Class 7: 2/10/14 Section 3.1

The idea of a limit of a function at a pt $x=c$ involves a very precise definition.

Calculus is the study of the properties of a function of one continuous variable. One of the fundamental properties is how a function varies as we vary the input variable.

The primary tool for studying how a function varies "at a pt" is the limit:

Let $f(x)$ be defined on an open interval containing a pt c, except possibly at c itself:

Def A function $f(x)$ as above is said to have a limit L at $x=c$ if for any choice of $\varepsilon>0$, there is a $\delta>0$ so that

$$|f(x)-L|<\varepsilon \text{ whenever } 0<|x-c|<\delta.$$

Notes ① You will not be responsible for actually using this to "calculate" limits. However, you should understand it.

② if the limit does exist at a pt $x=c$, (then L is a real number), we say

$$\lim_{n\to\infty} f(x) = L$$

Notes cont'd.

(3) Intuitive idea: Take a sequence of values for x that approach $x=c$.

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ex. if $c=2$, then choose $1, 1.9, 1.99, 1.999, 1.9999, \ldots$ as your infinite list of numbers (sequence)

Then see if the sequence of function values

$$f(1), f(1.9), f(1.99), f(1.999), \ldots$$

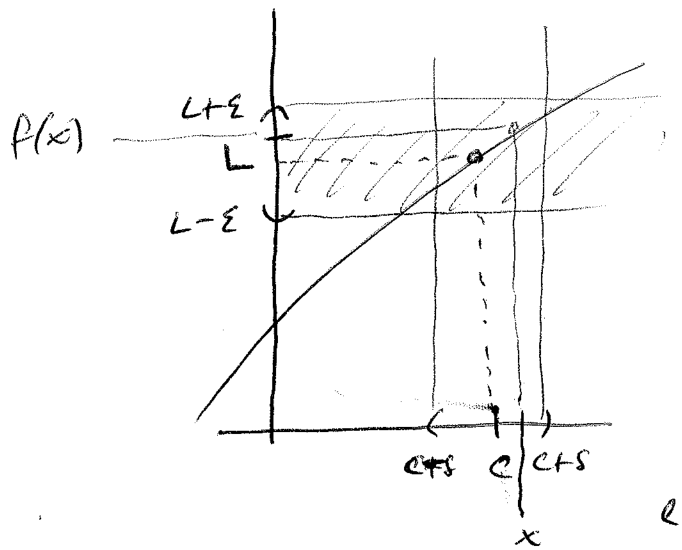
approaches any particular number in $\mathbb{R}$.

If it approaches a number L, and if every possible sequence you choose (on both sides of $x=c$) approaches L, then limit at $x=c$ of $f(x)$ exists and is L.

(4) Notice that c does not have to be in the domain of f for the limit to exist at $x=c$ (in the definition, $x \neq c$!). But the domain of f must be on both sides of $x=c$.

Notes cont'd

(5) Visually, limits are easy to see:

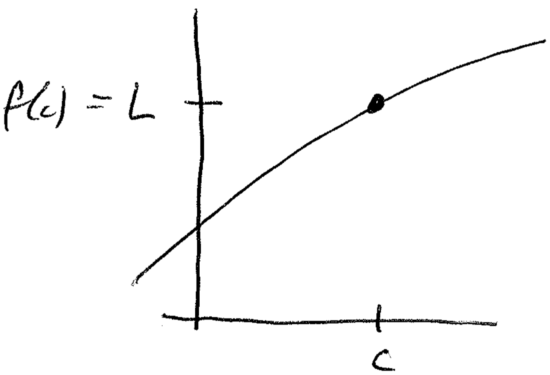


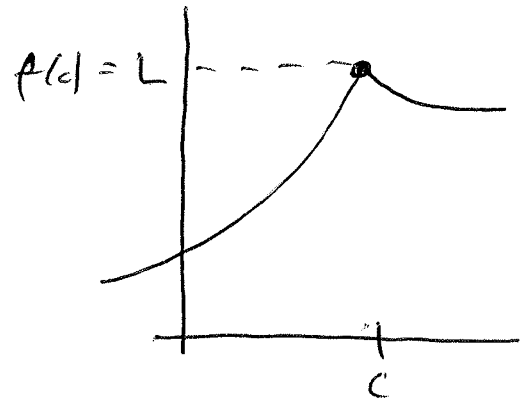
Given a small interval created using an $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ centered around L on vertical axis, can we create a small interval about $x=c$ on horiz. axis (except possibly at $x=c$), so that for every pt $x \in (c-\delta, c+\delta)$, its function value $f(x) \in (L-\varepsilon, L+\varepsilon)$. The answer in the above graph looks like yes, we can.

(6) Visually, limit are easy to see: If a function graph looks like a smooth curve, possibly with a corner, ~~...~~ through $f(c)$, or if there is a hole in the graph only at what would be $f(c)$, then the limit will exist and be $f(c)$ or where the hole is.

