Class 12: 2/24/14 Section 4.1 I

Last class, we showed that the polynomial $f(x) = 8x^5 - 4x^4 + 3x - 5$ has a root (place where $f(x) = 0$) inside the interval $[0,1]$.

We did this using the fact that f is continuous on $[0,1]$, and $f(0) < 0 < f(1)$

Hence by the Intermediate Value Theorem.

Q: The IVT says nothing about how to find an intermediate value, only its existence. So how can one find one?

A: (1) If the sought-for intermediate value is $f(a) < L < f(b)$ for $c \in (a,b)$, simply solve $f(c) = L$ for c.

However this is hard often and sometimes not possible.

Question Answer cont'd

(2) Use an approximation method like the <u>Bisection Method</u>. Really it is just a repeated use of the IVT on ever smaller intervals:

<u>ex</u> Approximate the root found in the previous example:

- Here we know $c \in [0,1]$, where $f(c) = 0$. and $f(0) = -5 < 0 < f(1) = 2$.
- Evaluate f at the midpoint of $[0,1]$:

$$f\left(\tfrac{1}{2}\right) = -\frac{57}{16} < 0.$$

- Then ~~~~ c must exist in the interval $[\tfrac{1}{2}, 1]$, since $f(\tfrac{1}{2}) = -\frac{57}{16} < 0 < f(1) = 2$. by again using the IVP. on this smaller interval.

ex (cont'd)

- Evaluate f on the midpoint of $[\frac{1}{2}, 1]$: $f(\frac{3}{4}) \cong -2.44$.
- Then c must be in the interval $[\frac{3}{4}, 1]$ since

  $f(\frac{3}{4}) \cong -2.44 < 0 < f(1) = 2$, by

  using IVT again on $[\frac{3}{4}, 1]$.
- Continue on ..., until desired accuracy is achieved.

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example It is obvious that $x=0$ solves $x = 2\sin x$. Is there another solution for $x > 0$?

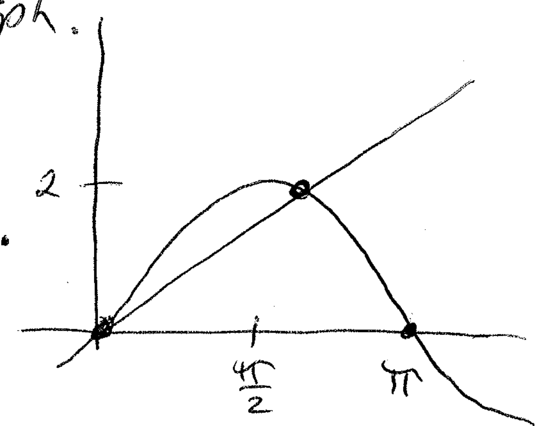
Strategy: Let $h(x) = (2\sin x) - x$ be a new function. Then $h(x) = 0$ precisely at places that satisfy $2\sin x = x$. (why?) Find 2 positive values of x where $h(x)$ has different signs. Then IVT implies a

Strategy cont'd.

new positive solution to $h(x)=0$.

Solution Graph the functions $2\sin x$ and x on same graph.

The graphs will cross precisely where $h(x)=0$. (why?)



By the graphs, it looks like there is a crossing on the interval $[\frac{\pi}{2}, \pi]$. Try these endpts:

$$h\left(\tfrac{\pi}{2}\right) = 2\sin\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = 2 - \left(\tfrac{3.14\cdots}{2}\right) > 0$$

$$h(\pi) = 2\sin(\pi) - \pi < 0$$

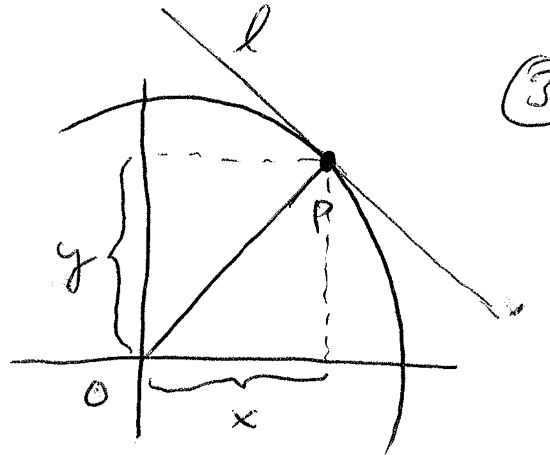
Hence $h(x)$ is cont on $[\frac{\pi}{2}, \pi]$, and $h(\pi) < 0 < h(\frac{\pi}{2})$. Hence by the IVP, there is a pt $c \in (\frac{\pi}{2}, \pi)$, where $h(c)=0$. But then

$$2\sin c = c.$$

Def. The derivative of a function $f(x)$ at a pt $x=c$ in its domain measures how the function values (output values) are changing as we vary the input values through the pt c.

Notes ① It can tell you if the function is increasing or decreasing and by how much.

② Geometrically (visually) it helps to define the line tangent to the graph of $f(x)$ at $x=c$. This is a generalization to the notion of a tangent line to the circle at a pt on the circle.



