

Pay close attention to the limit laws on page 74. They will be the primary means for you to "calculate" limits of new functions using limits of sequences you already know.

But in particular, pay attention to one crucial fact: Certain limits must exist for the laws to apply!

example: The Sum Law of limits

If $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ both exist,

$$\text{then } \lim_{n \rightarrow \infty} (a_n + b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) + \left(\lim_{n \rightarrow \infty} b_n \right)$$

Caution: The Law cannot be used if the 2 limits on the right hand side do not both exist!!

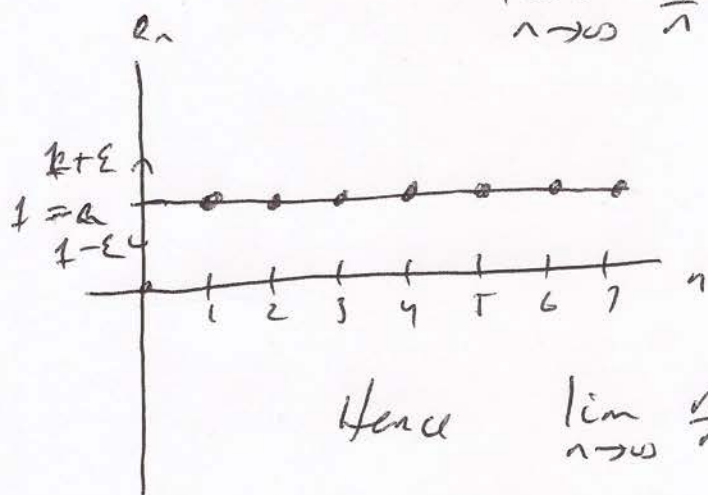
example Let $a_n = \frac{1}{n}$. Does $\{\frac{1}{n}\}_{n=1}^{\infty}$ have a limit? For $n > 0$, $\frac{1}{n} = 1$ for all $n \in \mathbb{N}$. Hence $\{\frac{1}{n}\} = \{1\} = \{1, 1, 1, \dots\}$.

By the Definition of a limit for sequences, choose $a = 1$, and check to see

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 1. \text{ Given any } \epsilon > 0, \text{ if I}$$

choose $N = 1$, then if $n > N = 1$,

$$I \text{ know } |a_n - a| < \epsilon \\ 0 = |\frac{1}{n} - 1| < \epsilon \quad \checkmark$$



$$\text{Hence } \lim_{n \rightarrow \infty} \frac{1}{n} = 1.$$

The Product Rule for limits: if $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ both exist, then

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \cdot \left(\lim_{n \rightarrow \infty} b_n \right)$$

$$\text{Try it: } \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} \left(n \cdot \frac{1}{n} \right) \stackrel{?}{=} \underbrace{\left(\lim_{n \rightarrow \infty} n \right)}_{\infty} \cdot \underbrace{\left(\lim_{n \rightarrow \infty} \frac{1}{n} \right)}_0 = ?$$

But since the 2 individual limits on the RHS do not both exist, the rule does not apply.

example Find $\lim_{n \rightarrow \infty} \frac{4n+1}{2n}$, if it exists.

Strategy: Use limit laws as they apply to find limits based on known limits.

Solution: $\lim_{n \rightarrow \infty} \frac{4n+1}{2n} = \lim_{n \rightarrow \infty} \left(\frac{4n}{2n} + \frac{1}{2n} \right)$

only if both exist (sum law) \parallel $\left(\lim_{n \rightarrow \infty} \frac{4n}{2n} \right) + \left(\lim_{n \rightarrow \infty} \frac{1}{2n} \right)$ constant multiple rule.

\parallel $\left(\lim_{n \rightarrow \infty} 2 \right) + \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \frac{1}{n}$

\parallel $2 + \frac{1}{2} (0) = 2 + 0 = \boxed{2}$.

