Class 13: 2/26/14 Section 4.1

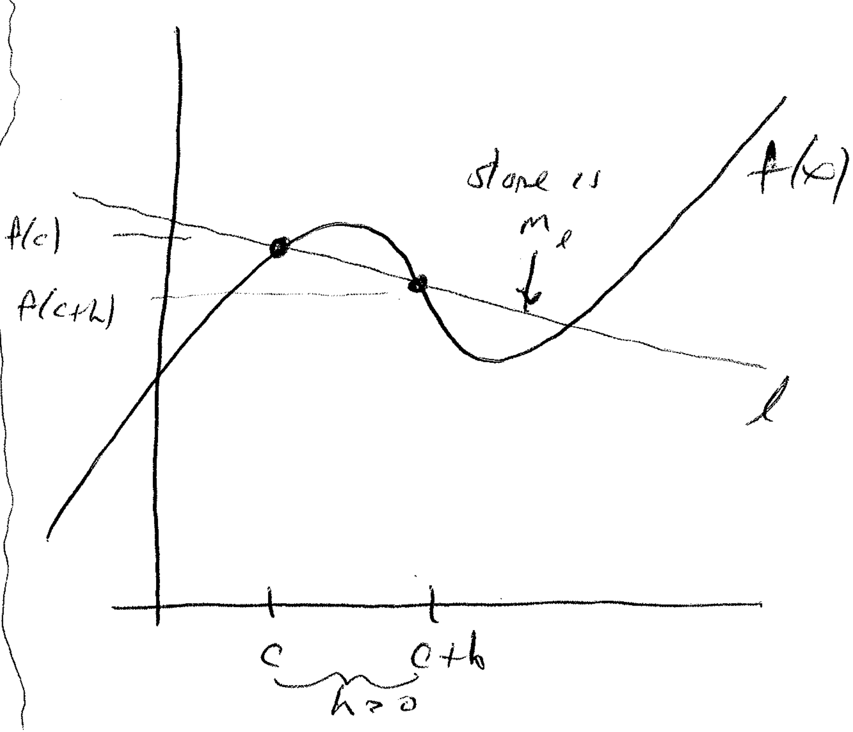
Last class, we defined a way to construct a tangent line to the graph of $f(x)$ at the point $x=c$ in its domain:

Choose a small number $h>0$ (when we want our definitions and/or constructions to utilize a small number but we do not want to lock ourselves in to any specific choice, we use a parameter like h or ε.)

Then for l the line through $(c, f(c))$ and $(c+h, f(c+h))$, its slope is

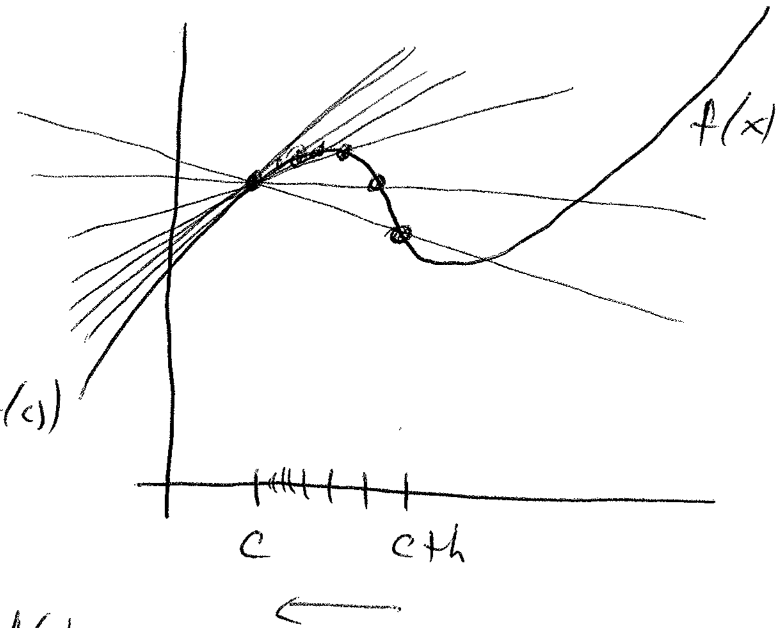
$$m_l = \frac{f(c+h)-f(c)}{(c+h)-c}$$

$$= \frac{f(c+h)-f(c)}{h}$$



If we then "push" h to 0, we get

for each choice of smaller h's going to 0, the line joining $(c, f(c))$ and $(c+h, f(c+h))$ has slope $\frac{f(c+h)-f(c)}{h}$,