Notes cont'd

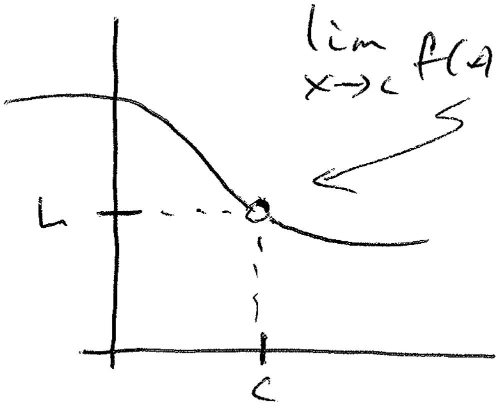
(7) Some visual examples:

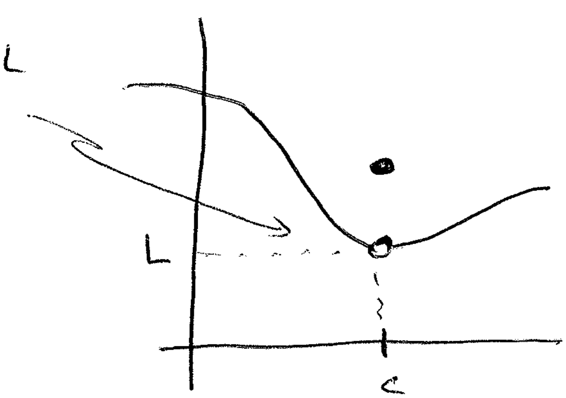




If you can draw graph of f(x) on an interval containing c without lifting your pencil from the paper, then $\lim_{n \to x} f(x) = L = f(c)$.

$\lim_{x \to c} f(x) = L$

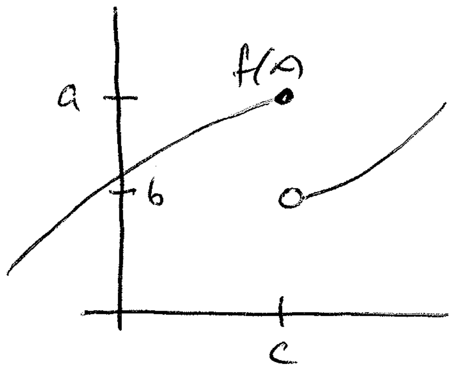


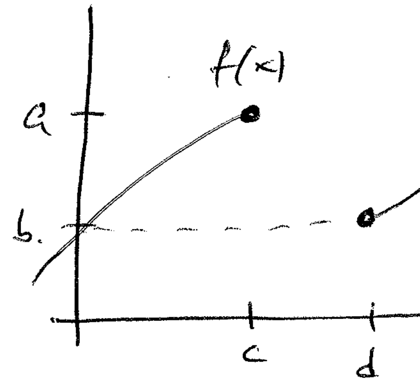


For a limit to exist of f(x) at $x=c$, it doesn't matter what happens at $x=c$, only what happens "near" $x=c$ (on both sides).

Notes cont'd,

(7) cont'd





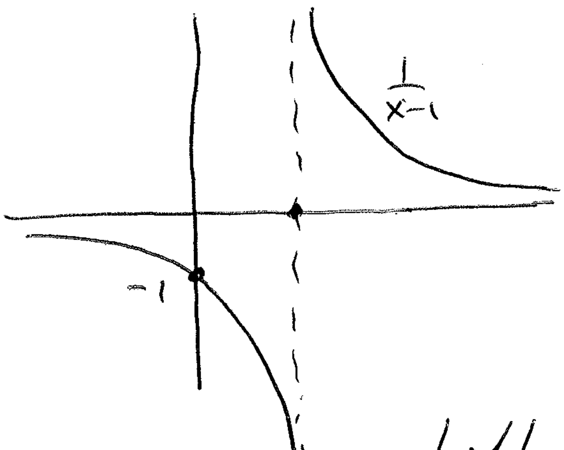
On the left, if we approach $x=c$ from pts lower than c, then the function values "go to" a, but from above c, function values go to b. Since $a \neq b$, $\lim_{x \to c} f(x)$ does not exist

On the right, one cannot create a sequence of input pts x approaching $x=c$ from numbers larger than c. Hence $\lim_{x \to c} f(x)$ cannot exist.

Notes cont'd

⑧ How else can limits fail to exist?

example $\lim_{x \to 1} \frac{1}{x-1}$ does not exist.



What happens to the function values if we choose a sequence of pts x approaching $x=1$ from the left? Try $\{1-.1, 1-.01, 1-.001, 1-.0001, \ldots\}$.

from the right? Try $\{1.1, 1.01, 1.001, 1.0001, \ldots\}$.