Hence $\boxed{\lim_{n \rightarrow \infty} \frac{4n+1}{2n} = 2}$

Note: In the previous example, the 2 individual limits did exist. Hence the Sum Rule was valid.

example ~~Find~~ $\lim_{n \rightarrow \infty} \frac{4n+1}{2n} \neq \frac{\lim_{n \rightarrow \infty} (4n+1)}{\lim_{n \rightarrow \infty} 2n}$

Q: Why does the Quotient Rule for limits not apply here?

example Explain why $\lim_{n \rightarrow \infty} \sin\left(\frac{n\pi}{2}\right)$
does not exist.

Back to recursion

Q: What is the difference between the 2
sequence types?

Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = r x$, $r \neq 0$

② $a_n = f(n)$

③ $b_{n+1} = f(b_n)$

① Write out the (or few) terms:

$$\{a_n\} = \{0, r, 2r, 3r, 4r, 5r, \dots\}$$

For b_n , we cannot start without a starting value b_0 :

Let $b_0 = 0$, then $\{b_n\} = \{0, 0, 0, 0, \dots\}$.

Let $b_0 = 1$, then $\{b_n\} = \{1, r, r^2, r^3, r^4, r^5, \dots\}$.

② For different values of r , do these sequences have limits?

$$\text{Let } r=3: \{a_n\} = \{0, 3, 6, 9, 12, 18, \dots, 3n, \dots\}$$

$$b_0=0 \quad \{b_n\} = \{0, 0, 0, 0, 0, \dots\}$$

$$b_0=1 \quad \{b_n\} = \{1, 3, 9, 27, 81, \dots, 3^n, \dots\}$$

$$\text{Let } r=\frac{1}{3}: \{a_n\} = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \dots, \frac{n}{3}, \dots\}$$

$$b_0=0 \quad \{b_n\} = \{0, 0, 0, \dots\}$$

$$b_0=1 \quad \{b_n\} = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots, \frac{1}{3^n} = (\frac{1}{3})^n, \dots\}$$

$$\text{Let } r=-2: \{a_n\} = \{0, -2, -4, -6, -8, \dots, (-2)n, \dots\}$$

$$b_0=0 \quad \{b_n\} = \text{well, you know ...}$$

$$b_0=1 \quad \{b_n\} = \{1, -2, 4, -8, 16, -32, 64, \dots, (-2)^n, \dots\}$$

$$\text{Let } r=-\frac{1}{2}: \{a_n\} = \{0, -\frac{1}{2}, 1, -\frac{3}{2}, \dots, -\frac{n}{2}, \dots\}$$

$$b_0=1: \{b_n\} = \{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \dots, (-\frac{1}{2})^n, \dots\}$$

③ How many of these sequences have limits?

At least for the positive values of r , if these sequences model populations, how many have fixed populations?

example Another recursively defined sequence.

Suppose $a_{n+1} = \frac{1}{4}a_n + \frac{3}{4}$ models a population of ants (a_n measured in millions), where $a_0 > 0$ (doesn't make much sense when $a_0 = 0$; why not?).

Choose ANY starting value (population) $a_0 > 0$.

Does $\lim_{n \rightarrow \infty} a_n$ exist? If so, find it.

(a) Choose $a_0 = 2$. Then $\{a_n\} = \{2, \frac{5}{4}, \frac{17}{16}, \frac{65}{64}, \frac{257}{256}, \dots\}$.

(b) Choose $a_0 = \frac{1}{2}$. Then $\{a_n\} = \{\frac{1}{2}, \frac{7}{8}, \frac{31}{32}, \frac{127}{128}, \dots\}$.

(c) Choose $a_0 = 10$. Then $\{a_n\} = \{10, \frac{13}{4}, \frac{25}{16}, \frac{73}{64}, \dots\}$.

Can you guess the limits if they exist in each case?

To calculate like we have been doing, we would need to find the n th term (as a function of n). This is not easy and sometimes impossible!