so that in the limit, as h goes to 0,

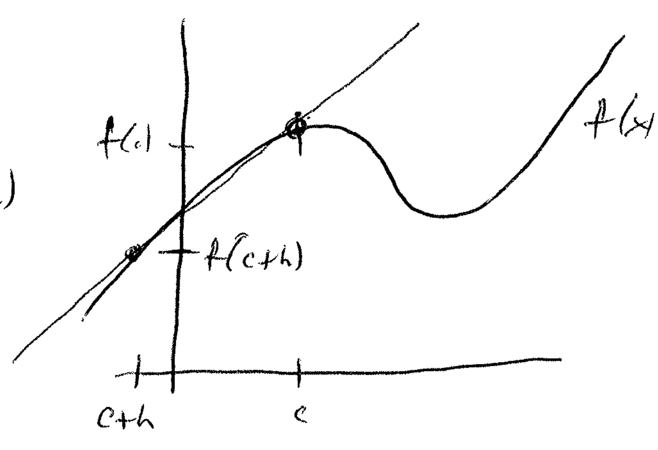
$$\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$$

If this right side limit exists, then the slopes "go" to the slope of a unique line.

Do the same for the other side (small $h<0$).

to get

$$\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}$$



If $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$

and both exist, then the full limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

<u>Def</u> The <u>derivative</u> of f with respect to x, at $x = c$, when it exists, is denoted

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

<u>Notes</u> ① If $f'(c)$ exists, we say f is <u>differentiable</u> at $x = c$.

② The quantity $\frac{f(c+h) - f(c)}{h}$ for any choice of h is called a <u>difference quotient</u>. Sometimes it is denoted $\left.\frac{\Delta f}{\Delta x}\right|_{x=c}$

$$\frac{\Delta f}{\Delta x} = \frac{\text{change in } f}{\text{change in } x} = \frac{f(c+h) - f(c)}{(c+h) - c}$$

(2) cont'd

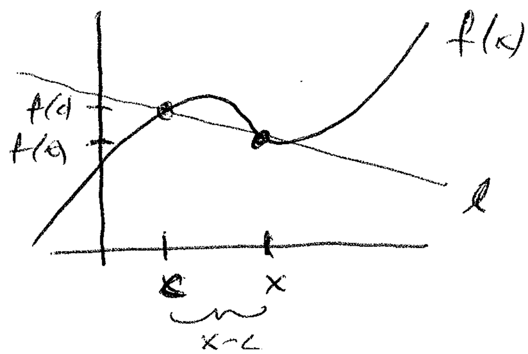
The difference quotient $\left.\frac{\Delta f}{\Delta x}\right|_{x=c}$ is called the average rate of change of f on Δx.

And $f'(c)$ is called the instantaneous rate of change of f at $x=c$.

(3) Another way to define the limit is to not bother with the h:

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

slope of l



(4) More Notation

(a) For $y=f(x)$, we sometimes write $y(x)$. Then derivative is $y'(x)$.

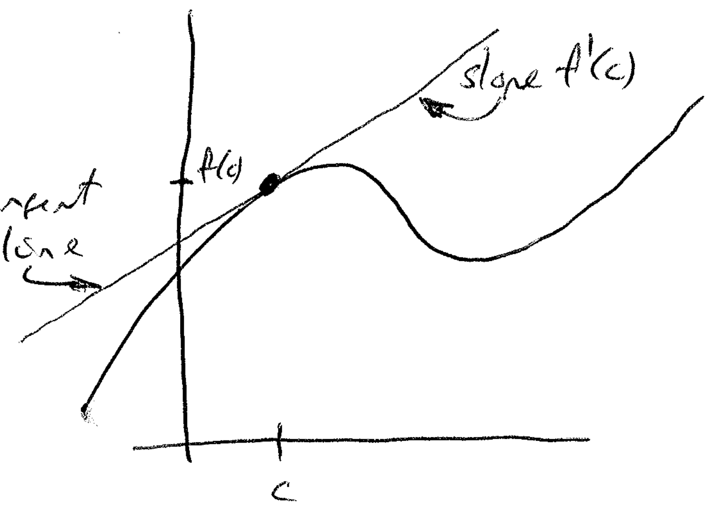
(b) Leibniz notation:

- $f'(x) = \frac{df}{dx} = \frac{d}{dx}(f(x))$, where $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$
- For $y=f(x)$, $y'(x) = \frac{dy}{dx}$
- $f'(c) = \left.\frac{df}{dx}\right|_{x=c}$

(5) Given $y = f(x)$, if $f'(c)$ exists at $x = c$, then the ~~slope of the~~ line passing through $(c, f(c))$ with slope $f'(c)$ is the tangent line to the graph of $f(x)$ at $x = c$:

Recall, the line with slope m passing through (x_0, y_0) has eqn:

$y - y_0 = m(x - x_0)$.



The eqn of the tangent line to graph of $f(x)$ at $x = c$ is then:

$$y - f(c) = f'(c)(x - c)$$

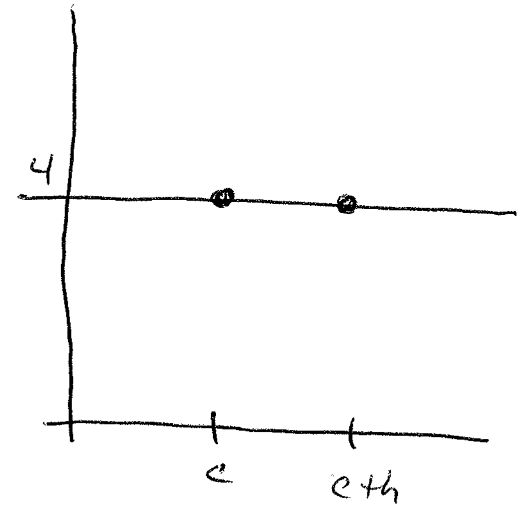
or

$$y = f'(c)(x - c) - f(c)$$

Q: Must f be continuous at $x = c$ for $f'(c)$ to exist? why or why not?

Here, the definition of derivative can be used to calculate:

<u>ex.</u> let $f(x) = 4$.



Here, for any choice of $c \in \mathbb{R}$, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$
$$= \lim_{h \to 0} \frac{4-4}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Hence $f(x) = 4$ is differentiable at every point in its domain, and its derivative is always 0. (the tangent line to the graph is the graph itself.)

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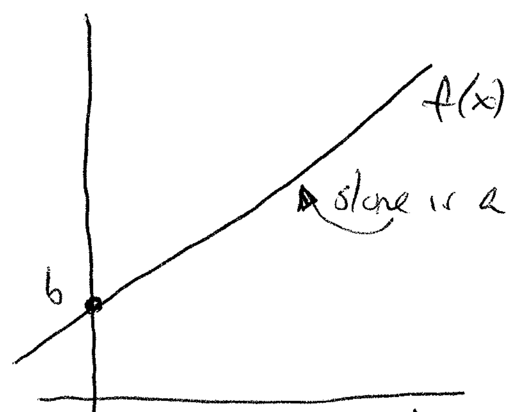
ex. Find $f'(c)$ for any choice of c where $f(x) = ax + b$.



Here for any choice of $c \in \mathbb{R}$,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$
$$= \lim_{h \to 0} \frac{a(c+h) + b - (a(c) + b)}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{ac} + ah + \cancel{b} - \cancel{ac} - \cancel{b}}{h}$$
$$= \lim_{h \to 0} \frac{ah}{h} = \lim_{h \to 0} a = a.$$

Again the slope of any tangent line to $f(x)$ at $x = c$ is a.