Class 11: 2/21/14 Section 3.5 I

There are other techniques for calculating limits of functions:

(I) Inspection of continuity: If you can discern that the function $f(x)$ is continuous at $x=c$, then by continuity

$$\lim_{x \to c} f(x) = f(c).$$

You can call this "plugging in the value".

ex: Calculate $\lim_{x \to 0} \frac{\sin x}{1+\cos x}$, if possible.

Strategy: Check for continuity. If cont. at $x=0$, then plug in value and evaluate.

Solution: Both $\sin x$ and $1+\cos x$ are continuous functions on all of $\mathbb{R}$. And since $1+\cos(0) \neq 0$, near $x=0$, the ratio $\frac{\sin x}{1+\cos x}$ is also cont as the quotient of 2 continuous functions.

Solution cont'd.

Thus $\frac{\sin x}{1+\cos x}$ is cont at $x=0$ and

$$\lim_{x \to 0} \frac{\sin x}{1+\cos x} = \frac{\sin(0)}{1+\cos(0)} = \frac{0}{1+1} = 0. \blacksquare$$

Note: The quotient $\frac{\sin x}{1+\cos x}$ is not continuous on all of $\mathbb{R}$, though. In fact, the largest interval containing $x=0$ where $\frac{\sin x}{1+\cos x}$ is continuous is $(-\pi, \pi)$. (Why?)

(II) More "clever multiplication by 1".

(Manipulation ~~by~~ of a function in a way that doesn't change the function, but makes it easier to work with.)

ex. Show $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$.

example (cont'd).

Strategy: Use the same "clever multiplication by 1" trick we used with rational functions to allow the Limit Rules to apply.

Solution: The function $f(x) = \frac{1-\cos x}{x}$ is not continuous at $x=0$. Hence we cannot "plug in" $x=0$. But it is cont "near 0". Hence the limit may exist.

We try a "clever mult by 1" via mult. of the numerator by its conjugate:
For an expression involving 2 terms like $a+b$, the conjugate expression is $a-b$. Multiplying an expression by its conjugate always yields the difference of its squares:

$$(a+b)(a-b) \overset{\text{FOIL}}{=\!=} a^2+ab-ab-b^2 = a^2-b^2$$

Solution cont'd.

Why mult by a conjugate expression?
It is a form of manipulation that has use: Like when rationalizing a denominator

ex. $\dfrac{2}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

ex. $\dfrac{4}{\sqrt{2}-\sqrt{3}} = \dfrac{4}{\sqrt{2}-\sqrt{3}}\left(\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right) = \dfrac{4(\sqrt{2}+\sqrt{3})}{2+3} = \dfrac{4}{5}(\sqrt{2}+\sqrt{3}).$

For us:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x}\left(\frac{1+\cos x}{1+\cos x}\right)$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

This only looks more complicated, yet

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right).$$

Both of the limits of the 2 fracs in parentheses exist by previous examples. Hence by the Product Rule for limits

Solution cont'd

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right) \underset{\text{limits}}{\overset{\text{Prod Rule}}{=\!=\!=}} \underbrace{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}_{1}\underbrace{\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right)}_{0}$$

$$= 1\cdot 0 = 0.$$

Hence $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$.

(III) Change of variable

ex. Calculate $\lim_{x\to 0} 3x$. Here $3x$ is a polynomial and continuous at $x=0$. Hence $\lim_{x\to 0} 3x = 0$.

Suppose a function $h(x)$ can be written as a composition of 2 other functions $h(x) = f(g(x))$. Then it is often possible to change the variable in $h(x)$ to make the function easier to study:

Let $z = g(x)$, the inside function. Then
$h(x) = f(g(x)) = f(z)$.

III cont'd.

16 the new variable (as a function) $z = g(x)$ is continuous at $x = c$,

$\lim_{x \to c} z = \lim_{x \to c} g(x) = g(c) = L$, then

$$\lim_{x \to c} h(x) = \lim_{x \to c} f(g(x)) = \lim_{z \to L} f(z)$$

we have changed the variable in the limit. Many times the limit on the right is easier to calculate.

ex. Show $\lim_{x \to 0} \frac{\sin 3x}{5x}$ exists and calculate.