Note: it is possible in this case, but not easy.

The book gives you the answer in the case of $a_0 = 2$, namely $a_n = \frac{4^n + 1}{4^n}$. Then they show you it is correct.

This form of "guess and check" is valid in mathematics, but requires cleverness. Not easy!

The other two are: (b) $a_0 = \frac{1}{2}$, $a_n = \frac{4^n - \frac{1}{2}}{4^n}$

(c) $a_0 = 10$, $a_n = \frac{4^n + 9}{4^n}$.

See a pattern??

For recursive sequences, there is an alternate strategy!

Look for fixed point populations. They sometimes act as limits for other sequences that start nearby.....

Suppose $a_{n+1} = f(a_n)$ for some function
 where there is a fixed point population
 at a .

Then if we start the sequence at $a_0 = a$, then

$$a_1 = f(a_0) = a, \quad a_2 = f(a_1) = f(a) = a, \quad \dots$$

and ~~the~~ the entire sequence satisfies

$$a = f(a)$$

↖ ↗
 same

Solve this equation for values of a . These are
 your fixed pt populations.

example For $a_{n+1} = \frac{1}{4}a_n + \frac{3}{4}$, we solve

$$a = \frac{1}{4}a + \frac{3}{4} \text{ to discover fixed pt pop.}$$

$$a = \frac{1}{4}a + \frac{3}{4} \Leftrightarrow \frac{3}{4}a = \frac{3}{4} \Leftrightarrow a = 1.$$

Test it: Let $a_0 = 1$, $a_1 = \frac{1}{4}(1) + \frac{3}{4} = 1$, $a_2 = 1$, $a_3 = 1, \dots$

So $a = 1$ is a fixed pt population.

Fact: If a recursively defined sequence converges, it must converge to a fixed pt population.

Go back to the 3 sequences we constructed:

$$a_0 = 2 : \{a_n\} = \left\{ 2, \frac{5}{4}, \frac{17}{16}, \frac{65}{64}, \dots, \frac{4^n + 1}{4^n}, \dots \right\}$$

$$a_0 = \frac{1}{2} : \{a_n\} = \left\{ \frac{1}{2}, \frac{7}{8}, \frac{31}{32}, \frac{127}{128}, \dots, \frac{4^n - \frac{1}{2}}{4^n}, \dots \right\}$$

$$a_0 = 10 : \{a_n\} = \left\{ 10, \frac{17}{4}, \frac{25}{16}, \frac{23}{64}, \dots, \frac{4^n + 9}{4^n}, \dots \right\}$$

By the fact above, if these sequences converge, they must converge to 1. They all do, though we won't construct the n for any choice of $\epsilon > 0$ here.

Q: Can you conclude here that

$$\lim_{n \rightarrow \infty} \frac{4^n + c}{4^n} = 1 \quad \text{for any } c \in \mathbb{R} ?$$

The answer is yes. Think about it.

This gives us a strategy to analyze recursively defined sequences:

- ① Immediately find the fixed pt populations.
- ② If your sequence approaches any one of them, it will converge to it.

That fixed pt population becomes the limit.

example Find $\lim_{n \rightarrow \infty} a_n$ if it exists, for

$$a_{n+1} = \sqrt{5a_n}, \quad a_0 = 2.$$

Solution: Any fixed pt population will satisfy

$$a = \sqrt{5a}. \quad \text{Solve this for } a:$$

$$a = \sqrt{5a} \Leftrightarrow a^2 = 5a \Leftrightarrow a^2 - 5a = 0 = (a-5)a.$$

This is solved by $a=0$, $a=5$. Sequence is $\{a_n\} = \{2, \sqrt{10}, \sqrt[4]{250}, \sqrt[8]{6250}, \dots\}$.

Hard to calculate, but sequence is increasing.

Hence by fact, $\{a_n\} \rightarrow 5$.