③ For a circle, the tangent line is really the unique line that touches the circle at only 1 pt (all lines through p touch the circle twice but 1). For $p=(x_0,y_0)$ on circle, the ray $\overline{OP}$ has slope $m_{\overline{OP}} = \frac{y_0}{x_0}$.
The tangent line l is perpendicular (why?) to $\overline{OP}$, so it has slope $m_l = \frac{-1}{m_{\overline{OP}}} = -\frac{x_0}{y_0}$.

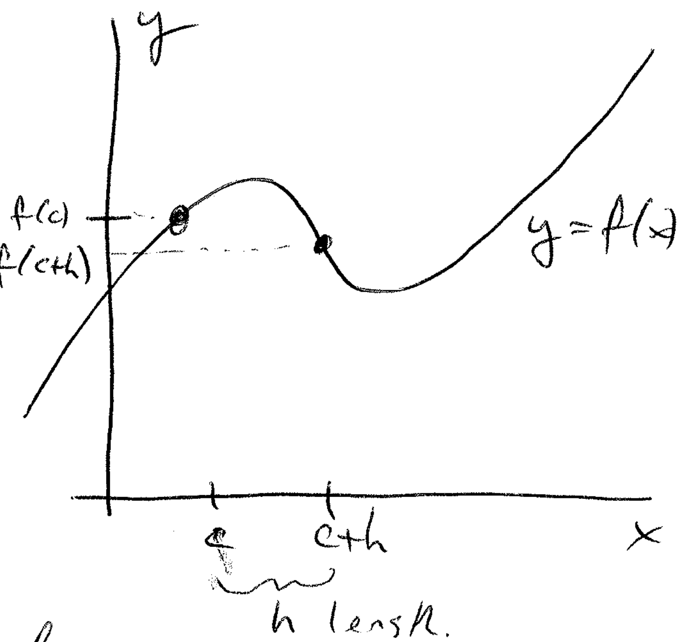
④ To do something like this for a general curve in the plane (say the graph of a function $f(x)$), we cannot use the "touch only once" criterion:

- Might touch many times, or at every pt., or just once.
- There is ~~not~~ no geometric way to define the tangent line like for the circle.

Idea

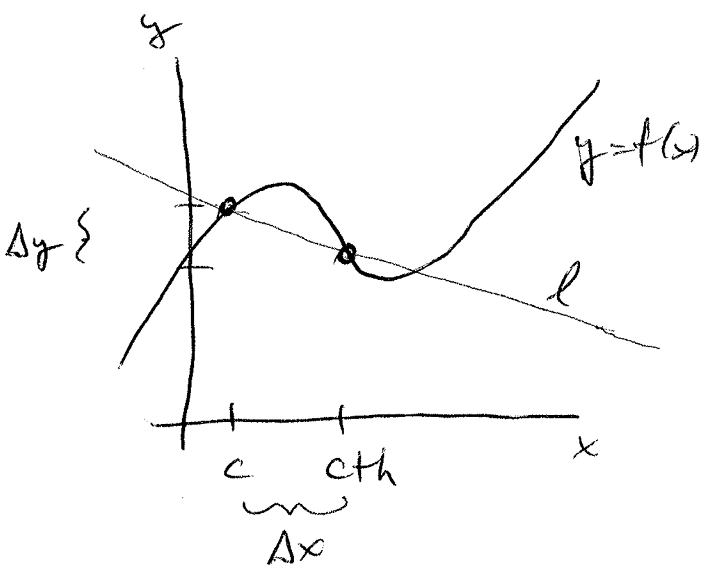
Given $f(x)$ defined and continuous on some open set containing a point c in its domain,

let $h>0$ be a small number, so that $c+h$ is still in the open set.



The average rate of change of f on the interval $[c, c+h]$ is

$$\frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{\Delta y}{\Delta x} = \frac{f(c+h) - f(c)}{(c+h) - c}$$



Note: This average rate of change is precisely the slope of the line l passing through $(c, f(c))$ and $(c+h, f(c+h))$

If we slowly "push" h to 0, making the interval $[c, c+h]$ smaller and smaller, and pushing Δx to 0, since $f(x)$ is continuous at $x=c$, the function values (the y-values) also come together (their difference goes to 0):

$$\Delta y = f(c+h) - f(c) \longrightarrow 0 \ldots$$

Q: So then where does $\dfrac{\Delta y}{\Delta x} = \dfrac{f(c+h) - f(c)}{(\cancel{c}+h) - \cancel{c}}$ go if both Δx and Δy are going to 0?

A: Maybe they go to the limit if it exists:

$$\lim_{h \to 0^+} \frac{\Delta y}{\Delta x} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{(\cancel{c}+h) - c}$$

$$= \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

<u>Note:</u> The quotient $\dfrac{f(c+h) - f(c)}{h}$ is <u>Never</u> defined at $h=0$. Hence we ALWAYS must be clever to evaluate this limit.

We can't just "plug in h" until we do some manipulation.

Visually, if this limit exists, then the slopes of the lines through $(c, f(c))$ and $(c+h, f(c+h))$ will tend to something even as the pts "come together".

Do the same thing for a small $h<0$ (on the other side of c.

What these slopes tend to is an extremely important property of the function $f(x)$ at $x=c$, and an extremely important concept in calculus.