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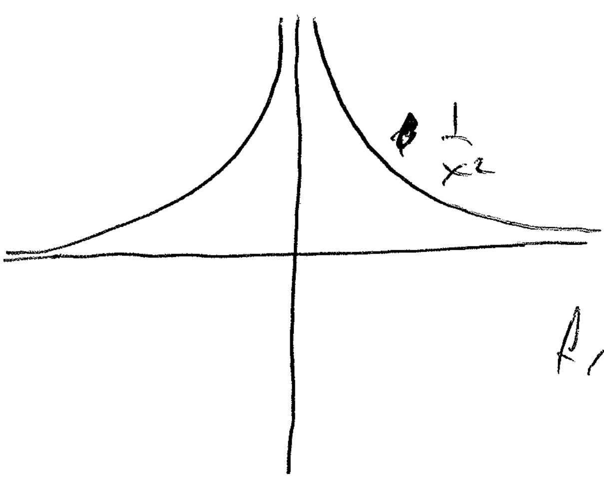
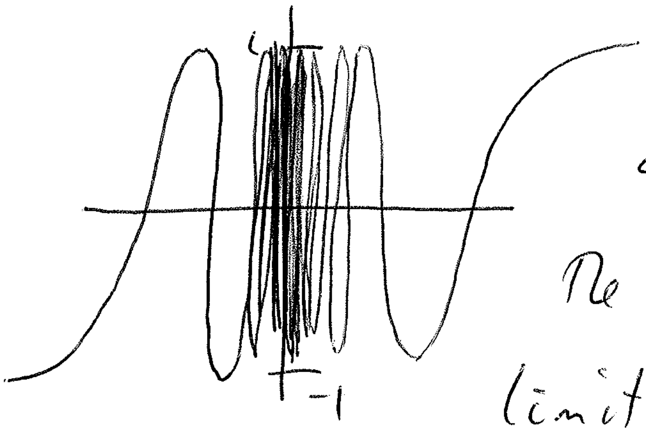
example $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.



Test the 2 sides here again. How does this case differ from the last one?

Notes cont'd

(8) cont'd $\lim_{x \to 0} \sin \frac{4\pi}{x}$ does not exist.



It is very hard to "see" the graph of this function near $x=0$. The domain is all $x \neq 0$. But the limit may exist?

Choose the sequence $a_n = \frac{1}{10^n}$. Here $\{a_n\} \to 0$. The sequence of function values

$$\left\{\sin \frac{4\pi}{a_n}\right\} = \{\sin 4\pi, \sin 10\pi, \sin 100\pi, \sin 1000\pi, \ldots\}$$
$$= \{0, 0, 0, 0, 0, \ldots\} \to 0.$$

Hence it is possible that $\lim_{x \to 0} \sin \frac{\pi}{x} = 0$.

But if instead, we chose $b_n = \frac{2}{10^n+1}$, then $\{b_n\} \to 0$

And the sequence of function values $\left\{\sin \frac{x}{b_n}\right\} =$

$$\left\{\sin \frac{\pi}{2}, \sin \frac{11\pi}{2}, \sin \frac{101\pi}{2}, \sin \frac{1001\pi}{2}, \sin \frac{10001\pi}{2}, \ldots\right\}$$
$$= \{1, -1, 1, 1, 1, 1, 1, \ldots\} \to 1.$$

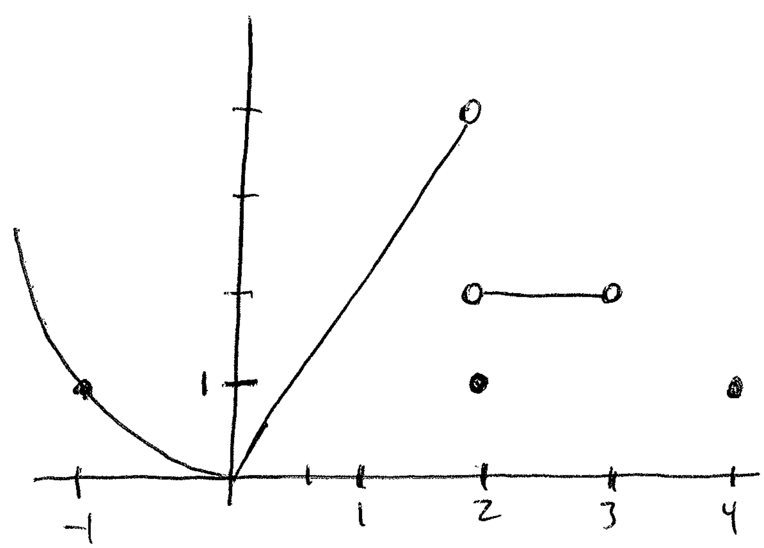
Notes cont'd.

(8) cont'd

For a limit to exist, it must be unique (else the definition will fail). Hence for $\sin(\frac{\pi}{x})$ there is no limit @ $x=0$.

(9) For piecewise defined functions, most of the time it is only on the pts where 2 pieces of the domain meet where one needs to check for limit issues:

$$f(x) = \begin{cases} x^2 & x<0 \\ 2x & 0 \le x < 2 \\ 1 & x=2 \\ 2 & x<2<3 \\ 1 & x=4 \end{cases}$$



$\lim_{x \to 3} f(x)$ does not exist

At edges of pieces

$\lim_{x \to 0} f(x) = 0$

$\lim_{x \to 2} f(x)$ does not exist

$\lim_{x \to 4} f(x)$ does not exist.