Strategy: This limit looks like $\lim_{x \to 0} \frac{\sin x}{x}$ which does exist. Use a change of variable to make it look like $\frac{\sin x}{x}$ and then calc.

Solution: The numerator in $h(x) = \frac{\sin 3x}{5x}$ looks like a composition $f \circ g$ where $f(x) = \sin x$ and $g(x) = 3x$.

Let $z = 3x$. Since $\lim_{x \to 0} z = \lim_{x \to 0} 3x = 0$, we change the variable:

$$\lim_{x \to 0} \frac{\sin 3x}{5x} = \lim_{x \to 0} \frac{\sin 3x}{\frac{5}{3}(3x)} = \lim_{z \to 0} \frac{\sin z}{\frac{5}{3} z}$$

$$\overset{\text{Const multiple Rule for limits.}}{=} \frac{3}{5} \lim_{z \to 0} \frac{\sin z}{z} = \frac{3}{5}(1) = \frac{3}{5}.$$

Hence $\lim_{x \to 0} \frac{\sin 3x}{5x}$ exists and equals $\frac{3}{5}$.

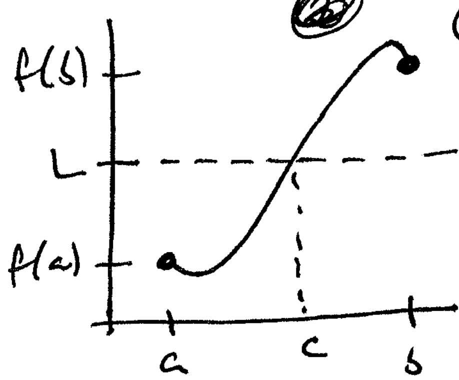
Section 3.5 Properties of Continuous Fncs.

One of the more profound and obvious theorems of calculus relies solely on the continuity of a function on a closed interval:

Intermediate Value Theorem (IVT)

Let $f(x)$ be continuous on $[a,b]$. If $L \in \mathbb{R}$ where either $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there is at least one pt $x \in (a,b)$, where $f(c) = L$.

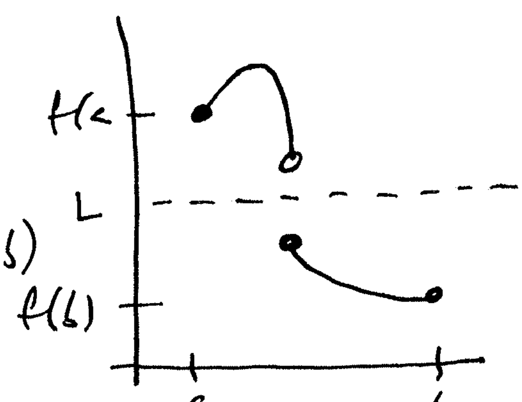
<u>Notes</u> ① In other words, a continuous function achieves <u>ALL</u> of its intermediate values on a closed interval.



② This is an example of an existence theorem. It does not and cannot help to actually find a value of c, where $f(c) = L$. It only establishes its existence.

③ Without continuity, the theorem fails.

Here there is no $x = c \in (a,b)$ where $f(c) = L$.



example Show $f(x) = 8x^5 - 4x^4 + 3x - 5$ has a root (an input value c, where $f(c) = 0$) on the interval $[0,1]$.

Strategy Use the Intermediate Value Thm.

Solution: $f(x)$ is a polynomial and hence is continuous everywhere. So $f(x)$ is continuous on $[0,1]$. At the end pts,

$$f(0) = 8(0)^5 - 4(0)^4 + 3(0) - 5 = -5 < 0$$

$$f(1) = 8(1)^5 - 4(1)^4 + 3(1) - 5 = 2 > 0$$

Since $f(x)$ is cont on $[0,1]$ and

$$-5 = f(0) < 0 < f(1) = 2$$

By the IVT, we know there must exist a pt $c \in (0,1)$ where $f(c) = 0$ (0 was our choice of L). Here c is a root of $f(x)$.