underbrace{\lim_{h \rightarrow 0} g(x+h)}_{g \text{ is cont.}} + \lim_{h \rightarrow 0} f(x) \underbrace{\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}}_{g \text{ is diff.}} \\ &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \end{aligned}$$

ex. Calculate $h'(x)$, for $h(x) = (2x^2)(x^3 - 3x + 1)$

Strategy: We use the Product Rule directly, and then combine first and use the power, sum and constant multiple Rules to check.

Solution:

$$\frac{d}{dx} [2x^2(x^3 - 3x + 1)] \stackrel{\text{Prod Rule}}{=} \frac{d}{dx} [2x^2] (x^3 - 3x + 1) + (2x^2) \frac{d}{dx} [x^3 - 3x + 1]$$

//
combine first
//

$$= (4x)(x^3 - 3x + 1) + (2x^2)(3x^2 - 3)$$

$$\frac{d}{dx} [2x^5 - 6x^3 + 2x^2]$$

//

$$10x^4 - 18x^2 + 4x$$

$$= 4x^4 - 12x^2 + 4x + 6x^4 - 6x^2$$

$$= 10x^4 - 18x^2 + 4x$$

equal

One more Rule Quotient Rule

Thm For f, g differentiable functions;

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

pt.

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x)}{g(x+h)g(x)} - \frac{f(x)g(x+h)}{g(x)g(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))g(x)}{h \cdot g(x+h)g(x)} - \frac{f(x)(g(x+h) - g(x))}{h \cdot g(x+h)g(x)} \\ &\stackrel{?}{=} \left(\lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) \left(\lim_{h \rightarrow 0} g(x) \right) \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)g(x)}. \quad \square \end{aligned}$$

- $\left(\lim_{h \rightarrow 0} f(x) \right) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right)$

Notes

① Calculate the derivative of $\frac{1}{x^2}$ three ways:

② Power Rule:

$$\frac{d}{dx} \left[\frac{1}{x^2} \right] = \frac{d}{dx} [x^{-2}] = -2x^{-3} = -\frac{2}{x^3}.$$

③ Product Rule

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{x^2} \right] &= \frac{d}{dx} \left[\frac{1}{x} \cdot \frac{1}{x} \right] = \frac{d}{dx} \left[\frac{1}{x} \right] \cdot \frac{1}{x} + \frac{1}{x} \cdot \frac{d}{dx} \left[\frac{1}{x} \right] \\ &= -\frac{1}{x^2} \cdot \frac{1}{x} + \frac{1}{x} \left(-\frac{1}{x^2} \right) = -\frac{2}{x^3}. \end{aligned}$$

④ Quotient Rule

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{x^2} \right] &= \frac{\frac{d}{dx} [1] \cdot x^2 - 1 \cdot \frac{d}{dx} [x^2]}{(x^2)^2} \\ &= \frac{0 \cdot x^2 - 2x}{x^4} = \frac{-2x}{x^4} = -\frac{2}{x^3}. \end{aligned}$$

② We can now use quotient Rule to ~~calculate~~ prove the power rule for all negative integers.